An introduction to nonlinear analysis of fuzzy control systems

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While the design methodology for fuzzy controllers has proven itself in certain commercial and industrial applications, there is a significant need to perform mathematical analysis of fuzzy control systems prior to implementation: (i) to verify and *certify* their behavior so that, for example, instabilities can be avoided for applications demanding highly reliable operation such as aircraft and nuclear reactor control, and (ii) to provide insight to the expert on how to modify the fuzzy controller to guarantee that performance specifications are met (e.g., to guarantee a specified rise-time or the absence of steady state tracking error). In this paper we provide a survey of, and an introduction to the area of nonlinear analysis of fuzzy control systems. We begin by overviewing several approaches to stability analysis including Lyapunov's Direct and Indirect Methods, and the Circle Criterion. We provide examples to illustrate how to design stable fuzzy control systems and test for stability, including an application of Lyapunov's direct method to Takagi–Sugeno fuzzy systems. Next, we introduce the idea of analyzing the steady state tracking error of a class of fuzzy control systems and provide examples of how to predict and reduce steady state error. Finally, we provide an introduction to the use of the describing function technique for the prediction of the existence, frequency, amplitude, and stability of limit cycles. We provide examples of limit cycle analysis and show how to design fuzzy controllers to avoid limit cycles. While our primary objective is to provide a control-theoretic introduction to, and survey of approaches to nonlinear analysis of fuzzy control systems where we utilize several existing results and provide useful tutorial examples, in the process we actually make contributions by providing, for example, the first results that show how to analyze steady state tracking error for fuzzy control systems.

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1. Introduction

We begin by explaining the role and current state-of-the-art of nonlinear analysis in fuzzy control system design. Next, we overview the work that has directly influenced the development of this tutorial survey and outline the contents of this paper.

1.1. Motivation for nonlinear analysis

The fuzzy controller design methodology primarily involves distilling human expert knowledge about how to control a system into a set of rules. While a significant amount of attention has been given to the advantages of the heuristic fuzzy control design methodology (which is similar to an expert system construction methodology), relatively little attention has been given to its possible disadvantages. For example, the following questions are cause for concern:

- Will the behaviors observed by a human expert include all possibly unforeseen situations that can occur due to disturbances, noise, or plant parameter variations?
- Can the human expert realistically and reliably foresee problems that could arise from *closed-loop* system instabilities or limit cycles?
- Will the expert really know how to incorporate stability criteria and performance objectives (e.g. rise-time, overshoot, and tracking specifications) into a rule-base to ensure that reliable operation can be obtained?

These questions may seem even more troublesome if: (i) the control problem involves a "critical environment" where the failure of the control system to meet performance objectives could lead to loss of human life or an environmental disaster (e.g., in aircraft or nuclear power plant control), or (ii) if the human expert's knowledge implemented in the fuzzy controller is somewhat inferior to that of a very experienced specialist that we expect to have designed the control system (different designers have different levels of expertise). Clearly then for some applications there is a need for a methodology to develop, implement, and evaluate fuzzy controllers to ensure that they are reliable in meeting their performance specifications.

The standard control engineering methodology involves repeatedly coordinating the use of:

1. modeling,
2. controller (re)design,
3. simulation,
4. mathematical analysis,
5. implementation/evaluation.

to develop control systems. Next, we will examine the relevance of this established methodology to the development of fuzzy control systems. Engineering a fuzzy control system uses many ideas from the standard control engineering methodology except in fuzzy control it is often said that a formal mathematical model is assumed unavailable so that mathematical analysis is impossible. While it is often the case that it is difficult, impossible, or cost-prohibitive to develop an *accurate* mathematical model for many processes, it is almost always possible for the control engineer to specify some type of *approximate* model of the process (afterall, we do know what physical object we are trying to control). Indeed, it has been our experience that most often the control engineer developing a fuzzy control system *does* have a mathematical model available. While it may not be used directly in controller design, it is often used in simulation to evaluate the performance of the fuzzy controller before it is implemented. Certainly there are some applications where one can design a fuzzy controller and evaluate its performance directly via an implemenation. In such applications one is not overly concerned with the failure of the control system (e.g., for some commercial products such as washing machines or a shaver) and therefore there may be no need for a mathematical model for conducting simulation-based evaluations before implementation. In other applications there is the need for a high level of confidence in the reliability of the fuzzy control system before it is implemented.

In addition to simulation-based studies, one approach to enhancing our confidence in the reliability of fuzzy control systems is to use the mathematical model of the plant and nonlinear analysis for (i) verification of stability and performance specifications and (ii) possible re-design of the fuzzy controller. Some may be confident that a true expert would (i) never need anything more than intuitive knowledge for rule-base design, and (ii) never design a faulty fuzzy controller. However, a true expert will certainly use all available information to ensure the reliable operation of a control system including approximate mathematical models, simulation, nonlinear analysis, and experimentation. We emphasize, however, that mathematical analysis cannot alone provide the definitive answers about the reliability of the fuzzy control system since *such analysis proves properties about the model of the process, not the actual physical process*. It can be argued that a mathematical model is never a perfect representation of a physical process; hence, while nonlinear analysis may appear to provide definitive statements about control system reliability, it is understood that such statements are only accurate to the extent that the mathematical model is accurate. Nonlinear anlaysis does not replace the use of common sense and evaluation via simulations and experimentation; it simply assists in providing a rigorous engineering evaluation of a fuzzy control system before it is implemented.

It is important to note that the advantages of fuzzy control often become most apparent for very complex problems where we have an intuitive idea about how to achieve high performance control. In such control applications an accurate mathematical model is so complex (i.e., high order, nonlinear, stochastic, with many inputs and outputs) that it is sometimes not very useful for the analysis and design of conventional control systems (since assumptions needed to apply conventional control are often violated). The conventional control engineering approach to this problem is to use an approximate mathematical model that is accurate enough to characterize the essential plant behavior, yet simple enough so that the necessary assumptions to apply the analysis and design techniques are satisfied. However, due to the inaccuracy of the model, upon implementation the developed controllers often need to be tuned via the "expertise" of the control engineer. The fuzzy control approach, where explicit characterization and utilization of control expertise is used earlier in the design process, largely avoids the problems with model complexity that are related to design (i.e., for the most part fuzzy control system design does not depend on a mathematical model[2]). However, the problems with

[2]Fuzzy control system design can depend on a mathematical model if one needs it to perform simulations to gain insight into how to choose the rule-base and membership functions.

model complexity that are related to analysis have not been solved (i.e., analysis of fuzzy control systems critically depends on the form of the mathematical model); hence, it is often difficult to apply nonlinear analysis techniques to the applications where the advantages of fuzzy control are most apparent! For instance, existing results for stability analysis of fuzzy control systems typically require that the plant model be deterministic, satisfy some continuity constraints, and sometimes require the plant to be linear or "linear-analytic". The only results for analysis of steady state tracking error of fuzzy control systems, which are introduced here, and the existing results on the use of describing functions for analysis of limit cycles essentially require a linear time-invariant plant (or one that has a special form so that the nonlinearities can be bundled into one nonlinear component in the loop). The current status of the field, as characterized by these limitations, coupled with the importance of nonlinear analysis of fuzzy control systems, make it an open area for research where an introductory survey can help establish the necessary foundations for a bridge between the fuzzy control and nonlinear analysis communities[3].

Overall, as an introduction and survey, the objectives of this paper are to:

1. Help teach sound techniques for the construction of fuzzy controllers by alerting the designer to some of the pitfalls (e.g., instabilities, limit cycles, steady state errors) that can occur if a rule-base is constructed improperly;
2. Provide insights into how to modify the fuzzy controller rule-base to guarantee that performance specifications are met (thereby helping make the fuzzy controller design process more systematic); and
3. Provide a motivation and foundation for future work in the area of using nonlinear analysis as a technique to enhance a rigorous control engineering evaluation for the verification and certification of fuzzy control systems that are to operate in critical environments.

[3]Clearly fuzzy control technology is leading the theory; the practioner will go ahead with the design and implementation of many fuzzy control systems without the aid of nonlinear analysis. In the mean time, theorists will attempt to develop a mathematical theory for the verification and certification of fuzzy control systems. This theory will have a synergistic effect by driving the development of fuzzy control systems for applications where there is a need for highly reliable implementations.

In particular, in this tutorial survey paper we will show how the use of the standard approach to fuzzy controller design can result in limit cycles, steady state tracking errors, and instabilities even for simple linear plants. We will provide an introduction to the use of nonlinear analysis techniques (i.e., stability analysis, describing function analysis, and steady state error analysis) for fuzzy control system analysis and show how they can aid in picking the membership functions in a fuzzy controller to avoid limit cycles, instabilities, and ultimately to meet a variety of closed-loop specifications. Since most fuzzy control systems are "hybrid" in that the controller contains a linear portion (e.g., an integrator or differentiator) as well as a nonlinear portion (a fuzzy system), we will show how to use nonlinear analysis to design both of these portions of the fuzzy control system.

Essentially, we will show in this paper how to exploit the information that can be obtained from the use of a mathematical model (even if the model is inexact or represents the linearization of a nonlinear plant) to enhance the performance of a fuzzy control system. We do not suggest that the conventional fuzzy control system design approach be discarded, but that it be augmented with information from mathematical modeling and analysis if such information is available and the application dictates its use. This paper takes a step in the direction of joining conventional nonlinear control analysis and design approaches with the standard approach to fuzzy control system design. We acknowledge the value of both approaches and are trying to find the best parts of both and combine them to improve the control system design process. For a more detailed discussion on (i) the general relationships between conventional and intelligent control, and (ii) mathematical modeling and nonlinear analysis of more general intelligent control systems (including expert control systems) see [3, 30, 35–37].

1.2. Literature overview and paper summary

As the reader will see, the introduction to nonlinear analysis of fuzzy control systems developed here requires a good knowledge of "classical control" (primarily the Laplace transform, Root locus, Bode and Nyquist plots, and system type and tracking error) and at least an introductory course on differential equations, but does not require the reader to have studied more advanced topics in mathematics and control.

Throughout the development of this paper it will become clear that we advocate the use of graphical approaches for nonlinear analysis of fuzzy control systems since such approaches (i) fit well with the overall philosophy of using intuition to design fuzzy controllers, and (ii) adopt a similar philosophy to what is used in classical control where graphs and intuition have played a major role in the successful construction of many control systems. In this section we overview the contents this paper and explain the relevance of other work to the tutorial survey that we present here. We begin by overviewing the relevant work in stability analysis.

The work in [11] and [14] presents Lyapunov methods for analyzing stability of fuzzy control systems. The authors in [27, 28] also use Lyapunov's Direct Method and the Generalized Theorem of Popov [32] to provide sufficient conditions for fuzzy control system stability. In [4], stability indices for fuzzy control systems are established using phase portraits (of course standard phase plane analysis [24] can be useful in characterizing and understanding the dynamical behavior of low order fuzzy control systems [19]). Related work is given in [16]. The Circle Criterion [32] is used in [38] and [39] to provide sufficient conditions for fuzzy control system stability. Related work is given in [7, 23, 41].

An area that is receiving an increasing amount of attention is in the area of stability analysis of fuzzy control systems where the fuzzy control system is developed using ideas from sliding mode control or where Takagi–Sugeno fuzzy systems [43] are used in a gain scheduling type of control [15, 33, 34]. Here, our treatment of the stability of Takagi–Sugeno fuzzy systems is based on the work in [44, 46, 48]. Extensions to this work that focus on robustness can be found in [45, 47] and work focusing on the use of linear matrix inequality (LMI) methods for analysis and controller construction is provided in [45, 50, 51, 53, 54].

The characterization and analysis of the stability of fuzzy dynamical systems is studied in [26]. Furthermore, approximate analysis of fuzzy systems is studied by the authors in [12, 13, 17] using the "cell-to-cell mapping approach" from [20, 21].

In Section 2 we overview the notation that we use to discuss fuzzy controllers and in Section 3 we introduce Lyapunov's direct and indirect methods and provide a tutorial example of how to use Lyapunov's indirect method for stability analysis of a fuzzy control system for an inverted pendulum. In addition we show how to use Lyapunov's direct method for stability analysis of Takagi–Sugeno fuzzy systems (due to space constraints we cannot cover the Takagi–Sugeno fuzzy system but refer the reader to [15, 43] for a nice introduction). In addition, in Section 3.2 we provide a tutorial example on the use of the Circle Criterion to study absolute stability of fuzzy control systems.

Improvement of steady state tracking error in control systems is of fundamental importance. To date, for fuzzy control systems the best one could do is attempt to reduce tracking error by heuristic re-design of the fuzzy controller rule-base. In Section 4, we will show that the approach in [40] for analysis of steady state errors in nonlinear control systems can be applied to fuzzy control systems. A tutorial example will be used to illustrate how to predict and reduce steady state errors in fuzzy control systems.

Nonlinearities in control systems, such as fuzzy controllers, sometimes cause limit cycles (isolated periodic orbits [24]) in the system response. Using the Circle Criterion we can sometimes prove the absolute stability of a system and therefore the absence of limit cycles. Likewise, using a steady state error prediction procedure we can sometimes prove the absence of limit cycles if we show that the steady state error is zero. However, neither of these techniques will predict the amplitude or frequency of limit cycles when they exist. In Section 5 we will illustrate how describing function analysis [5] can be used to predict the existence, stability, frequency, and amplitude of limit cycles in fuzzy control systems. This type of analysis has already been examined in [25] and very recently in [6]. Even though the author in [6] uses a quantized fuzzy controller and mathematically determined describing functions while we use continuous fuzzy controllers and experimentally determined describing functions, there is little conceptual difference between what we do in this paper and what is done in [6]. However, the work in this paper was done simultaneously to and independent of the work in [6]. Our work will differ from that in [25] in that we will use experimentally determined describing functions whereas in [25] the describing function is determined for a "multilevel relay" model of a specific class of fuzzy controllers. Very recently there has been some additional work on the use of the describing function to analyze and design fuzzy control systems presented in [1, 2]. In Section 5, after providing a brief overview of the theory of describing functions, we an several example of how to use the describing function method for the prediction of limit cycles and for designing the fuzzy controller to eliminate limit cycles.

In Section 6 we provide some concluding remarks where we overview the contents and contributions of this paper, outline the limitations of the nonlinear analysis techniques, and provide several research directions.

2. Fuzzy control

In this section we will introduce the fuzzy control system to be investigated and briefly examine the nonlinear characteristics of the fuzzy controller. For the applications in this paper (except some in Section 3.1), the closed-loop systems will be as shown in Fig. 1 (where we assume that $G(s)$ is a single input single output (SISO) linear system) or they will be modified slightly so that the fuzzy controller is in the feedback path. We will be using both SISO and MISO (multiple input single output) fuzzy controllers as they are defined in the next subsections. The intent of this section is to merely provide an explanation of the notation that we use in discussing fuzzy controllers. For an introduction to fuzzy control see [29, 55]. For a detailed comparative anlaysis of fuzzy controllers and linear controllers and more details on the nonlinear characteristics of fuzzy controllers see [8–10, 52].

2.1. Proportional fuzzy controller

For the "proportional fuzzy controller" (as the SISO fuzzy controller is sometimes called) shown in Fig. 2 the rule-base can be constructed in a symmetric fashion typically with rules of the following form:

1. If e is NB then u is NB.
2. If e is NM then u is NM.
3. If e is NS then u is NS.
4. If e is ZE then u is ZE.
5. If e is PS then u is PS.
6. If e is PM then u is PM.
7. If e is PB then u is PB.

where NB, NM, NS, ZE, PS, PM, and PB are "linguistic values". The membership functions for the premises and consequents of the rules are symmetric, normal, and uniformly distributed triangular membership functions as shown in Fig. 3. Notice in Fig. 3 that the widths of the membership functions are parameterized by A and B. Throughout this paper, unless it is indicated otherwise, the same rule-base and similar uniformly distributed membership functions will be used for all applications (where if the number of input and output membership functions and rules increase our analysis approaches work in a similar manner). The fuzzy controller will be adjusted by changing the values of A and B. The manner in which these values affect the nonlinear map that the fuzzy controller implements will be discussed below. The fuzzy inference block operates by using the product to combine the conjunctions in the premise of the rules and in the representation of the fuzzy implication. Singleton fuzzification is used and defuzzification is performed using the centroid method (for further explanation see [29, 55]).

The SISO fuzzy controller described above implements a static nonlinear input-output (I/O) map between its input $e(t)$ and output $u(t)$ (we assume throughout the paper that the fuzzy controller is designed so that the existence and uniqueness of the solution of the differential equation describing the closed-loop system is guaranteed). The particular shape of the nonlinear map depends on the rule-base, inference strategy, fuzzification, and defuzzification strategy utilized by the fuzzy controller. Two I/O maps for different rule-bases implemented in the fuzzy controller are shown in Fig. 4 with $A = B = 1$. Rule-Base 1 is of the symmetric form described at the beginning of this subsection while Rule-Base 2 uses the same values of A and B as Rule-Base 1 but the consequents of the rule-base were chosen differently. Modifications to the fuzzy controller can provide an infinite variety of such I/O maps. Notice, however, that there is a marked similarity between the I/O map in Fig. 4(a) and the standard $saturation$ nonlinearity. In fact, the parameters A and B from the fuzzy controller are equivalent to the saturation parameters of the standard saturation nonlinearity, i.e., B is the level at which the output saturates and A is the value of $e(t)$ at which the saturation of $u(t)$ occurs. Because the I/O map is odd, $-B$ is the saturation level for $e(t) \leq -A$ and $-A$ is the value of $e(t)$ where the saturation occurs. By modifying A and B (and hence moving the input and output membership functions) we can change the I/O map nonlinearity and its effects on the system. Throughout this introductory paper we will always use rules in the form of Rule-Base 1. We emphasize, however, that the nonlinear analysis techniques used in this paper will work in the same manner

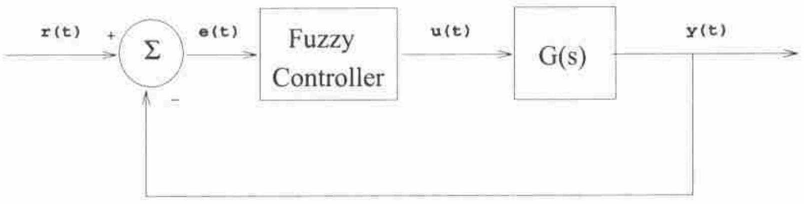


Fig. 1. Fuzzy control system.

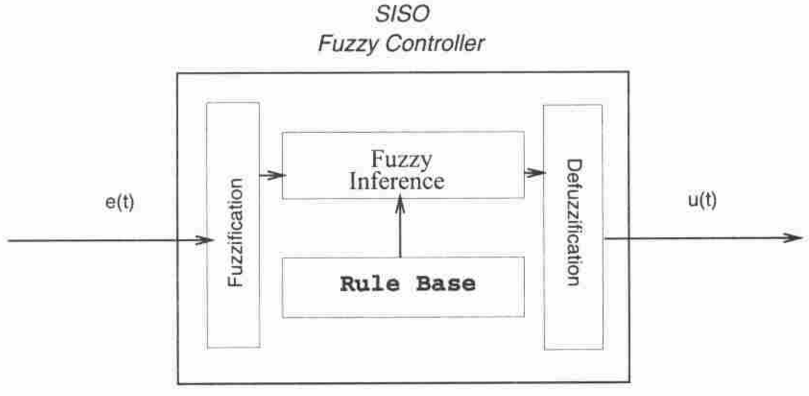


Fig. 2. Fuzzy controller.

for other types of rule-bases (and different fuzzification, inference, and defuzzification techniques).

2.2. *Proportional-derivative fuzzy controller*

There are many different types of fuzzy controllers we could examine for the multiple input single output (MISO) case. We will constrain ourselves to the two input "proportional-derivative fuzzy controller" (as it is sometimes called). This controller, shown in Fig. 5, is similar to our SISO fuzzy controller with the addition of the second input, $\dot{e}$. In fact, the membership functions on the universes of discourses and linguistic values NB, NM, NS, ZE, PS, PM, and PB for e and u are the same as shown in Fig. 3 and will still be adjusted using the parameters A and B respectively. The membership functions on the universe of discourse and the linguistic values for the second input, $\dot{e}$, are the same as for e with the exception that the adjustment parameter will be denoted by D. Therefore there are now three parameters for changing the fuzzy controller: A, B, and D. Assuming that there are seven membership functions on each input universe of discourse, there are 49 possible rules that can be put in the rule-base. A typical rule will take on the form:

IF e is NB **AND** $\dot{e}$ is NB **THEN** u is NB.

The complete set of rules is shown in tabulated form in Fig. 6. In Fig. 6 the premises for the input e are represented by the linguistic values found in the top row, the premises for the input $\dot{e}$ are represented by the linguistic values in the left most column, and the linguistic value representing the consequent for each of the 49 rules can be found at the intersection of the row and column of the appropriate premises. The shaded section of Fig. 6 is the representation of the above rule "**IF** e is NB **AND** $\dot{e}$ is NB **THEN** u is NB". The remainder of the MISO fuzzy controller is similar to the SISO fuzzy controller (i.e., singleton fuzzification, the product for the premise and fuzzy implication, and centroid defuzzification are used).

We can also construct an I/O map for this MISO fuzzy controller. While a three dimensional plot of this map is possible, we show a two dimensional plot of $u(t)$ for $A = B = D = 1$ in Fig. 7. In this figure each line represents $u(t)$ for a different value of $\dot{e}(t)$. The parameters A and B have the same effect as with the SISO fuzzy controller. By changing the values A, B, and D we will be able to change the effect of the MISO fuzzy controller on the closed-loop system

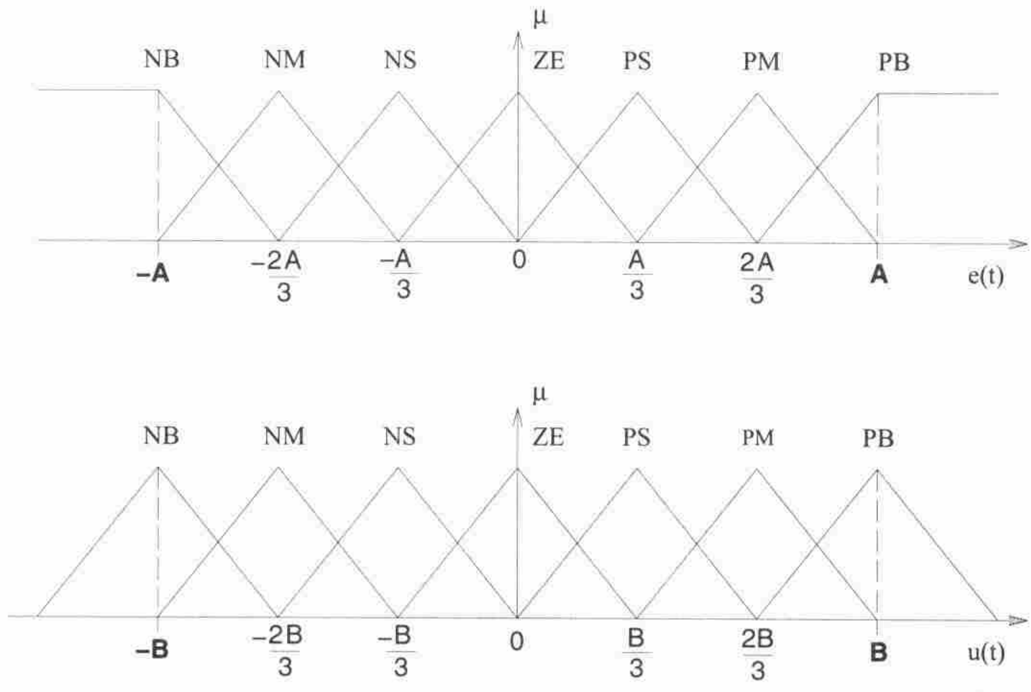


Fig. 3. Membership functions for $e(t)$ and $u(t)$.

without reconstructing the rule-base or any other portion of the fuzzy controller. Again, we emphasize that while we use this particular fuzzy controller which is conveniently parameterized by A, B, and D, the approaches to nonlinear analysis in this paper work in a similar manner for fuzzy controllers that use other membership functions, rule-bases, inference engines, and fuzzification and defuzzification strategies.

3. Stability analysis

Often the designer is first concerned about investigating the stability properties of a fuzzy control system since it is often the case that if the system is unstable there is no chance that any other performance specifications will hold. For example, if the fuzzy control system for a nuclear reactor is unstable, one would be more concerned with the possibility of a melt-down rather than with the efficiency of energy production. In this section we overview three approaches to stability analysis of fuzzy control systems: Lyapunov's direct and indirect methods, and the Circle Criterion approach to the analysis of absolute stability.

3.1. Lyapunov stability analysis

3.1.1. Theory

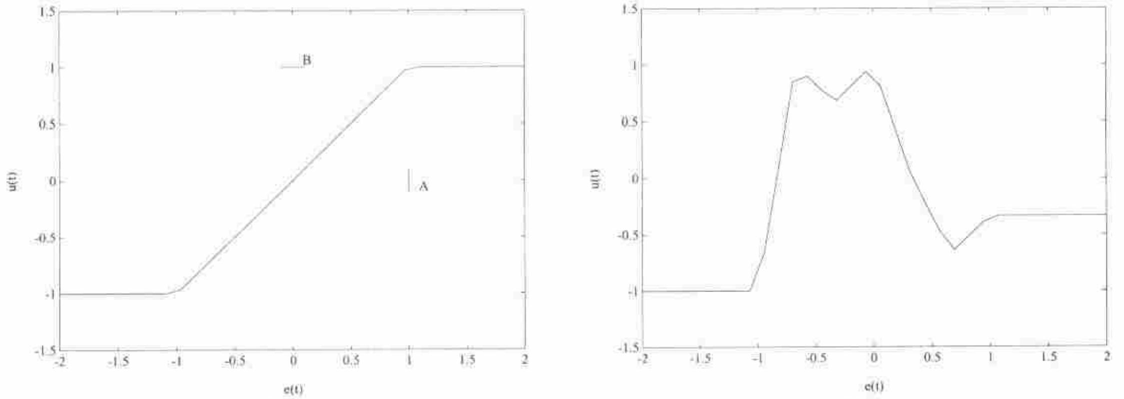
Following [24, 31] suppose that a dynamical system is represented with

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $x \in \Re^n$ and $f : D \to \Re^n$ with $D = \Re^n$ or $D = B(h)$ for some $h > 0$ where $B(h) = \{x \in \Re^n : |x| < h\}$ and $|\cdot|$ is a norm on $\Re^n$ (e.g., $|x| = \sqrt{(x^t x)}$). Assume that for every x_0 the initial value problem

$$\dot{x}(t) = f(x(0)), \quad x(0) = x_0 \tag{2}$$

possesses a unique solution $\phi(t, x_0)$ which depends continuously on x_0. A point $x_e \in \Re^n$ is called an "equilibrium point" of (1) if $f(x_e) = 0$ for all $t \geq 0$. An equilibrium point x_e is an "isolated equilibrium point" if there is an $h' > 0$ such that $B(x_e, h') = \{x \in \Re^n : |x - x_e| < h'\} \subset \Re^n$ contains no other equilibrium points besides x_e. As is standard we will assume that the equilibrium of interest is an isolated equilibrium located at the origin of $\Re^n$. This assumption results in no loss of generality since if $x_e \neq 0$ is an equilibrium of (1) and we let $\bar{x}(t) = x(t) - x_e$ then $\bar{x} = 0$ is an equilibrium of the transformed system $\dot{\bar{x}}(t) = \bar{f}(\bar{x}(t)) = f(\bar{x}(t) + x_e)$.



(a) (b)

Fig. 4. I/O Map of the Proportional Fuzzy Controller: (a) Rule-Base 1, (b) Rule-Base 2.

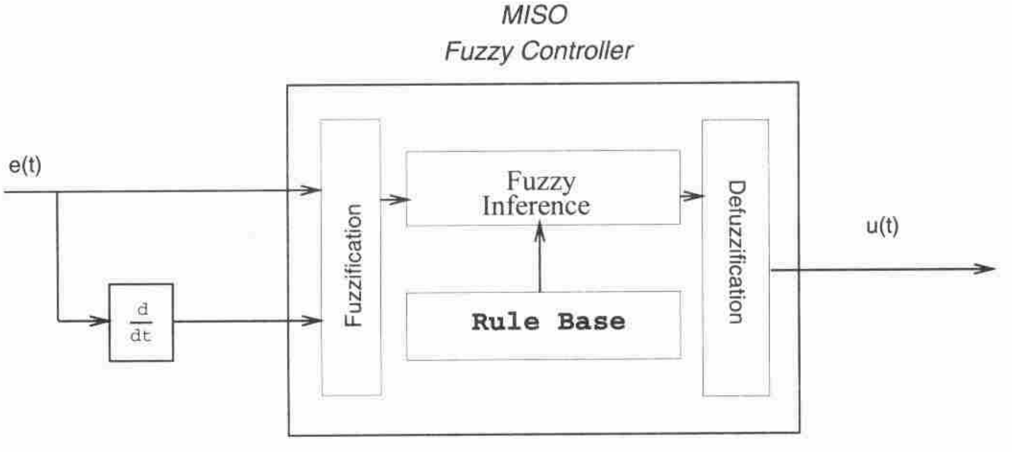


Fig. 5. Block diagram for the MISO fuzzy controller.

The equilibrium $x_e = 0$ of (1) is "stable" (in the sense of Lyapunov) if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $|\phi(t, x_0)| < \epsilon$ for all $t \geq 0$ whenever $|x_0| < \delta(\epsilon)$. A system that is not stable is called "unstable". The equilibrium $x_e = 0$ of (1) is said to be asymptotically stable if it is stable and there exists $\eta > 0$ such that $\lim_{t\to\infty} \phi(t, x_0) = 0$ whenever $|x_0| < \eta$. The set $X_d \subset \Re^n$ of all $x_0 \in \Re^n$ such that $\phi(t, x_0) \to 0$ as $t \to \infty$ is called the "domain of attraction" of the equilibrium $x_e = 0$ of (1). The equilibrium $x_e = 0$ is said to be "globally asymptotically stable" if $X_d = \Re^n$.

The stability results for an equilibrium $x_e = 0$ of (1) that we provide next depend on the existence of an appropriate "Lyapunov function" $V : D \to \Re$ where $D = \Re^n$ for global results and $D = B(h)$ for some $h > 0$, for local results. If V is continuously differentiable with respect to its arguments then the derivative of V with respect to t along the solutions of (1) is

$$\dot{V}_{(1)}(x(t)) = \nabla V(x(t))^t f(x(t)) \tag{3}$$

where $\nabla V(x(t)) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right]^t$ is the gradient of V with respect to x.

Lyapunov's direct method:

1. Let $x_e = 0$ be an equilibrium for (1). Let $V : B(h) \to \Re$ be a continuously differentiable function on $B(h)$ such that $V(0) = 0$ and $V(x) > 0$ in $B(h) - \{0\}$, and $\dot{V}_{(1)}(x) \leq 0$ in $B(h)$. Then $x_e = 0$ is stable. If in addition, $\dot{V}_{(1)}(x) < 0$ in $B(h) - \{0\}$ then $x_e = 0$ is asymptotically stable.

$\dot{e}$ \ e	NB	NM	NS	ZE	PS	PM	PB
NB	NB	NS	PS	PB	PB	PB	PB
NM	NB	NM	ZE	PM	PM	PB	PB
NS	NB	NM	NS	PS	PM	PB	PB
ZE	NB	NM	NS	ZE	PS	PM	PB
PS	NB	NB	NM	NS	PS	PM	PB
PM	NB	NB	NM	NM	ZE	PM	PB
PB	NB	NB	NB	NB	NS	PS	PB

Fig. 6. Table of P-D fuzzy controller rules.

2. Let $x_e = 0$ be an equilibrium for (1). Let $V : \Re^n \to \Re$ be a continuously differentiable function such that $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$, $|x| \to \infty$ implies that $V(x) \to \infty$, and $\dot{V}_{(1)}(x) < 0$ for all $x \neq 0$, then $x_e = 0$ is a globally asymptotically stable.

Let $\frac{\partial f}{\partial x} = \left[\frac{\partial f_i}{\partial x_j}\right]$ denote the $n \times n$ "Jacobian matrix". For the next result assume that $f : D \to \Re^n$ where $D \subset \Re^n$, that $x_e \in D$, and that f is continuously differentiable.

Lyapunov's indirect method:

Let $x_e = 0$ be an equilibrium point for the nonlinear system (1). Let the $n \times n$ matrix

$$\bar{A} = \frac{\partial f}{\partial x}(x)\bigg|_{x=x_e=0} \tag{4}$$

then

1. The origin $x_e = 0$ is asymptotically stable if $\text{Re}[\lambda_i] < 0$ (the real part of λ_i) for all eigenvalues λ_i of $\bar{A}$;
2. The origin $x_e = 0$ is unstable if $\text{Re}[\lambda_i] > 0$ for one or more eigenvalues of $\bar{A}$; and
3. If $\text{Re}[\lambda_i] \leq 0$ for all i with $\text{Re}[\lambda_i] = 0$ for some i where the λ_i are the eignevalues of $\bar{A}$ then we cannot conclude anything about the stability of $x_e = 0$ from Lyapunov's indirect method.

3.1.2. Example

As outlined in the Introduction there have been several researchers who have investigated the use of Lyapunov stability theory for analysis of fuzzy control systems. Here we will illustrate the use of Lyapunov's indirect method for stability analysis of a simple inverted pendulum.

A simple model of the inverted pendulum is given by [24]

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{\ell}\sin(x_1) - \frac{k}{m}x_2 + \frac{1}{m\ell^2}T, \end{aligned} \tag{5}$$

where $g = 9.81$, $\ell = 1.0$, $m = 1.0$, $k = 0.5$, x_1 is the angle (in radians) the pendulum makes relative to its vertical (downward position), x_2 is the angular

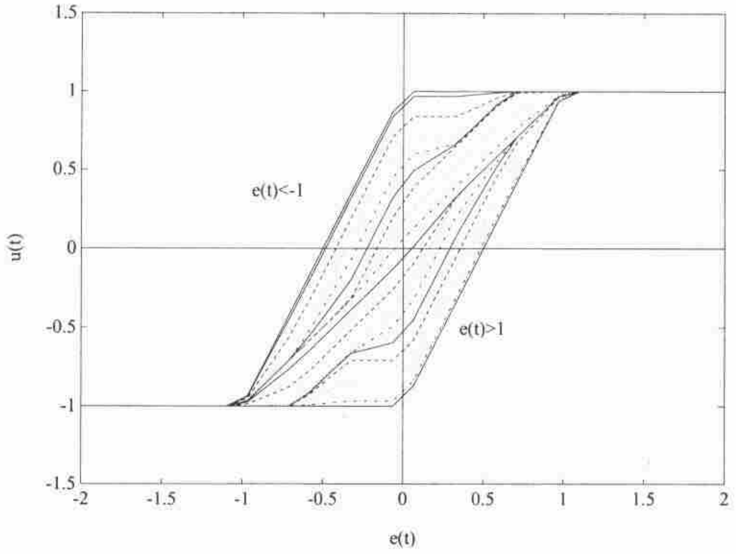


Fig. 7. 2-Dimensional P-D fuzzy controller I/O map.

velocity (in radians per second), and T is the control input.

If we assume that $T = 0$ then there are two distinct isolated equilibrium points, one in the downward position $[0\ 0]^t$ and one in the inverted position $[\pi\ 0]^t$. Since we are interested in the control of the pendulum about the inverted position we need to translate the equilibrium by letting $\bar{x} = x - [\pi\ 0]^t$. From this we obtain

$$\begin{aligned}\dot{\bar{x}}_1 &= \bar{x}_2 = \bar{f}_1(\bar{x}),\\ \dot{\bar{x}}_2 &= \frac{g}{\ell}\sin(\bar{x}_1) - \frac{k}{m}\bar{x}_2 + \frac{1}{m\ell^2}T = \bar{f}_2(\bar{x}),\end{aligned} \tag{6}$$

where if $T = 0$ then $\bar{x} = 0$ corresponds to the equilibrium $[\pi\ 0]^t$ in the original system (5) so studying the stability of $\bar{x} = 0$ corresponds to studying the stability of the fuzzy control system about the inverted position. Now, it is traditional to omit the cumbersome bar notation in (6) and study the stability of $x = 0$ for the system

$$\begin{aligned}\dot{x}_1 &= x_2 = f_1(x),\\ \dot{x}_2 &= \frac{g}{\ell}\sin(x_1) - \frac{k}{m}x_2 + \frac{1}{m\ell^2}T = f_2(x),\end{aligned} \tag{7}$$

with the understanding that we are actually studying the stability of (6). Assume that the fuzzy controller denoted by $\Phi(x_1, x_2)$, which utilizes x_1 and x_2 as inputs to generate T as an output, is designed so that f is continuously differentiable and that D is a neighborhood of the origin. For (7)

$$\begin{aligned}\bar{A} &= \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=0} \\ &= \left.\begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} + \frac{1}{m\ell^2}\frac{\partial T}{\partial x_1} & -\frac{k}{m} + \frac{1}{m\ell^2}\frac{\partial T}{\partial x_2} \end{bmatrix}\right|_{x=0}.\end{aligned} \tag{8}$$

To ensure that the eigenvalues λ_i, $i = 1, 2$, of $\bar{A}$ are in the left half of the complex plane it is sufficient that

$$\begin{aligned}&\lambda^2 + \left(\frac{k}{m} - \frac{1}{m\ell^2}\frac{\partial T}{\partial x_2}\right)\lambda \\ &\qquad + \left(-\frac{g}{\ell} - \frac{1}{m\ell^2}\frac{\partial T}{\partial x_1}\right) = 0\end{aligned} \tag{9}$$

where $x = 0$ has its roots in the left half plane. Equation (9) will have its roots in the left half plane if each of its coefficients are positive (this can be seen by use of the Routh Criterion [18]); hence if we substitute the values of the model parameters we need

$$\left.\frac{\partial T}{\partial x_2}\right|_{x=0} < 0.5, \quad \left.\frac{\partial T}{\partial x_1}\right|_{x=0} < -9.81 \tag{10}$$

to ensure asymptotic stability. To show that these conditions are met we design a fuzzy controller (different from the one that is given in Section 2) and simply plot the controller surface and check these conditions graphically. We omit the details due to space constraints.

3.1.3. Stability analysis of Takagi–Sugeno fuzzy systems

While in the remainder of this paper we consider nonlinear analysis of fuzzy control systems where the fuzzy controller is the "standard" one, in this one subsection we consider the case where the plant and controller are Takagi–Sugeno fuzzy systems (due to space constraints we cannot provide a detailed introduction to the Takagi–Sugeno fuzzy system and hence must refer the reader unfamiliar with these to [15, 43, 46]). While here we will study the continuous-time case, in [48] the authors focus on the discrete-time case.

The fuzzy system defined in the previous section will be referred to as a "standard fuzzy system." In this section we will define a Takagi–Sugeno fuzzy system [43]. For the Takagi–Sugeno fuzzy system we use singleton fuzzification and the i-th MISO rule has the form

If $\tilde{x}_1$ is $\tilde{A}_1^j$ **and** $\tilde{x}_2$ is $\tilde{A}_2^k$ **and**, …, **and** $\tilde{x}_n$ is $\tilde{A}_n^l$
Then $c_i = a_{i,0} + a_{i,1}x_1 + \cdots + a_{i,n}x_n$

(where the $a_{i,j}$ are real numbers). The premise of this rule is defined the same as for a MISO rule for a standard fuzzy system. The consequents of the rules are different. Instead of a linguistic term with an associated membership function, in the consequent we use a *function* $c_i = g_i(\cdot)$ that does not have an associated membership function. For a Takagi–Sugeno fuzzy system the consequent mappings are linear (actually "affine" due to the $a_{i,0}$ term).

It is important to note that a Takagi–Sugeno fuzzy system may have any linear mapping (affine mapping) as its output function and this contributes to the generality of the Takagi–Sugeno fuzzy system. One mapping that has proven to be particularly useful is to have a linear dynamical system as the output function so that the i-th rule has the form

If $\tilde{x}_1$ is $\tilde{A}_1^j$ **and** $\tilde{x}_2$ is $\tilde{A}_2^k$ **and**, …, **and** $\tilde{x}_p$ is $\tilde{A}_p^l$
Then $\dot{x}^i(t) = A_i x(t) + B_i u(t)$.

Here, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^t$ is the n-dimensional state, $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^t$ is the m-dimensional model input, and A_i and B_i, $i = 1, 2, \ldots, R$ are the state and input matrices of appropriate dimension. This fuzzy system can be thought of as a nonlinear interpolator between R linear systems.

Let μ_i denote the certainty of the premise of the i-th rule and define

$$\xi_i(x(t)) = \frac{\mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))}.$$

With this, the Takagi–Sugeno fuzzy system can be written as

$$\dot{x}(t) = \frac{\sum_{i=1}^R (A_i x(t) + B_i u(t))\mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))}$$

or

$$\dot{x}(t) = \left(\sum_{i=1}^R A_i \xi_i(x(t))\right) x(t) + \left(\sum_{i=1}^R B_i \xi_i(x(t))\right) u(t). \tag{11}$$

If $R = 1$ we get a standard linear system. Generally, for $R > 1$ and a given value of $x(t)$ only certain rules will turn on and contribute to the output.

We could let $u(t) = 0$, $t \geq 0$ and study the stability of Eq. (11). Instead, we will consider the case where we use a controller to generate $u(t)$. Assume that we can measure $x(t)$ and that the controller is another Takagi–Sugeno fuzzy system with R rules (the same number of rules as was used to describe the plant) of the form

If $\tilde{x}_1$ is $\tilde{A}_1^j$ **and** $\tilde{x}_2$ is $\tilde{A}_2^k$ **and**, …, **and** $\tilde{x}_n$ is $\tilde{A}_n^l$
Then $u^i = K_i x(t)$

where K_i, $i = 1, 2, \ldots, R$, are $1 \times n$ vectors of control gains and the premises of the rules are identical to the premises of the plant rules that were used to specify Eq. (11). In this case

$$u(t) = \sum_{j=1}^R K_j \xi_j(x(t)) x(t). \tag{12}$$

If we connect the controller to the plant in Eq. (11) we get a closed-loop system

$$\dot{x}(t) = \Big(\sum_{i=1}^R A_i \xi_i(x(t)) + \Big(\sum_{i=1}^R B_i \xi_i(x(t))\Big)\Big(\sum_{j=1}^R K_j \xi_j(x(t))\Big)\Big) x(t) \tag{13}$$

which is in the form of Eq. (1). We assume that μ_i and hence ξ_i are defined so that Eq. (13) possesses a unique solution that is continuously dependent on $x(0)$.

For stability analysis we use the direct method of Lyapunov. Choose a (quadratic) Lyapunov function

$$V(x) = x^t P x$$

where P is a "positive definite matrix" (denoted by $P > 0$) that is symmetric (i.e., $P = P^t$). Given a symmetric matrix P we can easily test if it is positive definite. You simply find the eigenvalues of P and if they are all strictly positive then P is positive definite. If P is positive definite then for all $x \neq 0$, $x^t P x > 0$. Hence, we have $V(x) > 0$ and $V(x) = 0$ only if $x = 0$. Also, if $|x| \to \infty$, then $V(x) \to \infty$.

To show that the equilibrium $x = 0$ of the closed-loop system in Eq. (13) is globally asymptotically stable we need to show that $\dot{V}(x) < 0$ for all x. Notice that

$$\dot{V}(x) = x^t P \dot{x} + \dot{x}^t P x$$

so that since

$$\xi_i(x(t)) = \frac{\mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))}$$

we have

$$\begin{aligned}
\dot{V}(x) &= x^t P \left[\frac{\sum_{i=1}^R A_i \mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))} + \left(\frac{\sum_{i=1}^R B_i \mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))} \right) \left(\frac{\sum_{j=1}^R K_j \mu_j(x(t))}{\sum_{j=1}^R \mu_j(x(t))} \right) \right] x \\
&+ x^t \left[\frac{\sum_{i=1}^R A_i \mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))} + \left(\frac{\sum_{i=1}^R B_i \mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))} \right) \left(\frac{\sum_{j=1}^R K_j \mu_j(x(t))}{\sum_{j=1}^R \mu_j(x(t))} \right) \right]^t P x \\
&= x^t P \left[\frac{\sum_{i=1}^R A_i \mu_i(x(t)) \sum_{j=1}^R \mu_j(x(t))}{\sum_{i=1}^R \mu_i(x(t)) \sum_{j=1}^R \mu_j(x(t))} + \left(\frac{\sum_{i=1}^R B_i \mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))} \right) \left(\frac{\sum_{j=1}^R K_j \mu_j(x(t))}{\sum_{j=1}^R \mu_j(x(t))} \right) \right] x \\
&+ x^t \left[\frac{\sum_{i=1}^R A_i \mu_i(x(t)) \sum_{j=1}^R \mu_j(x(t))}{\sum_{i=1}^R \mu_i(x(t)) \sum_{j=1}^R \mu_j(x(t))} + \left(\frac{\sum_{i=1}^R B_i \mu_i(x(t))}{\sum_{i=1}^R \mu_i(x(t))} \right) \left(\frac{\sum_{j=1}^R K_j \mu_j(x(t))}{\sum_{j=1}^R \mu_j(x(t))} \right) \right]^t P x.
\end{aligned}$$

Now, if we let $\sum_{i,j}$ denote the sum over all possible combinations of i and j, $i = 1, 2, \ldots, R$, $j = 1, 2, \ldots, R$ we get

$$\begin{aligned}
\dot{V}(x) &= x^t P \left[\frac{\sum_{i,j} A_i \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} + \frac{\sum_{i,j} B_i K_j \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} \right] x \\
&+ x^t \left[\frac{\sum_{i,j} A_i \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} + \frac{\sum_{i,j} B_i K_j \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} \right]^t P x \\
&= x^t P \left[\frac{\sum_{i,j} (A_i + B_i K_j) \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} \right] x + x^t \left[\frac{\sum_{i,j} (A_i + B_i K_j) \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} \right]^t P x \\
&= x^t \left[P \left[\frac{\sum_{i,j} (A_i + B_i K_j) \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} \right] + \left[\frac{\sum_{i,j} (A_i + B_i K_j) \mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} \right]^t P \right] x \\
&= x^t \left[\sum_{i,j} \mu_i(x(t)) \mu_j(x(t)) \left[P(A_i + B_i K_j) + (A_i + B_i K_j)^t P \right] \left[\sum_{i,j} \mu_i(x(t)) \mu_j(x(t)) \right]^{-1} \right] x.
\end{aligned}$$

Now, since

$$0 \leq \frac{\mu_i(x(t)) \mu_j(x(t))}{\sum_{i,j} \mu_i(x(t)) \mu_j(x(t))} \leq 1$$

we have

$$\dot{V}(x) \leq \sum_{i,j} x^t \left(P(A_i + B_i K_j) + (A_i + B_i K_j)^t P \right) x.$$

Hence, if

$$x^t \left(P(A_i + B_i K_j) + (A_i + B_i K_j)^t P \right) x < 0, \quad (14)$$

then $\dot{V}(x) < 0$.

Let

$$Z = P(A_i + B_i K_j) + (A_i + B_i K_j)^t P.$$

Notice that since P is symmetric Z is symmetric so that $Z^t = Z$. Equation (14) holds if Z is a "negative definite matrix." For a symmetric matrix Z we say that it is negative definite (denoted $Z < 0$) if $x^t Z x < 0$ for all $x \neq 0$. If Z is symmetric then it is negative definite if the eigenvalues of Z are all strictly negative. Hence, to show that the equilibrium $x = 0$ of Eq. (13) is globally asymptotically stable we must find a *single* $n \times n$ positive definite matrix P such that

$$P(A_i + B_i K_j) + (A_i + B_i K_j)^t P < 0 \qquad (15)$$

for all $i = 1, 2, \ldots, R$ and $j = 1, 2, \ldots, R$.

Notice that in Eq. (15) finding the common P matrix such that the R^2 matrices are negative definite is not trivial to compute by hand if n and R are large. Fortunately, "linear matrix inequality" (LMI) methods can be used to find P if it exists, and there are functions in a Matlab toolbox for solving LMI problems (see the literature overview for some methods).

As a simple example of how to use the stability test in Eq. (15) assume that $n = 1$, $R = 2$, $A_1 = -1$, $B_1 = 2$, $A_2 = -2$, and $B_2 = 1$. These provide the parameters describing the plant. We do not provide the membership functions as any you choose (provided they result in a differential equation with a unique solution that depends continuously on $x(0)$) will work for the stability analysis that we provide.

Equation (15) says that to stabilize the plant with the Takagi–Sugeno fuzzy controller in Eq. (12) we need to find a scalar $P > 0$ and gains K_1 and K_2 such that

$$P(-1 + 2K_1) + (-1 + 2K_1)P < 0,$$
$$P(-1 + 2K_2) + (-1 + 2K_2)P < 0,$$
$$P(-2 + K_1) + (-2 + K_1)P < 0,$$
$$P(-2 + K_2) + (-2 + K_2)P < 0.$$

Choose any $P > 0$ such as $P = 0.5$. The stability test indicates that we need K_1 and K_2 such that $K_1 < 0.5$ and $K_2 < 2$ to get a globally asymptotically stable equilibrium $x = 0$. If you simulated the closed-loop system for some $x(0) \neq 0$ you would find that $x \to 0$ as $t \to \infty$.

3.2. *Analysis of absolute stability*

In this section we will examine the use of the Circle Criterion for testing and designing to insure the stability of a fuzzy control system. Our work will differ from that of [38] and [39] in that we will use a form of the Circle Criterion that gives sufficient *and* necessary conditions (in relation to the class of nonlinearities in a sector) for stability. We will use the Circle Criterion theory found in [24] and [49] to achieve the desired results. Of course there are other frequency domain based criteria for stability that can be utilized for fuzzy control system analysis (e.g., Popov's Criterion and the multivariable circle criterion [24, 32]).

3.2.1. *The Circle Criterion*

Figure 8 shows a basic regulator system. In this system $G(s)$ is the transfer function of the plant and is equal to $C(sI - A)^{-1}B$ where (A, B, C) is the state space description of the plant (x is the n-dimensional state vector). Furthermore, (A, B) is controllable and (A, C) is observable [18]. The function $\Phi(t, y)$, represents a memoryless, possibly time varying nonlinearity (in our case the fuzzy controller) with $\Phi : [0, \infty) \times \Re \to \Re$. Even though the fuzzy controller is in the feedback path rather than the feedforward path in this system, we will be able to use the same fuzzy controller described in Section 2 since it represents an odd function (i.e., $\Phi(-y) = -\Phi(y)$). It is assumed that $\Phi(t, y)$ is piecewise continuous in t and locally Lipschitz [31]. If Φ is bounded within a certain region as shown in Fig. 9 so that there exist $\alpha, \beta, a, b, (\beta > \alpha, a < 0 < b)$ for which

$$\alpha y \leq \Phi(t, y) \leq \beta y \qquad (16)$$

for all $t \geq 0$ and all $y \in [a, b]$ then Φ is a "sector nonlinearity". If Eq. (16) is true for all $y \in (-\infty, \infty)$ then the sector condition holds globally and the system is "absolutely stable" (i.e., $x = 0$ is (uniformly) globally asymptotically stable). For the case where Φ only satisfies Eq. (16) locally (i.e., for some a and b), if certain conditions (to be listed below) are met then the system is "absolutely stable on a finite domain" (i.e., $x = 0$ is (uniformly) asymptotically stable). Recall that in Section 2 we explain how the fuzzy controller is often similar to a saturation nonlinearity. Clearly the fuzzy controller can be sector bounded in the same manner as the saturation nonlinearity with either $\alpha = 0$ for the global case or for local stability $\alpha > 0$. Lastly, $D(\alpha, \beta)$ is a closed disk in the complex

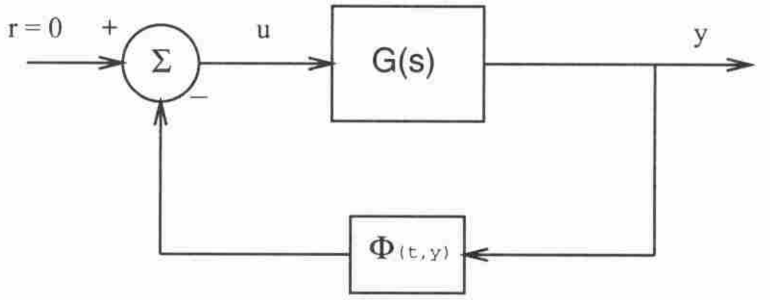


Fig. 8. Regulator system.

plane whose diameter is the line segment connecting the points $\frac{-1}{\alpha} + j0$ and $\frac{-1}{\beta} + j0$.

The Circle Criterion states that with Φ satisfying the sector condition in Eq. (16) the regulator system in Fig. 8 is absolutely stable if one of the following three conditions is met:

1. If $0 < \alpha < \beta$, the Nyquist Plot of $G(j\omega)$ is bounded away from the disk $D(\alpha, \beta)$ and encircles it m times in the counterclockwise direction where m is the number of poles of $G(s)$ in the open right half plane (RHP).
2. If $0 = \alpha < \beta$, $G(s)$ is Hurwitz (poles in the open LHP) and the Nyquist Plot of $G(j\omega)$ lies to the right of the line $s = \frac{-1}{\beta}$.
3. If $\alpha < 0 < \beta$, G(s) is Hurwitz and the Nyquist Plot of $G(j\omega)$ lies in the interior of the disk $D(\alpha, \beta)$ and is bounded away from the circumference of $D(\alpha, \beta)$.

If Φ satisfies Eq. (16) only on the interval $y \in [a, b]$, then the above conditions ensure absolute stability on a finite domain [24]. Is is Circle Criterion conditions such as these that have been used in the past of stability analysis of fuzzy control systems. It is important to note that the above conditions are sufficient conditions for stability only and hence there is the concern that they are conservative. However, in [49] it is shown how the Circle Criterion can be adjusted such that the conditions are sufficient and necessary; it is this set of conditions that we will introduce for stability analysis of fuzzy control systems as it is explained next.

It is necessary to begin by providing some mathematical preliminaries. For each real $p \in [1, \infty)$, the set L_p consists of functions $f(\cdot) : [0, \infty) \to \Re$ such that

$$\int_0^\infty |f(t)|^p \, dt < \infty. \tag{17}$$

Let

$$f_T(t) = \begin{cases} f(t), & 0 \le t \le T, \\ 0, & T < t. \end{cases} \tag{18}$$

Let the set L_{pe}, the *extension of* L_p, consist of all functions $f_T : [0, \infty) \to \Re$, such that $f_T \in L_p$ for all finite T. Finally, let

$$\| f(\cdot) \|_p = \left[\int_0^\infty |f(t)|^p \, dt \right]^{1/p}, \tag{19}$$

$$\| f(\cdot) \|_{Tp} = \| f(\cdot)_T \|_p \,. \tag{20}$$

If R is a binary relation on L_{pe} then R is said to be *L_p-stable* if

$$(x, y) \in R, \; x \in L_p \Rightarrow y \in L_p \,. \tag{21}$$

R is *L_p-stable with finite gain (wfg)* if it is L_p-stable, and in addition there exist finite constants γ_p and b_p such that

$$(x, y) \in R, \; x \in L_p \Rightarrow \|y\|_p \le \gamma_p \|x\|_p + b_p \,. \tag{22}$$

R is *L_p-stable with finite gain and zero bias (wb)* if it is L_p-stable, and in addition there exists a finite constant γ_p such that

$$(x, y) \in R, \; x \in L_p \Rightarrow \|y\|_p \le \gamma_p \|x\|_p \,. \tag{23}$$

For more details see [49].

Assume that we are given the regulator system shown in Fig. 8 (with G defined as above) except that now Φ is in general defined by $\Phi : L_{2e} \to L_{2e}$. Φ belongs to the open sector (α, β) if it belongs to the sector $[\alpha + \epsilon, \beta - \epsilon]$ for some $\epsilon > 0$ with the sector bound defined as

$$\|\Phi x - [(\beta + \alpha)/2] x\|_{T2} \le (\beta - \alpha)/2 \|x\|_{T2}, \quad \forall T \ge 0, \forall x \in L_{2e}. \tag{24}$$

In actuality, this definition of the sector $[\alpha, \beta]$ is consistent with our previous definition in Eq. (16) if Φ

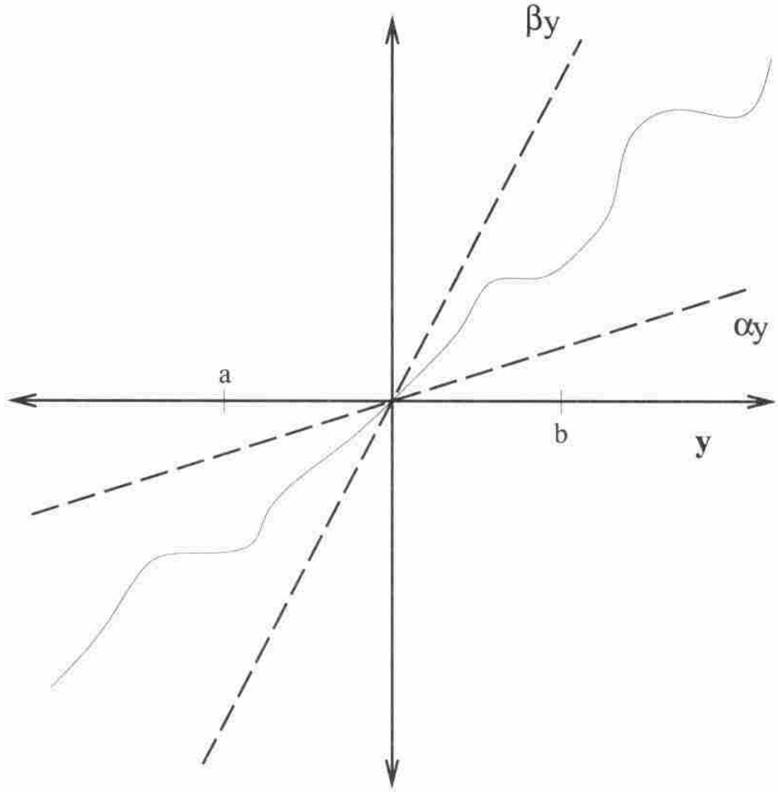


Fig. 9. Sector bounded nonlinearity.

is memoryless. Next, we state a slightly different version of the Circle Criterion that we will call the Circle Criterion with Sufficient and Necessary Conditions (SNC).

For the system of Fig. 8 with Φ defined as $\Phi : L_{2e} \rightarrow L_{2e}$ which satisfies Eq. (24) and α, β two given real numbers with $\alpha < \beta$ the following two statements are equivalent [49]:

1. The feedback system is L_2-stable with finite gain and zero bias for every Φ belonging to the sector (α, β).
2. The transfer function G satisfies one of the following conditions as appropriate:
   (a) If $\alpha\beta > 0$ then the Nyquist plot of $G(j\omega)$ does not intersect the interior of the disk $D(\alpha, \beta)$ and encircles the interior of the disk $D(\alpha, \beta)$ exactly m times in the counterclockwise direction, where m is the number of poles of G with positive real part.
   (b) If $\alpha = 0$, then G has no real poles with positive real part, and $\mathrm{Re}G(j\omega) \geq -\frac{1}{\beta}, \forall\omega$.
   (c) If $\alpha\beta < 0$, then G is a stable transfer function and the Nyquist plot of $G(j\omega)$ lies inside the disk $D(\alpha, \beta)$ for all ω.

If the conditions in Statement 2 are satisfied, the system is L_2-stable and the result is similar to the Circle Criterion with sufficient conditions only. Negation of Statement 2 infers negation of Statement 1 and we can state that the system will not be L_2-stable for $every$ nonlinearity in the sector (it may not be apparent which of the nonlinearities in a sector will cause the instability).

3.2.2. Example

For an example we will use a plant with transfer function $G(s) = \frac{1}{s^3+7s^2+7s+15}$. This plant is chosen because it illustrates the problems with stability that can arise when designing fuzzy controllers. The Nyquist Plot of $G(j\omega)$ for this plant is shown in Fig. 10. It would be impossible to choose A and B for the fuzzy controller without more information about the physical system that the above transfer function represents. However, with conditions such as amplifiers whose gain, K, is limited or performance speci-

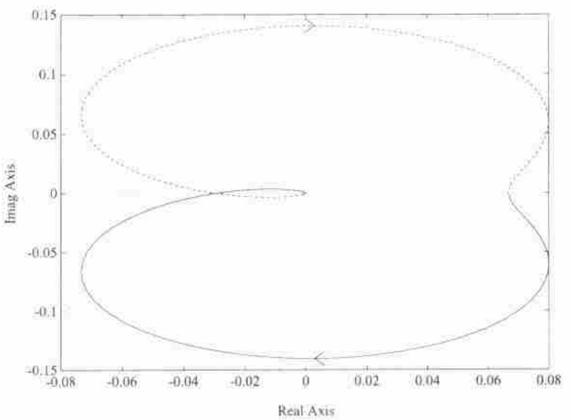


Fig. 10. Nyquist plot of example plant.

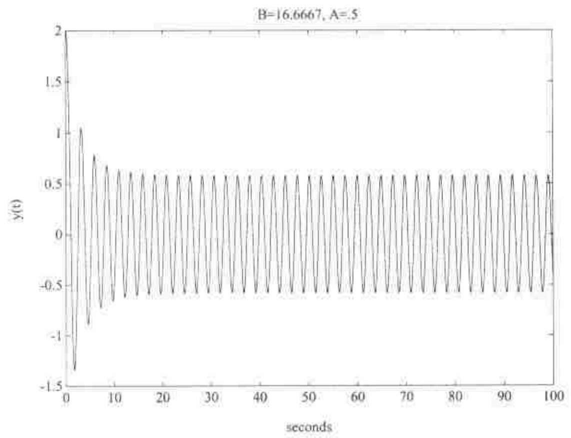


Fig. 11. Oscillations from the fuzzy control system.

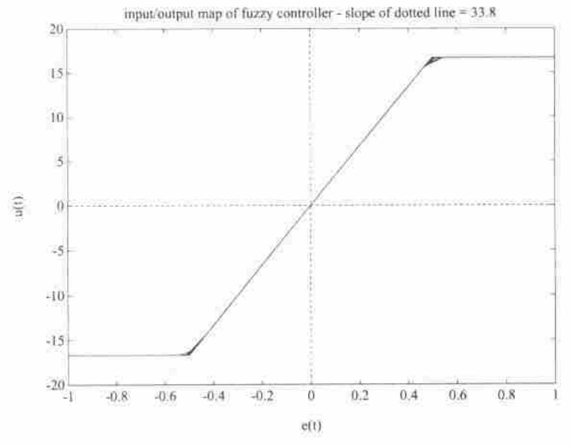


Fig. 12. The sector bounded nonlinearity Φ.

fications of some type, a possible fuzzy controller for this system (i.e., one that some expert could construct) would be one with $A = 0.5$ and $B = 16.6667$. However, if we simulate this system with initial conditions $x(0) = [0\ 0\ 2]^T$ the system has sustained oscillations as shown in Fig. 11. We can see that the *ad hoc* design procedure for fuzzy controllers can cause problems that cannot, perhaps, be foreseen via an intuitive analysis. If we consider the fuzzy controller as a nonlinearity, Φ, we can find a sector (α, β) in which Φ lies and use the Circle Criterion to determine why the instability is occurring and perhaps determine how to tune the fuzzy controller so that it does not cause sustained oscillations. Figure 12 shows the fuzzy controller nonlinearity and we can see that it lies in the sector $(0, 33.8)$. Because $\alpha = 0$ we will use the second condition of the Circle Criterion/SNC from [49]. For our current system the plant has poles at -6.2648 and $-0.3676 \pm j1.5031$. However, by looking at the Nyquist plot in Fig. 10 we can see that a line drawn at $-\frac{1}{\beta} = -0.0296$ intersects the Nyquist plot and violates the second part of the condition. Hence the Circle Criterion/SNC predicts that not all of the nonlinearities within this sector will be stable. We have found a fuzzy controller which verifies this statement by producing sustained oscillations in the closed loop system as shown in Fig. 11. Next we use condition (b) of the Circle Criterion/SNC to provide ideas on how to tune the fuzzy controller.

To meet the second part of the condition, we will have to adjust β so that $-\frac{1}{\beta} < -0.0733$, (i.e., so that $\beta < 13.64$). As there are many different choices for A and B so that the fuzzy controller will fit inside the sector, more about the system would have to be known (e.g. what the saturation limits at the input of the plant are) to know whether to tune A or B. If we desired to change A and not B then we would leave $B = 16.6667$ and make $A > 1.222$ so that $\frac{B}{A} < 13.64$. If we desired to change B and not A then we would leave $A = 0.5$ and make $B < 6.82$. As an example, we will choose the first case, leave $B = 16.6667$, and change A to 1.3. A simulation of the resulting fuzzy control system with $x(0) = [0\ 0\ 2]^T$ is shown in Fig. 13. Notice that there is no sustained oscillation and we have used the Circle Criterion/SNC to redesign the fuzzy controller to avoid the instability.

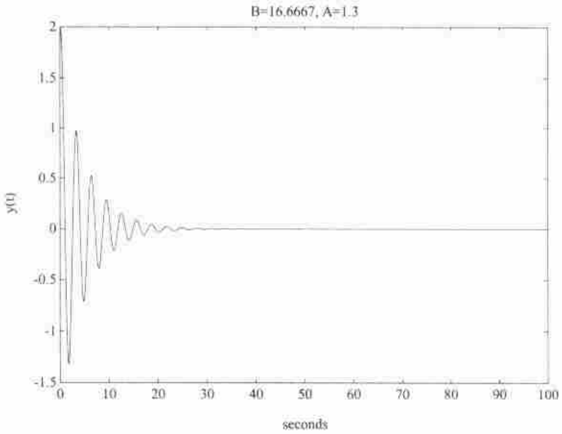


Fig. 13. Re-designed fuzzy control system.

4. Analysis of steady state tracking error

A terrain following, terrain avoidance aircraft control system uses an altimeter to provide a measurement of the distance of the aircraft from the ground to decide how to steer the aircraft to follow the earth at a pilot-specified height. If a fuzzy controller is employed for such an application and it consistently seeks to control the height of the plane to be lower than what the pilot specifies there will be a steady state tracking error (an error between the desired and actual heights) that could result in a crash. In this section we show how to use the results in [40] for predicting and eliminating steady state tracking errors for fuzzy control systems so that such problems can be avoided.

4.1. Theory

We must make several assumptions. First the system is assumed to be of the configuration shown in Fig. 14 where r, e, u, and y belong to $L_{\infty e}$ and $\Phi(t,e)$ is the SISO fuzzy controller described in Section 2. We will call $e_{ss} = \lim_{t\to\infty} e(t)$ the steady state tracking error. $G(s)$ has the form

$$G(s) = \frac{p(s)}{s^\rho q(s)} \tag{25}$$

where ρ, a nonnegative integer, is the number of poles of $G(s)$ at $s = 0$, and $p(s)$ and $s^\rho q(s)$ are relatively prime polynomials such that $\deg(p(s)) < \deg(s^\rho q(s))$. Furthermore, we assume that $(\Phi z)(t) = \eta[z(t)]$, $t \geq 0$, for $z \in L_{\infty e}$, where $\eta : \Re \to \Re$, $\eta(0) = 0$, and η is bounded by α and β according to

$$\alpha \leq \frac{\eta(a) - \eta(b)}{a - b} \leq \beta \tag{26}$$

for all $a \neq b$. Notice that this sector bound is different from the sector bound in Section 3.2. This new sector bound is determined by the maximum and minimum slopes of Φ at any point and is sometimes not as easy to determine as the graphical sector bound of Section 3.2.

Finally we assume that one of the following three Circle Criterion type conditions is met:

1. $0 < \alpha < \beta$, and the locus of $G(j\omega)$ for $-\infty < \omega < \infty$, with the usual Nyquist-locus indentations where needed:
   (a) is bounded away from the circle C_1 of radius $\frac{1}{2}(\alpha^{-1} - \beta^{-1})$ centered on the real axis of the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}), 0]$.
   (b) encircles C_1 in the counterclockwise direction n_p times where n_p is the number of poles of $G(s)$ with positive real parts.
2. $0 = \alpha < \beta$, $G(s)$ has no poles in the open Right Half Plane, and $Re[G(j\omega)] > -\beta^{-1}$ for all real ω for which $Re[G(j\omega)]$ is finite.
3. $\alpha < 0 < \beta$, $G(s)$ has no poles in the closed Right Half Plane (RHP), and the locus of $G(j\omega)$ for $-\infty < \omega < \infty$ is contained within the circle C_2 of radius $\frac{1}{2}(\beta^{-1} - \alpha^{-1})$ centered on the real axis of the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}), 0]$.

To predict the value of e_{ss} several definitions must be made. First, an "average gain" for Φ, c_0, is defined as $c_0 = \frac{1}{2}(\alpha + \beta)$ and we assume that $c_0 \neq 0$. In [40] the authors show that for this c_0, $1 + c_0 G(s) \neq 0$ for $Re(s) \geq 0$. Therefore, the rational function $H(s) = \frac{G(s)}{1 + c_0 G(s)}$, at all points s at which $G(s)$ is regular, is strictly proper and has no poles in the closed RHP. Defined in this manner, $H(s)$ is the closed loop equation for the system shown in Fig. 14 with c_0 as an average gain of Φ. Finally, we will define $\tilde{\eta} : \Re \to \Re$ by $\tilde{\eta}(a) = \eta(a) - c_0 a$, $a \in \Re$. That is, $\tilde{\eta}$ is the difference between the actual value of Φ at some point a and a predicted value found by using the average gain, c_0.

Suppose that the above assumptions are met. It is proven in [40] that for each $\gamma \in \Re$, there exists a unique $\xi \in \Re$ such that

$$\gamma = \xi + H(0)\tilde{\eta}(\xi) \tag{27}$$

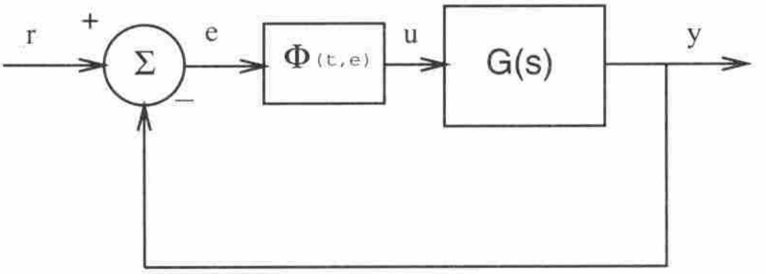


Fig. 14. The steady state error system configuration.

and

$$\xi = \lim_{k\to\infty} \xi_k \tag{28}$$

where

$$\xi_{k+1} = \gamma - H(0)\tilde{\eta}(\xi_k) \tag{29}$$

and $\xi_0 \in \Re$ is arbitrary (Eq. (29) is an iterative algorithm that will be used to find e_{ss}). Furthermore, if we define c as

$$c = \frac{1}{2}(\beta - \alpha)|H(0)| \tag{30}$$

and assume that $c < 1$ then the equation

$$|\xi - \xi_k| \leq \frac{c^k}{1-c}|\xi_0 - \gamma + H(0)\tilde{\eta}(\xi_0)|,\ k \geq 1, \tag{31}$$

must be true for the iterative algorithm, Eq. (29) to converge. Finally, let $\Theta : \Re \to \Re$ be the map that takes γ into ξ (i.e. $\Theta(\gamma) = \xi$). Note that this map is defined by the above algorithm.

The Theorem for finding the steady state error of a fuzzy control system from [40] is as follows:

Theorem. *Assuming that all the described assumptions are satisfied:*

1. *If r approaches a limit l as $t \to \infty$ then $e_{ss} \equiv \lim_{t\to\infty} e(t)$ exists. Moreover, $e_{ss} \neq 0$ if and only if $l \neq 0$ and $\rho = 0$, and then $e_{ss} = \Theta(\gamma)$ where $\gamma = \frac{l}{1+c_0 G(0)}$.*
2. *Assuming that*

$$r(t) = \sum_{j=0}^{\nu} a_j t^j,\ t \geq 0 \tag{32}$$

in which the a_j are real, ν is a positive integer, and $a_\nu \neq 0$, the following holds:

(a) e is unbounded if $\nu > \rho$.

(b) if $\nu \leq \rho$ then e approaches a limit as $t \to \infty$. If $\nu = \rho$ this limit is $\Theta(\gamma)$ where

$$\gamma = \frac{a_\nu \nu! q(0)}{c_0 p(0)}. \tag{33}$$

If $\nu < \rho$ then the limit is zero.

An examination of the above Theorem reveals that in actuality the proposed method for finding the steady state error for fuzzy control systems is similar to the equations used in conventional linear control systems. The Theorem performs the function of identifying an appropriate equation for e_{ss} based on the input type and the "system type". Notice that the two equations for γ in the Theorem are analogous to the equations for the "error constants" [18], $\frac{1}{1+K_p}$, $\frac{1}{K_v}$, and $\frac{1}{K_a}$ and provide an initial estimate for e_{ss}. The final prediction of e_{ss} is found by using the iterative equation defined by (29). This equation uses the initial guess, γ and $\tilde{\eta}$ (the difference between the actual value of Φ and the estimate using c_0) to iteratively determine e_{ss} if it exists. In our next section we will apply this process to several examples. Then we will show how to use the result in design.

4.2. Example

For an example we will examine a plant of the form $G(s) = \frac{1}{s^2+4s+3}$. First we must choose a fuzzy controller and determine the α and β describing the sector in which it lies. We will use the fuzzy controller whose I/O map is shown in Fig. 15. To find α and β numerically we could perform an exhaustive search by inserting all possible values of a and b into the equation $\alpha \leq \frac{\eta(a)-\eta(b)}{a-b} \leq \beta$ and determining the maximum and minimum values. However, it is obvious from the horizontally flat section of the I/O map that α must be 0. Furthermore, since there are no large oscillations at any point of the I/O map, a fairly accurate

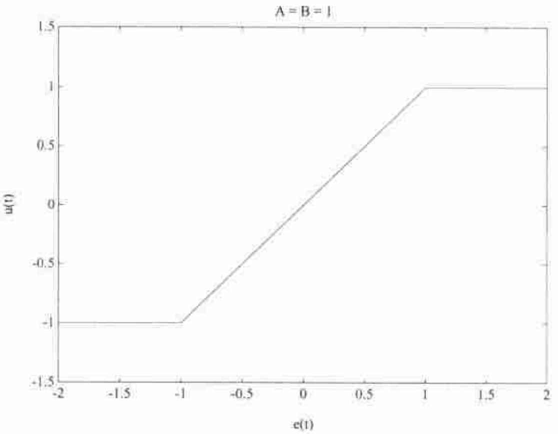


Fig. 15. I/O map of a fuzzy controller.

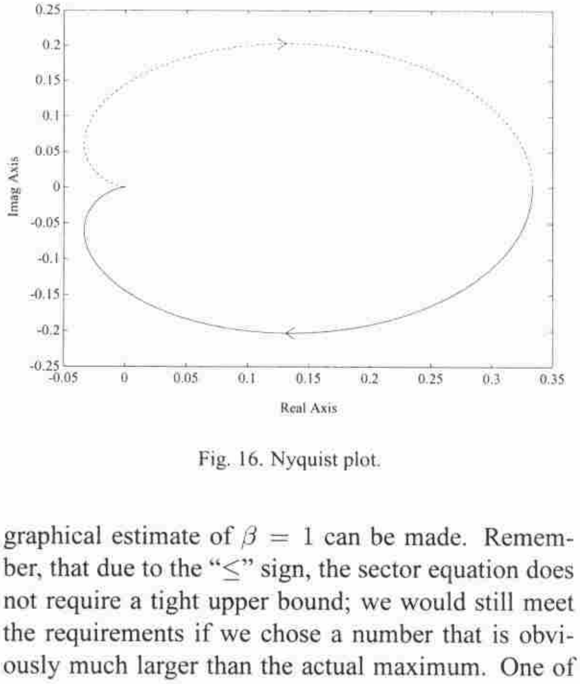


Fig. 16. Nyquist plot.

graphical estimate of $\beta = 1$ can be made. Remember, that due to the "$\leq$" sign, the sector equation does not require a tight upper bound; we would still meet the requirements if we chose a number that is obviously much larger than the actual maximum. One of the assumptions for this procedure to work is that our system must satisfy one of the Circle Criterion type conditions. With $\alpha = 0$, we must satisy condition 2. Our plant has no poles in the RHP and by examining the Nyquist plot of $G(j\omega)$ shown in Fig. 16 we see that since $\beta = 1$ then the second part of the condition is also satisfied. All the assumptions are now valid.

With $\alpha = 0$ and $\beta = 1$ we can solve for $c_0 = 0.5(\alpha + \beta) = 0.5$. We find

$$H(s) = \frac{G(s)}{1 + c_0 G(s)} = \frac{1}{s^2 + 4s + 3.5} \qquad (34)$$

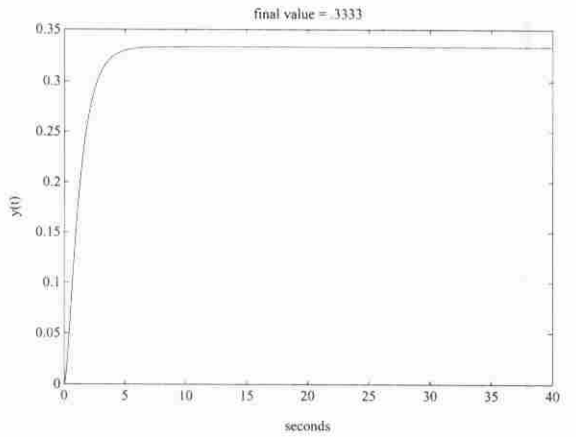


Fig. 17. Step response.

and $H(0) = \frac{1}{3.5} = 0.2857$. All that remains is to determine what type of input we will have, apply the Theorem from [40] to determine γ, and then solve the recursive equation from (29). If we choose a step input of size 3, then we will use condition 1 of the Theorem. The input will have a limit $l = 3$ as $t \to \infty$ and $\rho = 0$. Therefore e_{ss} exists, is not equal to zero, and can be found from

$$e_{ss} = \Theta(\gamma) = \Theta(\frac{l}{1 + c_0 G(0)}) = \Theta(2.5714). \quad (35)$$

We now have the values $\gamma = 2.5714$, $c_0 = 0.5$, and $H(0) = 0.2857$ and can recursively solve the equation $\xi_{k+1} = \gamma - H(0)\tilde{\eta}(\xi_k)$ to find e_{ss}. When these calculations are made, we find that our steady state error for a step of size 3 will be 2.6667. In Fig. 17 we can see that a simulation of this system does indeed have the predicted steady state error of 2.6667.

If this steady state error was considered to be excessive, it would be convenient to be able to redesign our fuzzy controller using the steady state error prediction procedure as part of the design process. Intuitively, we would expect that if we increased the "gain of the fuzzy controller", the steady state error would decrease. In terms of the e_{ss} prediction procedure this would mean changing α and β. Because of the inherent saturation of the fuzzy controller, α will always equal 0. Therefore, we will have to adjust by changing β only. If the fuzzy controller is changed so that it has the I/O map shown in Fig. 18, then $\beta = 20$ and $e_{ss} = 0.3911$. Further simulations show these predicted values of e_{ss} to be valid.

If we desire to examine the response of this system to a ramp input, we must use condition 2 of the

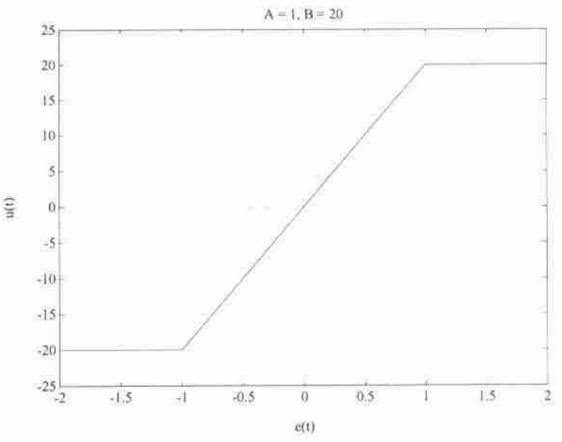


Fig. 18. I/O map for redesigned fuzzy controller.

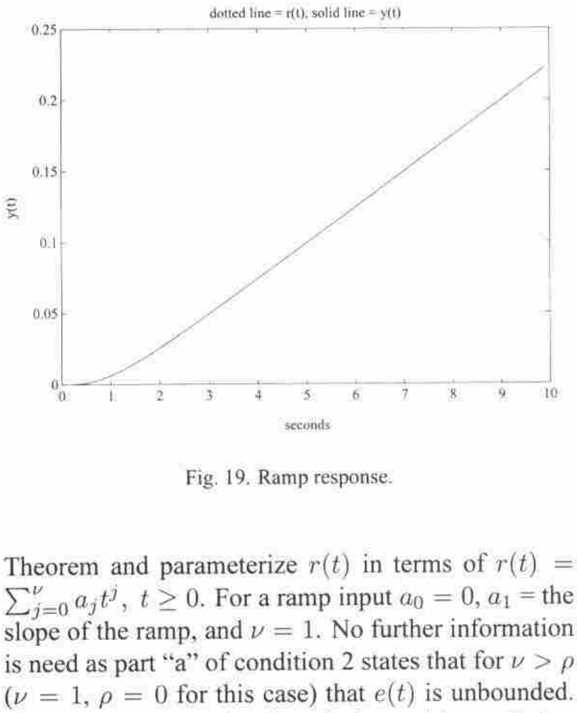


Fig. 19. Ramp response.

Theorem and parameterize $r(t)$ in terms of $r(t) = \sum_{j=0}^{\nu} a_j t^j,\ t \geq 0$. For a ramp input $a_0 = 0$, $a_1 =$ the slope of the ramp, and $\nu = 1$. No further information is need as part "a" of condition 2 states that for $\nu > \rho$ ($\nu = 1$, $\rho = 0$ for this case) that $e(t)$ is unbounded. The ramp response in Fig. 19 shows this prediction to be accurate: $e(t)$ grows without bound. Unfortunately, we cannot change this by redesigning our fuzzy controller because we cannot change the value of ρ (the plant type) by changing the fuzzy controller (unless, of course, we added a dynamic pre- or post-compensator). This clearly exhibits an inherent limitation on fuzzy control that is sometimes overlooked in the heuristic construction of the fuzzy controller.

The conventional solution to the problem of unbounded error to a ramp is to add an integrator to

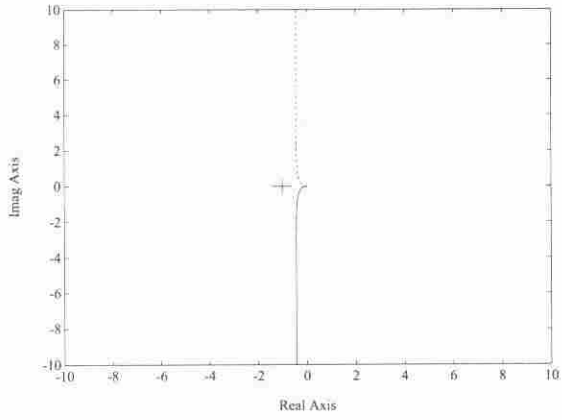


Fig. 20. Nyquist plot for example with integrator.

change the system type. To show the steady state error prediction capabilities of the procedure from [40] we will do this for our current plant. With the addition of an integrator, we now use $G(s) = \frac{1}{s(s^2+4s+3)}$. This plant has no poles in the RHP but we see from the Nyquist plot in Fig. 20 that the conditions are only satisfied when $\beta < 2.25$. Therefore we will use the original fuzzy controller from above with $\alpha = 0$ and $\beta = 1$. Once again we must use condition 2 from the Theorem since a ramp input results in an unbounded $e(t)$. However, since $\rho = 1 = \nu$ we can use part "b" of this condition and find e_{ss}. The equation for γ in this case is $\gamma = \frac{a_\nu \nu! q(0)}{c_0 p(0)}$. If we choose a ramp with slope 0.1, then $c_0 = 0.5$, $\nu = 1$, $a_1 = 0.1$, $q(0) = 3$, $p(0) = 1$, and $\gamma = 0.6$. The final value we need is $H(0)$ which for this case is 2. Up until now, the convergence requirement $c < 1$ has been met. However, for these values of α, β, and $H(0)$, c =1 and Eq. (29) (the iterative equation) will oscillate and not converge. It is easy to show that this will be true any time $\rho > 0$ and $\alpha = 0$. Pole shifting [24] can be useful to change α but it cannot help meet the $c < 1$ requirement (this can be seen by simple analysis of the effect of pole shifting on the conditions for convergence of the algorithm (29)). To overcome the problem with convergence of the algorithm in Eq. (29) we will move the pole of $G(s)$ which is located at zero slightly into the left half plane so that $\rho = 0$, calculate the new $H(0)$ for this system (call it $\hat{H}(0)$), analyze the steady state error for the original system using $\hat{H}(0)$, and use the results to predict e_{ss} for the actual system. We move the poles of $G(s)$ by inserting K into the system as shown in Fig. 21. For this system the adjusted plant,

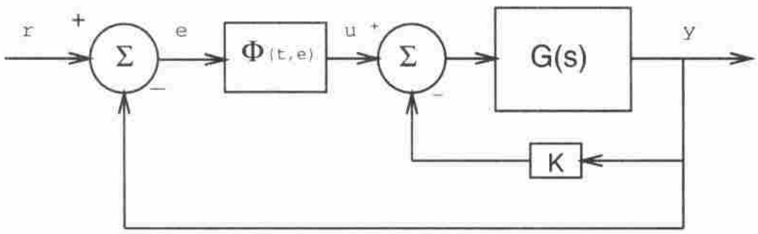


Fig. 21. System adjusted for convergence.

$G_t(s)$, is found by

$$G_t(s) = \frac{G(s)}{1 + KG(s)} . \quad (36)$$

By choosing K so that $\rho = 0$ we can use the same values of α and β but c will be less than 1 and the iterations in Eq. (29) will converge.

For the current system we will choose $K = 0.01$ for which $G_t(s) = \frac{1}{s^3+4s^2+3s+.01}$ and $\rho = 0$. For this plant $\hat{H}(0) = 1.9608$ and $c = 0.9804 < 1$. The values we need to find e_{ss} are c_0, $H(0)$, and γ. Because we have not changed α or β, c_o has not changed and still equals 0.5. We have already calculated the new $\hat{H}(0) = 1.9608$ and only lack γ. To find the proper value for γ we must return to the original plant, $G(s)$, and use the corresponding values of $p(0)$ and $q(0)$ in the equation for γ found in part 2*b* of the Theorem (notice that γ would not exist for G_t for a ramp input as $\rho > \nu$). Therefore γ is also the same as before and is equal to 0.6. We can now use the Eq. (29) to estimate e_{ss}. The estimate for these values is $e_{ss} = 0.3028$. Figure 22 shows a plot of the ramp response for this system. The actual steady state error from the simulation is 0.3003. Our prediction had 0.5% error, but this could be lowered by using smaller values of K. For example, for $K = 0.001$ we obtain $e_{ss} = 0.3001$; a closer estimate.

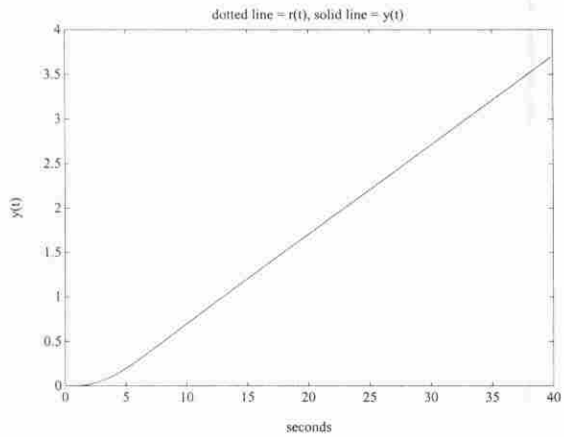


Fig. 22. Ramp response for plant with integrator.

5. Describing function analysis

Autopilots used for cargo ship steering seek to achieve a smooth response by appropriately actuating the rudder to steer the ship. The presence of unwanted oscillations in the steering angle results in loss of fuel efficiency and a less comfortable ride. While such oscillations, sometimes called "limit cycles", result from certain inherent nonlinearities in the control loop, it is possible to carefully construct a controller so that such undesirable behavior is avoided. In this section we investigate the use of the describing function method for the prediction of the existence, frequency, amplitude, and stability of limit cycles. We will first present describing function theory following the format in [42]. Secondly, we will use several examples to show how describing function analysis can be used in the design of SISO and MISO fuzzy controllers of the form described in Section 2. Finally, we will use describing function analysis to design fuzzy controllers for an underwater vehicle and a tape drive servo.

5.1. Theory

Since limit cycles are by definition periodic they can be described as a sum of sinusoids. Furthermore, since almost all physical systems are low pass systems, the higher frequency sinusoids are filtered out and only the lower frequency components remain. This allows us to model the limit cycle as a simple sinusoid and based upon the system equations predict when such a sinusoid might exist in the system.

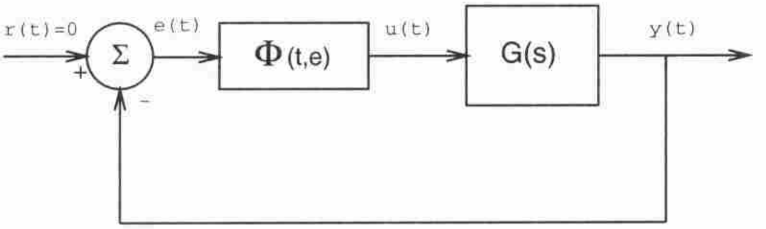


Fig. 23. Nonlinear system for describing function analysis.

5.1.1. Basic assumptions

There are several assumptions that need to be satisfied for the describing function method. These assumptions are as follows:

1. There is only a single nonlinear component and the system can be rearranged into the form shown in Fig. 23.
2. The nonlinear component is time-invariant.
3. Corresponding to a sinusoidal input $e(t) = sin(\omega t)$, only the fundamental component $u_1(t)$ in the output $u(t)$ must be considered.
4. The nonlinearity Φ (which will represent the fuzzy controller) is an odd function.

The first assumption requires that nonlinearities associated with the plant or output sensors be rearranged to appear in Φ as shown in Fig. 23. The second assumption originates from the use in this method of the Nyquist criterion which can only be applied to linear time-invariant systems. The third assumption implies that the linear component following the nonlinearity has characteristics of a low pass filter, i.e.

$$|G(j\omega)| \gg |G(nj\omega)| \text{ for } n = 2, 3, ... \quad (37)$$

and therefore the higher-frequency harmonics, as compared to the fundamental component, can be neglected in the analysis. This is the *fundamental assumption* of describing function analysis and represents an approximation as there will usually be higher-frequency components in the signal from the nonlinearity. The fourth assumption simplifies the analysis of the system by allowing us to neglect the static term of the Fourier expansion of the output. While relaxation of the above assumptions has been studied extensively [5], we focus here only on situations where such assumptions are satisfied.

5.1.2. Defining and computing the describing function

For an input $e(t) = C\sin(\omega t)$ to the nonlinearity, $\Phi(e)$, there will be an output $u(t)$. This output will often be periodic, though generally non-sinusoidal. Expanding $u(t)$ into a Fourier series results in

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega t) + b_n \sin(n\omega t)]. \quad (38)$$

The Fourier coefficients (a_i's and b_i's) are generally functions of C and ω and are determined by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)d(\omega t), \quad (39)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\cos(n\omega t)d(\omega t), \quad (40)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\sin(n\omega t)d(\omega t). \quad (41)$$

Because of our assumptions $a_0 = 0$, $n = 1$, and

$$\begin{aligned} u(t) \approx u_1(t) &= a_1\cos(\omega t) + b_1\sin(\omega t) \\ &= M(C,\omega)\sin(\omega t + \phi(C,\omega)) \end{aligned} \quad (42)$$

where

$$M : \Re \times \Re^{+} \rightarrow \Re \quad (43)$$

and

$$M(C,\omega) = \sqrt{a_1^2 + b_1^2} \quad (44)$$

and where

$$\phi : \Re \times \Re^{+} \rightarrow \Re \quad (45)$$

and

$$\phi(C,\omega) = \arctan(\frac{a_1}{b_1}) . \quad (46)$$

From the above equations we can see that the fundamental component of the output corresponding to a sinusoidal input is a sinusoid of the same frequency and can be written in complex representation as

$$u_1 = M(C,\omega)e^{j(\omega t+\phi(C,\omega))} = (b_1 + ja_1)e^{j\omega t}. \quad (47)$$

We will now define the *describing function* of the nonlinear element to be the complex ratio of the fundamental component of the nonlinear element by the input sinusoid, i.e.

$$\begin{aligned} N(C,\omega) &= \frac{u_1}{C\sin(\omega t)} \\ &= \frac{M(C,\omega)e^{j(\omega t+\phi(C,\omega))}}{Ce^{j\omega t}} = \frac{1}{C}(b_1 + ja_1). \end{aligned} \quad (48)$$

By replacing the nonlinear element, $\Phi(e)$, with its describing function, $N(C,\omega)$, the nonlinear element can be treated as if it where a linear element with a frequency response function. Generally, the describing function depends on the frequency and amplitude of the input signal. However for some special cases this is not the case. For example, if the nonlinearity is time-invariant and memoryless, $N(C,\omega)$ is real and frequency independent. For this case, $N(C,\omega)$ is real because evaluating Eq. (40) gives $a_1 = 0$. Furthermore, in the same equations, the integration of the single-valued function $u(t)\sin(\omega t) = [C\sin(\omega t)]\sin(\omega t)$ is done for the variable ωt, implying that ω does not explicitly appear in the integration and the function $N(C,\omega)$ is frequency independent.

There are several ways to compute describing functions. The describing function can be computed analytically if $u = \Phi(e)$ is known and the integrations to find a_1 and b_1 can be easily carried out. If the I/O relationship of $\Phi(e)$ is given by graphs or tables, then numerical integration can be used. The third method, and the one which we will use, is "experimental evaluation". We will excite the input of the fuzzy controller with sinusoidal inputs, save the related outputs, and then use the input and output waveforms to determine the gain and phase shift at the frequency of the input sinusoid. By varying the amplitude and frequency (or just the amplitude if the fuzzy controller is SISO, time-invariant, and memoryless) of the input sinusoid, we can find u_1 at several points and plot the corresponding describing function.

5.1.3. Predicting the existence and stability of limit cycles

In Fig. 23 if we replace $\Phi(e)$ with $N(C,\omega)$ and assume that a self-sustained oscillation of amplitude C and frequency ω exists in the system then for $r = 0$, $y \neq 0$, and

$$G(j\omega)N(C,\omega) + 1 = 0\,. \quad (49)$$

This equation, called the *harmonic balance equation* in [24], can be rewritten as

$$G(j\omega) = -\frac{1}{N(C,\omega)}\,. \quad (50)$$

If any limit cycles exist in our system, and the four assumptions are satisfied, then the amplitude and frequency of the limit cycles can be predicted by solving the harmonic balance equation. If there are no solutions to the harmonic balance equation then the system will have no limit cycles (under the above assumptions).

However, solving the harmonic balance equation is not trivial; for higher order systems the analytical solution is very complex. The usual method, therefore, is to plot $G(j\omega)$ and $-1/N(C,\omega)$ on the same graph and find the intersection points. For each intersection point there will be a corresponding limit cycle. The amplitude and frequency of each limit cycle can then be determined by finding the particular C and ω that give the value of $-1/N(C,\omega)$ and $G(j\omega)$ at the intersection point.

Along with the amplitude and frequency of the limit cycles, we wish to determine whether the limit cycles are stable or unstable. A limit cycle is considered stable if the system moves to the limit cycle when it is within a certain neighborhood of the limit cycle in the state plane. Therefore once the system is in a limit cycle, the system will return to the limit cycle when perturbations move the system off of the limit cycle. For an unstable limit cycle there is no neighborhood in the state plane within which the system moves to the limit cycle when the system starts near the limit cycle. Instead the system will move away from the limit cycle. Therefore if a system is perturbed from an unstable limit cycle the oscillations will either die out, increase until the system goes unstable or move to a stable limit cycle. The stability of limit cycles can be determined from the same plot used to predict the existence of the limit cycles and is determined by the following criterion from [42]:

Limit Cycle Criterion. Each intersection point of the curve $G(j\omega)$ and the curve $-1/N(C,\omega)$ corresponds to a limit cycle. If points near the intersection and along the increasing-C side of the curve $-1/N(C,\omega)$

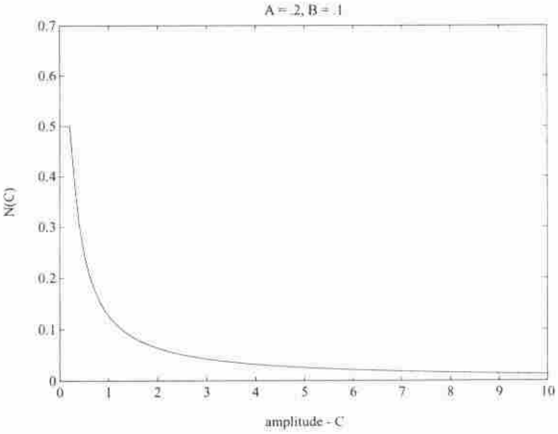


Fig. 24. $N(C,\omega)$ for a fuzzy controller with $A = 0.2$ and $B = 0.1$.

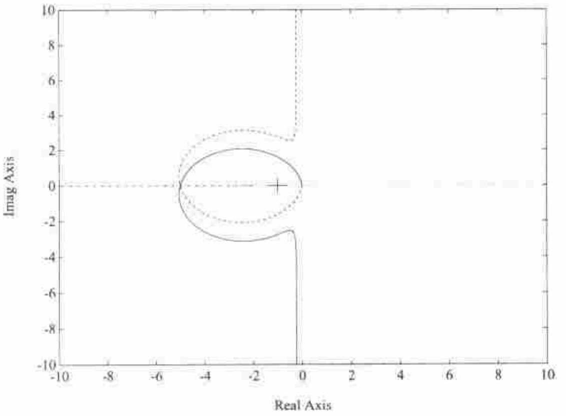


Fig. 25. Plot of $G(j\omega)$ and $-1/N(C,\omega)$.

are not encircled by the curve $G(j\omega)$, then the corresponding limit cycle is stable. Otherwise, the limit cycle is unstable.

5.2. *SISO fuzzy controller example*

Our example will consist of a fuzzy controller with $A = 0.2$ and $B = 0.1$ and a plant with transfer function $G(s) = \frac{1}{s(s^2+0.2s+1)}$ configured in the form of Fig. 23. To predict the limit cycles of this system using describing function analysis, we must first find the describing function of the fuzzy controller. Without using a set of equations for the fuzzy controller we have no choice but to find the describing function experimentally. We do know that the fuzzy controller is SISO, odd, time-invariant, and memoryless. Therefore, the describing function is frequency independent and can be found by varying the amplitude only. Using a sinusoidal input with constant frequency and different amplitude for each data point, we construct the describing function shown in Fig. 24 ($N(C)$ in this figure represents a variable gain dependent on $e(t)$). The next step in our procedure is to plot $G(j\omega)$ and $-1/N(C,\omega)$ on the same plot and find the intersection points. This plot is shown in Fig. 25 and there is an intersection point at $-5 + j0$. This point corresponds to $C = 0.636$ and $\omega = 1$ rad/s. Therefore there will be a limit cycle with amplitude 0.636 at a frequency of 1 rad/s. From the same figure, we can determine that this limit cycle is stable because the points near the intersection and along the increasing-C side of the curve $-1/N(C,\omega)$ are not encircled by the curve $G(j\omega)$. The last step of this process is to verify by simulation that the limit cycle does exist. This simulation with $r(t) = 1$ is shown in Fig. 26. We can see that the limit cycle does exist though the amplitude of 0.62 and frequency of 0.98 rad/s are slightly different then predicted.

Now that we have predicted the existence of a limit cycle for our system, we desire to redesign the fuzzy controller so that there are no limit cycles. Examining Fig. 25 again, we see that if $-1/N(C,\omega) < -5$ there would be no intersection point and no limit cycle. This means that we must change the fuzzy controller so that $N(C,\omega) < 0.2$. From Fig. 24 we can determine that the maximum value of $N(C,\omega)$ is equal to B/A of the fuzzy controller. Therefore if we change the fuzzy controller so that $B/A < 0.2$ the conditions will be met. We will choose $A = 0.6$ and $B = 0.05$. The plot of $G(j\omega)$ and $-1/N(C,\omega)$ in Fig. 27 shows that now there is no intersection point and Fig. 28 shows that there is no limit cycle in the simulation. We were successful in using describing function analysis to redesign our fuzzy controller to eliminate a limit cycle. Notice however, that we have also changed the system performance by increasing the rise time.

5.3. *MISO fuzzy controller example*

All of our describing function analysis to this point has been for SISO fuzzy controllers whose describing functions are not dependent on ω. However, it is important that we also examine how this type of analysis can be applied to MISO fuzzy controllers. While for a MISO fuzzy controller the basic theory is still

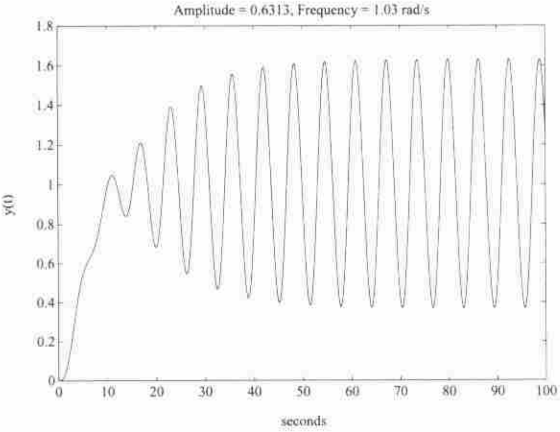


Fig. 26. Simulation to verify existence of limit cycles.

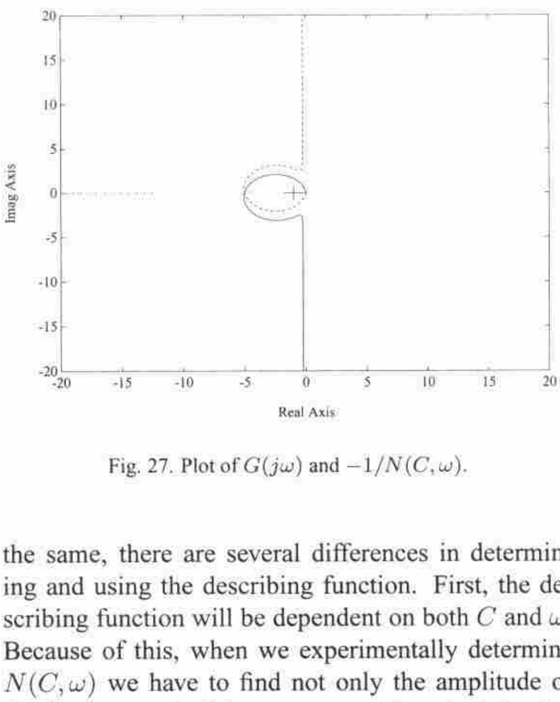


Fig. 27. Plot of $G(j\omega)$ and $-1/N(C,\omega)$.

the same, there are several differences in determining and using the describing function. First, the describing function will be dependent on both C and ω. Because of this, when we experimentally determine $N(C,\omega)$ we have to find not only the amplitude of the fundamental of the output waveform but also the phase of the fundamental for inputs of different amplitude and frequency. Methods for doing this can be found in [5]. This also means that there will be more lines to plot as we will have to plot $-1/N(C,\omega)$ as C changes for each value of ω, i.e. there will be a curve for each value of ω for which $N(C,\omega)$ is calculated. Second, not all intersections of $G(j\omega)$ and $-1/N(C,\omega)$ will be limit cycles. For an intersection to predict a limit cycle the values of ω for $G(j\omega)$ and $-1/N(C,\omega)$ at the intersection must be the same. It

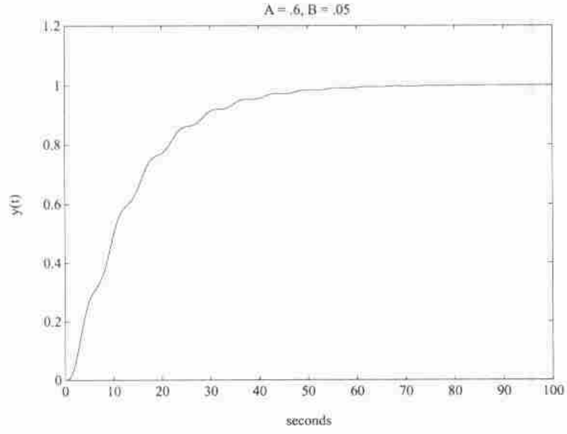


Fig. 28. Simulation of system with redesigned fuzzy controller.

can be seen that, as would be expected, the limit cycle prediction procedure using describing functions is slightly more complex for MISO systems. However, with the adjustments mentioned above the procedure follows the same format as before. This will be shown in the following example.

The plant for our example has the transfer function $G(S) = (s+1)^2/s^3$. Our fuzzy controller is the two input fuzzy controller with inputs e and $\dot{e}$ described in Section 2 with parameters A, B, D. It is our desire to use describing function analysis to help us choose the parameters A, B, and D for the fuzzy controller such that no limit cycles occur. Our first attempt will be with $A = B = D = 1$. To determine if there will be any limit cycles when this controller is used with our current plant, we will first have to determine $N(C,\omega)$ for the controller. We will experimentally determine the describing function for $0 \leq C \leq 10$ and $\omega = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90$, and 100. We then plot $G(j\omega)$ and $-1/N(C,\omega)$ on the same plot and search for intersection points. This plot is shown in Fig. 29. Notice that there are several intersection points. However, only at the intersections at points 1 and 2 is ω the same for $G(j\omega)$ and $-1/N(C,\omega)$. Therefore these are the only points that correspond to solutions of the harmonic balance equation and therefore limit cycles. At point 1, $\omega = 0.97$ rad/s and $C = 2.8$. However, because $G(j\omega)$ has several encirclements at infinity, point 1 is encircled and therefore represents an unstable limit cycle and the amplitude will not be 2.8. At point 2, $\omega = 1.14$ and $C = 1.9$ but once again this is an unstable limit cycle due to the encirclements at infinity.

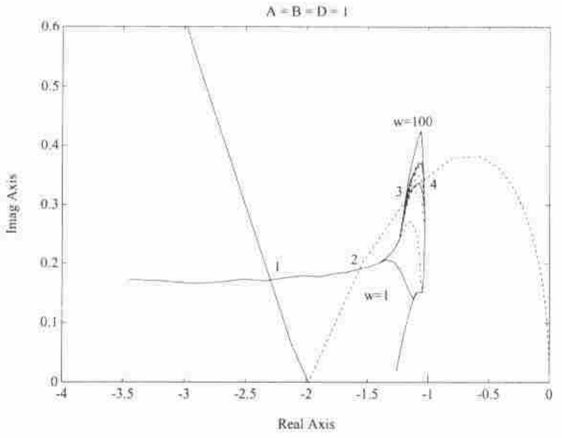


Fig. 29. Plot of $G(j\omega)$ and $-1/N(C,\omega)$ for $A=B=D=1$.

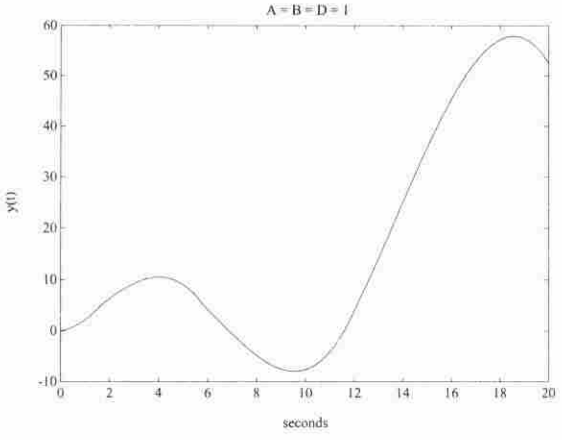


Fig. 30. Simulation with $A = B = D = 1$ and $r = 5$.

Therefore we can predict that there are two possible limit cycles for the system. A simulation of this system with $r = 5$ is shown in Fig. 30. It can be seen that an unstable limit cycle does indeed exist.

To remove the limit cycles we must change the controller so that $-1/N(C,\omega)$ no longer intersects $G(j\omega)$. For the SISO fuzzy controller we accomplished this by picking A/B less than the value of $Re[G(j\omega)]$ when $G(j\omega)$ crosses the real axis. However, for our MISO controller $-1/N(C,\omega)$ does not lie along the real axis and we have the extra parameter, D, which also affects the describing function. We will proceed by choosing values for A, B, and D and then checking the plot of $G(j\omega)$ and $-1/N(C,\omega)$ for intersection points. Our first attempt will be for $A = 3$, $B = 1$, and $D = 0.1$; we hope that these values will move $-1/N(C,\omega)$ to the left of $G(j\omega)$. It can be seen in Fig. 31 that these values do not give the desired result and that a limit cycle is still predicted. Next we will try $A = 1$, $B = 10$, and $D = 0.1$. Figure 32 shows that for this controller we again have a predicted limit cycle. However, the intersection occurs not at $C = 2.8$ or $C = 1.9$ but at $C = 33.3$; for $r \ll 33.3$ the system will be stable. If we simulate our system with this fuzzy controller and $r = 5$ we obtain the result shown in Fig. 33. The system is stable for $r = 5$. However, from further simulations we find that for $r \approx 25$ and greater the system enters an unstable limit cycle. If larger values of r are anticipated a fuzzy controller will have to be found for which there are no intersections of $G(j\omega)$ and $-1/N(C,\omega)$ at any point.

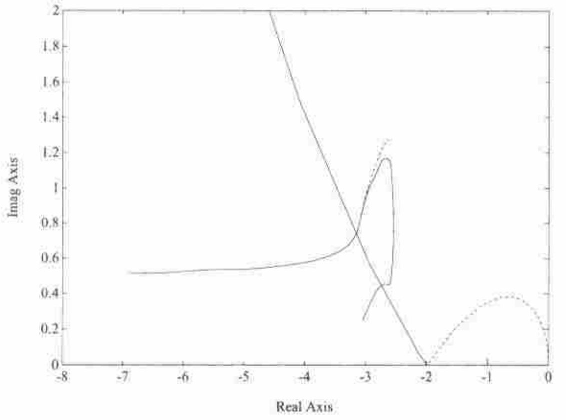


Fig. 31. Plot of $G(j\omega)$ and $-1/N(C,\omega)$ for $A = 3$, $B = 1$, and $D = 0.1$.

6. Concluding remarks

Overall, in this survey paper we have provided a tutorial introduction to nonlinear analysis of fuzzy control systems. In particular we have introduced and provided tutorial examples for (i) stability analysis, (ii) prediction of steady state tracking capabilities, and (iii) prediction of limit cycle amplitude, frequency, and stability. The approaches to nonlinear analysis provide convenient techniques to verify and certify fuzzy control systems and can often provide insights into how to design fuzzy controllers. In relation to work previously done in this area we contribute the following: (i) minor extensions to the work done in [4, 13, 14, 17, 20, 21, 26–28, 38, 39] by applying the

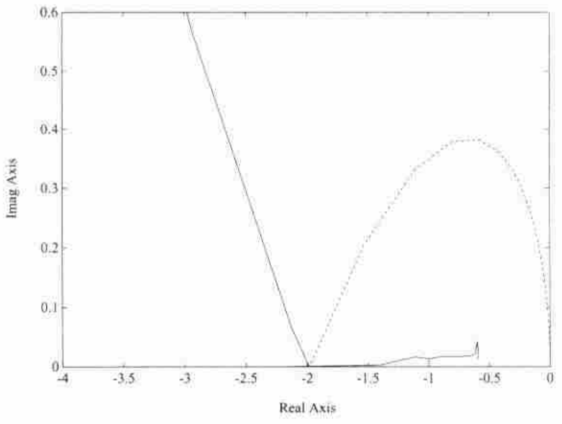


Fig. 32. Plot of $G(j\omega)$ and $-1/N(C,\omega)$ for $A = 1$, $B = 10$, and $D = 0.1$.

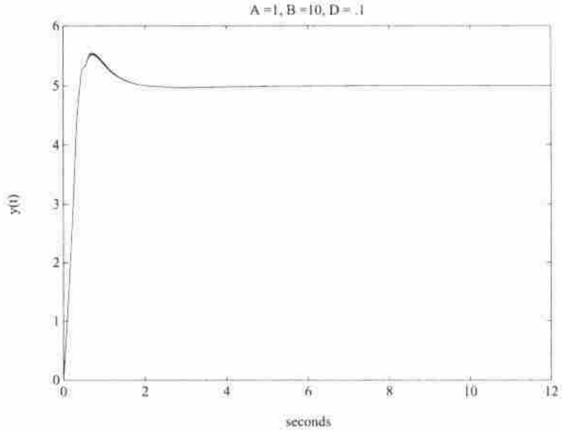


Fig. 33. Simulation with $A = 1$, $B = 10$, and $D = 0.1$.

Circle Criterion with sufficient and necessary conditions for stability from [49] to fuzzy control systems, (ii) introduction of the steady state error prediction method for systems with sector bounded nonlinearities from [40] and application of it to fuzzy control systems, and (iii) minor extension to the work of [25] by using describing functions found experimentally from the actual fuzzy controller rather than from models (simultaneous to and independent of the work done in [6]).

It is important to note that there are limitations to the approaches that we survey in this paper. In general, except for Lyapunov's methods in Section 3, we have only examined linear plant models or nonlinear plants that can be manipulated to be in the form of Fig. 8. While the results in Sections 3.2, 4, and 5 can certainly be applied to models linearized about operating points in a nonlinear system, such results are only local in nature. Furthermore, we have limited ourselves throughout the entire paper to SISO and MISO fuzzy controllers (except in the Lyapunov approaches). In addition to these general limitations, there are also limitations specific to each section. In Section 3.2 on absolute stability we have only examined the SISO fuzzy controller and not the MISO case (of course extension to the multivariable case is not difficult using, e.g., the development in [24]). Furthermore, although the Circle Criterion conditions are sufficient and necessary, the necessary conditions are for a class of nonlinearities and do not identify which of the nonlinearities (i.e., which fuzzy controller) within the class will cause the system to become unstable. In Section 4 the circumvention of the convergence problem by moving poles requires that we make the plant model a less reliable representation of the plant. Our describing function technique in Section 5, even though it can be applied to SISO and MISO fuzzy controllers and nonlinear plant models, is limited by the fact that the use of the approach for more than three inputs to the fuzzy controller becomes prohibitive.

To address these and other problems we suggest the following directions for further research:

- expansion of all the presented procedures to a wider class of nonlinear plants where a mathematical characterization of the fuzzy controller is used (while work along these lines has been addressed in [27, 28] we utilize a *graphical approach* to nonlinear analysis throughout the paper where we, e.g., plot the I/O map of the fuzzy controller and read off pertinent information such as the sector bounds, or use a graphical technique for describing function analysis[4]);
- development of a method for mathematically calculating the describing function for a wide class of fuzzy controllers which have different membership functions, rule-bases, inference strategies, and fuzzification and defuzzification techniques;

[4]We feel that the incorporation of graphical techniques for the nonlinear analysis of fuzzy control systems offers: (i) a more intuitive approach that ties in better with the fuzzy control design procedure, and (ii) some of the same advantages as have been realized in classical control via the use of graphical techniques (such as the Nyquist plot).

- investigation of extending the results of [40] to the MIMO case for analysis of steady state errors;
- development of a convergence algorithm for the steady state tracking analysis procedure of [40] that will converge for $\alpha = 0$ and $\rho \geq \nu$;
- further development of the "method of equivalent gains" for the design of fuzzy control systems (initial work in this area is contained in [22]);
- determination of how to use all the analysis approaches to gain insights into tuning individual membership functions (e.g., their shapes); and
- exploration of the use of the small gain theorem [24] and stability analysis results from variable structure control and differential geometric methods [24, 49] for fuzzy control systems (see, e.g., [33]).

Overall, it is hoped that this survey/tutorial paper serves (i) to provide an introduction to the important area of nonlinear analysis of fuzzy control systems and (ii) to motivate researchers to utilize rigorous engineering evaluations for the verification and certification of fuzzy control systems when this is dictated by the application.

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