Spatial Descriptions and Transformations

Read Chapter 2
Spatial representation (use a legged robot as an example)

- Use coordinate system (frame) to represent spatial positions and orientation of objects
  - \((\hat{X}_E, \hat{Y}_E, \hat{Z}_E)\) set of three orthogonal unit vectors used to define an earth-fixed coordinate system
  - \((\hat{X}_B, \hat{Y}_B, \hat{Z}_B)\) set of three orthogonal unit vectors used to define a body-fixed coordinate system, original at the Center of Gravity (COG)
  - \((\hat{X}_F, \hat{Y}_F, \hat{Z}_F)\) set of three orthogonal unit vectors used to define a foot-fixed coordinate system
Fundamentals

• When we manipulate objects as we do in robotics, we need a way of describing positions and orientations of objects and the spatial relationship between them: body ↔ ground; foot (hand) ↔ body
• Positions and orientations are equally important
• Standard approach describing the position/orientation of objects:
  – Attach a coordinate system (frame) to each object
  – Vectors which position its original in space to give directions of its unit vectors
  – Frame is a description for each object which carries all the position/orientation information
  – Define the position/orientation of the frame with respect to another
Homogeneous Transformation

- Use a $4 \times 4$ matrix
- Gives position/orientation information of one frame with respect to another
- First used in graphics, also in computer vision
- Applied in robotics to describe spatial relationship
- A free body in space is said to have 6 degrees of freedom (DOF) – 3 for position and 3 for orientation
- A homogeneous transformation in general has 6 independent pieces of information for specifying these 6 values
Position vectors

• A position vector may be represented by its coordinates in any given frame:

\[ \mathbf{P} = 5\mathbf{X}_B + 4\mathbf{Y}_B + 3\mathbf{Z}_B \]

\[
\begin{bmatrix}
B \mathbf{P} \\
= \\
\begin{bmatrix}
5 \\
4 \\
3 \\
\end{bmatrix} \\
\end{bmatrix}
\]

– A leading superscript indicates the coordinate system of the reference \{B\}

• Homogeneous coordinates (H.C.):

\[
\begin{bmatrix}
B \mathbf{P} \\
= \\
\begin{bmatrix}
5 \\
4 \\
3 \\
1 \\
\end{bmatrix} \\
= \\
\begin{bmatrix}
10 \\
8 \\
6 \\
2 \\
\end{bmatrix} = \\
\begin{bmatrix}
p_x \\
p_y \\
p_z \\
w \\
\end{bmatrix}
\end{bmatrix} \\
\mathbf{P} = \frac{p_x}{w} \mathbf{X}_B + \frac{p_y}{w} \mathbf{Y}_B + \frac{p_z}{w} \mathbf{Z}_B
Position vectors (continued)

• H.C. are handy because multiplication by a constant does not change the associated vector (we will use $w=1$ always)
• Dot product $u$ and $v$: $u \cdot v = ?$ a scalar
• Cross product of $u$ and $v$: $u \times v = ?$ a vector

$$u = u_x \hat{X}_B + u_y \hat{Y}_B + u_z \hat{Z}_B$$
$$v = v_x \hat{X}_B + v_y \hat{Y}_B + v_z \hat{Z}_B$$

$$u \cdot v = u_x v_x + u_y v_y + u_z v_z = |u||v|\cos\theta$$

$$w = (u_y v_z - u_z v_y) \hat{X}_B + (u_z v_x - u_x v_z) \hat{Y}_B + (u_x v_y - u_y v_x) \hat{Z}_B$$

$$|w| = |u||v|\sin\theta$$
Homogeneous transformation - position

• A point represented in one frame carries information in 4 vectors
  – 3 for directions of unit vector and 1 for P origin of the frame
• Assume the coordinates of the point P in the body frame to be determined in the earth-fixed frame. A $4 \times 4$ matrix $^{E}T_{B}$ will do the job:

$$^{E}P = ^{E}T_{B} \cdot ^{B}P$$

• What is $^{E}T_{B}$?
Homogeneous transformation – position
(continued)

\[
\begin{align*}
E_T^B &= \begin{bmatrix}
1 & 0 & 0 & {^E}_P^x_{BO RG} \\
0 & 1 & 0 & {^E}_P^y_{BO RG} \\
0 & 0 & 1 & {^E}_P^z_{BO RG} \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

\(^{E}P_{BO RG} = \) position vector from the origin of the earth-fixed frame to the origin of body-fixed frame expressed in the earth frame.

For the diagram shown earlier, \(^{E}P_{BO RG} = [0 \ -10 \ -3 \ 1]^T\).
Homogeneous transformation – position (continued)

- Then we have:

\[ \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -10 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
5 \\
4 \\
3 \\
1
\end{bmatrix} = \begin{bmatrix}
5 \\
-6 \\
0 \\
1
\end{bmatrix} \]

\[ eP = \begin{bmatrix} \{A\} \end{bmatrix} \]

\[ AP = AT_B BP \]

Equivalent vector to \[ AP \] is a vector from \( A \) to \( B \), expressed in frame \( A \)