Spatial Descriptions and Transformations

Read Chapter 2
Spatial representation (use a legged robot)

- Use coordinate system (frame) to represent spatial positions and orientation of objects
  - \((\hat{X}_E, \hat{Y}_E, \hat{Z}_E)\) set of three orthogonal unit vectors used to define an earth-fixed coordinate system
  - \((\hat{X}_B, \hat{Y}_B, \hat{Z}_B)\) set of three orthogonal unit vectors used to define a body-fixed coordinate system, original at the Center of Gravity (COG)
  - \((\hat{X}_F, \hat{Y}_F, \hat{Z}_F)\) set of three orthogonal unit vectors used to define a foot-fixed coordinate system
Fundamentals

- When we manipulate objects as we do in robotics, we need a way of describing positions and orientations of objects and the spatial relationship between them: body ↔ ground; leg ↔ body
- Positions and orientations are equally important
- Our approach to describe the position/orientation of objects:
  - Attach a coordinate system (frame) to each object
  - Vectors which position its original in space to give directions of its unit vectors
  - Frame is a description for each object which carries all the position/orientation information
  - Define the position/orientation of the frame with respect to another
Homogeneous Transformation

- Use a $4 \times 4$ matrix
- Gives position/orientation information of one frame with respect to another
- First used in graphics, also computer vision
- Applied in robotics to describe spatial relationship
- A free body in space is said to have 6 degrees of freedom (DOF)
- A *homogeneous transformation* in general has 6 independent pieces of information for specifying these 6 values
Position vectors

- A position vector may be represented by its coordinates in any given frame:
  - \( \mathbf{P} = 5\mathbf{\hat{X}}_B + 4\mathbf{\hat{Y}}_B + 3\mathbf{\hat{Z}}_B \)

\[
\mathbf{bP} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}
\]

- A leading superscript indicates the coordinate system of reference \{B\}

- Homogeneous coordinates:

\[
\mathbf{bP} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ w \end{bmatrix}
\]

\[
\mathbf{P} = \frac{p_x}{w} \mathbf{\hat{X}}_B + \frac{p_y}{w} \mathbf{\hat{Y}}_B + \frac{p_z}{w} \mathbf{\hat{Z}}_B
\]
Position vectors (continued)

• H.C. are handy because multiplication by a constant does not change the associated vector (we will use w=1 always)

• Dot product $u$ and $v$: $u \cdot v = ?$ a scalar

• Cross product of $u$ and $v$: $u \times v = ?$ a vector

$$u = u_x \hat{X_B} + u_y \hat{Y_B} + u_z \hat{Z_B}$$

$$v = v_x \hat{X_B} + v_y \hat{Y_B} + v_z \hat{Z_B}$$

• $u \cdot v = u_x v_x + u_y v_y + u_z v_z = |u||v|\cos \theta$

• $w = (u_y v_z - u_z v_y) \hat{X_B} + (u_z v_x - u_x v_z) \hat{Y_B} + (u_x v_y - u_y v_x) \hat{Z_B}$

• $|w| = |u||v|\sin \theta$
Homogeneous transformation - position

• A point represented in one frame carries information in 4 vectors
  – 3 for directions of unit vector and 1 for P origin of the frame

• Assume the coordinates of the point P in the body frame to be determined in the earth-fixed frame. A $4 \times 4$ matrix $^E{T}_B$ will do the job:

$$^E{P} = ^E{T}_B {^B}{P}$$

• What is $^E{T}_B$?
Homogeneous transformation – position (continued)

\[
ET_B = \begin{bmatrix}
1 & 0 & 0 & EP^x_{BORG} \\
0 & 1 & 0 & EP^y_{BORG} \\
0 & 0 & 1 & EP^z_{BORG} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[EP_{BORG} = \text{position vector from the origin of the earth-fixed frame to the origin of body-fixed frame expressed in the earth frame.}\]

For the diagram shown earlier, \[EP_{BORG} = [0 \hspace{1cm} -10 \hspace{1cm} -3 \hspace{1cm} 1]^T.\]
Homogeneous transformation – position (continued)

• Then we have:

\[
E_P= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -10 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
5 \\
4 \\
3 \\
1
\end{bmatrix} = \begin{bmatrix}
5 \\
-6 \\
0 \\
1
\end{bmatrix}
\]

\[\begin{align*}
AP &= AT_B BP \\
AP &= BP + AP_{BORG}
\end{align*}\]

Equivalent

Fourth vector of \(AT_B\) is a vector from \(A\) to \(B\) expressed in frame \(A\)