QUALITATIVE EQUIVALENCE OF DYNAMICAL SYSTEMS WITH APPLICATIONS TO DISCRETE EVENT SYSTEMS

A. N. Michel* and K. Wang*
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556

K. M. Passino*
Department of Electrical Engineering
The Ohio State University
2015 Neil Ave.
Columbus, OH 43210

Let $R$ denote the set of real numbers, let $R^+ = [0, \infty)$, and let $N$ denote the set of nonnegative integers, i.e., $N = \{0, 1, 2, \ldots\}$.

Let $R^n$ denote real $n$-space. If $x \in R^n$, then $x^T = (x_1, x_2, \ldots, x_n)$ denotes the transpose of $x$. Also, if $x, y \in R^n$, then $x \leq y$ signifies $x_1 \leq y_1, x < y$ signifies $x_i < y_i$, and $z > 0$ signifies $x_i > 0$ for all $i = 1, \ldots, n$.

We let $(X, d)$ be a metric space, where $X$ denotes the underlying set and $d$ denotes the metric. When $d$ is clear from context, we will frequently denote the metric space simply by $X$ instead of $(X, d)$.

The distance between a point $x \in X$ and a set $M \subset X$ is defined by

$$d(x, M) = \inf_{y \in M} d(x, y).$$

If $Y$ and $Z$ are metric spaces and if $f : Y \to Z$, and if $f$ is continuous, we write $f \in C(Y, Z)$, i.e., $C(Y, Z)$ denotes the set of all continuous mappings from $Y$ to $Z$.

For a function $f : R \to R$, we denote the upper right hand, upper left hand, lower right hand and lower left hand Dini derivatives by $D^+ f, D^- f, D^+ f$, and $D^- f$, respectively. When we have a fixed Dini derivative of $f$ in mind, we will simply write $D f$, in place of the preceding notation.

3. Background Material

The concept of general dynamical system which we will employ is more general than the conventional notion of dynamical system used in the literature (cf. [1]–[3]). For this reason, it will be necessary that we provide precise definitions of a general dynamical system and of its qualitative properties.

Definition 3.1 Let $(A, \rho)$ and $(X, d)$ be two metric spaces, let $T$ denote either $R^+$ or $N$, and let $p : T \times A \times X \to X$. For fixed $a \in A, t_0 \in T$, assume that $p(t, a, t_0) : T_{t_0} \to X$, where $T_{t_0} = [t_0, \infty) \cap T$. We call the mapping $p(t, a, t_0)$ a general motion. We say that a family of motions, $S$, forms a general dynamical system if $S$ is a subset of

$$\bigcup_{t_0 \in T} \{T_{t_0} \times \{a\} \times \{t_0\} \to X\}$$

satisfying the following properties:

(i) Let $S_{t_0} = S \cap (T_{t_0} \times \{a\} \times \{t_0\} \to X)$ and let $S_0 = \bigcup_{t_0 \in T} S_{t_0}$. For any $p \in S_a, p(t_0, a, t_0)$ depends only on $a$, i.e.,

$$p(t_0, a, t_0) = p(t_1, a, t_1) \quad \text{for any } p, p' \in S_a.$$  

We denote $S(a) = p(t_0, a, t_0)$ for all $p \in S_a$.

(ii) In (i), $S(a) \in C(A, X)$.

Remarks 3.1 (a) Henceforth, we will refer to the general dynamical system of Definition 3.1 as "system $S$" or as "general dynamical system $S$".

(b) In the important special case when $A \subset X$ and $\rho$ and $d$ are equivalent, the conditions (i) and (ii) in Definition 1 are usually replaced by

$$p(t_0, a, t_0) = \alpha, \quad \text{for all } p \in S,$$

In the present paper we address several qualitative aspects of general dynamical systems.

Definition 3.2 A set $M \subset X$ is said to be invariant with respect to system $S$ if

(i) $M \subset S(a) \Leftrightarrow \{x \in X : x = S(a), a \in A\}$, and

(ii) $S(a) \in M$ implies that $p(t, a, t_0) \in M$ for all $t \in T_{t_0}$, and all $p \in S$.  

The work of A. N. Michel and K. Wang was supported in part by the National Science Foundation, Grant ECS81-07728.

The work of K. M. Passino was supported in part by an Engineering Foundation Research Initiation Grant.
For purposes of brevity, we will frequently replace the phrase “M is invariant with respect to S” by “(S, M) is invariant.”

**Definition 3.3.** If (S, x₀) is invariant, we call x₀ an equilibrium (point).

**Definition 3.4.** For an invariant set M ⊂ X, D ⊂ A is called a primitive set of M, if

\[ D = \{ a ∈ A : S(a) ∈ M \}. \]

Throughout this paper, unless explicitly stated otherwise, we always assume that M ⊂ X is invariant with respect to system S, i.e., (S, M) is invariant. We note that if (S, M) is invariant, then M = S(D), where D is the primitive set of M.

**Definition 3.5.** (S, M) is stable if for every ϵ > 0 and any t₀ ∈ T, there exists a δ = δ(t₀, ϵ) > 0 such that

\[ d(p(t, a, t₀), M) < ϵ \]

for all t ∈ T₀, for every a ∈ S, whenever ρ(a, D) < δ, where D denotes the primitive set of M.

In the above definition, δ depends on ϵ and t₀. If δ is independent of the choice of t₀, i.e., δ = δ(ϵ), then (S, M) is said to be uniformly stable.

**Definition 3.6.** (S, M) is asymptotically stable if

(i) it is stable, and

(ii) there exists an η = η(t₀) > 0 such that

\[ \lim_{t→∞} d(p(t, a, t₀), M) = 0 \]

for all a ∈ S, whenever ρ(a, D) < η, where D is the primitive set of M.

**Definition 3.7.** (S, M) is uniformly asymptotically stable if

(i) it is uniformly stable, and

(ii) for every ϵ > 0 and any t₀ ∈ T, there exists a δ > 0, independent of t₀ and ϵ, such that

\[ d(p(t, a, t₀), M) < ϵ \]

for all t ∈ T₀, whenever ρ(a, D) < δ, where D is the primitive set of M.

**Definition 3.8.** (S, M) is exponentially stable if there exists a α > 0, and for every ϵ > 0 and γ > 0, independent of t₀ and ϵ, such that

\[ d(p(t, a, t₀), M) ≤ e^{-α(t-t₀)} \]

for all t ∈ T₀, whenever ρ(a, D) < δ, where D is the primitive set of M.

**Definition 3.9.** (S, M) is unstable if it is not stable.

**Definition 3.10.** A motion p(t, a, t₀) in S is bounded if there exists a β > 0 such that d(p(t, a, t₀), S(a)) < β for all t ∈ T₀, where β may depend on each motion.

**Definition 3.11.** A general dynamical system S is uniformly bounded if for any α > 0 and t₀ ∈ T, there exists a β = β(α) > 0 (independent of t₀) such that d(S(a), x₀) < α, then d(p(t, a, t₀), x₀) < β for all t ∈ T₀, where x₀ is a fixed point in X.

**Definition 3.12.** A general dynamical system S is uniformly ultimately bounded if there exists a B > 0 and if corresponding to any α > 0 and t₀ ∈ T, there exists a τ = τ(α) > 0 (independent of t₀) such that d(p(t, a, t₀), x₀) < B for all t ∈ T₀ + τ, whenever d(S(a), x₀) < α.

In the above two definitions, the constants β and B may depend on the choice of x₀ ∈ X. However, the definitions themselves are independent of the choice of x₀. Furthermore, we may replace x₀ ∈ X in these definitions by any bounded set in X.

We will utilize several types of comparison functions (cf.[1] and [6]) defined in the following.

**Definition 3.13.** A continuous function ϕ : [0, r₁] → R⁺ (or ϕ : [0, ∞) → R⁺) is said to belong to class Λ, i.e., ϕ ∈ Λ, if ϕ(0) = 0 and ϕ is strictly increasing on [0, r₁] (or on [0, ∞)). If ϕ : R⁺ → R⁺, ϕ ∈ Λ, and limₜ→∞ ϕ(ₜ) = ∞, then ϕ is said to belong to class Λ⁺, i.e., ϕ ∈ Λ⁺.

**Definition 3.14.** A vector valued function h(y, t) = (h₁(y, t), ..., hₙ(y, t))ᵀ is said to be quasi-monotone if for each component hᵢ, j = 1, ..., l, the inequality hᵢ(y, t) ≤ hᵢ(x, t) is true whenever yᵢ ≤ xᵢ, for all i ≠ j and yⱼ = xⱼ.

4. Stability Preserving Mappings

The principal aim of the present section is to identify dynamical systems with similar qualitative properties. We will accomplish this by utilizing stability preserving mappings. Such mappings were originally introduced by Thomas [4] and have subsequently been studied by Hahn [5] and by Michel and Miller [6]. The various notions of stability preserving mappings employed in the present paper are more general than the corresponding concepts used in [4]-[6].

In the sequel, for two general dynamical systems (Aᵢ, Xᵢ, Sᵢ), i = 1, 2, the term “the stability of (Sᵢ, Mᵢ) is equivalent to the stability of (S₂, M₂)” will mean that (Sᵢ, Mᵢ) is stable if and only if (S₂, M₂) is stable. The equivalence of uniform stability, asymptotic stability, uniform asymptotic stability, uniform boundedness of solutions, uniform ultimate boundedness of solutions, and so forth, for (Sᵢ, Mᵢ) and (S₂, M₂) are defined similarly.

**Definition 4.1.** Let (Aᵢ, Xᵢ, Sᵢ) be general dynamical systems, i = 1, 2. Assume that A₁ ⊂ X₂ and that the metric on A₁ is induced by the metric on X_2. We say that V : X₁ × T → X₂ is a stability preserving mapping from (S₁, M₁) to (S₂, M₂) if V satisfies

(i) S₂ = V(S₁) = \{ q(t, t₀) : q(t, t₀) = V(p(t, a, t₀), t) with τ = V(S₁(a), t₀) \};

(ii) M₂ = V(M₁) = \{ x₂ ∈ X₂ : x₂ = V(x₁(t), t) \};

(iii) the stability of (S₁, M₁) is equivalent to the stability of (S₂, M₂).

We say that V is a strongly stability preserving mapping, if V satisfies conditions (i), (ii), and (iii) and if (iv) the asymptotic stability, uniform stability, and uniform asymptotic stability of (S₁, M₁) and (S₂, M₂) are equivalent, respectively.

In the next result, we establish sufficient conditions for strongly stability preserving mappings.

**Theorem 4.1.** Let (Aᵢ, Xᵢ, Sᵢ), i = 1, 2 be two general dynamical systems and let A₁ ⊂ X₂ with the metric on A₁ induced by the metric on X₂. Assume that V : X₁ × T → X₂, V(x,t) is continuous in x ∈ X₁ for each fixed t ∈ T, and V satisfies

(i) S₂ = V(S₁);

(ii) there exist ψ₁, ψ₂ ∈ K such that

\[ ψ₁(d₁(x, M₁)) ≤ d₂(V(x, t), M₂) ≤ ψ₂(d₁(x, M₁)) \]

for x ∈ X₁, t ∈ T, where d₁, d₂ are metrics on X₁, X₂, respectively; and

(iii) there exist ψ₁ ∈ K such that

\[ ρ(a, D₁) ≤ ψ₁(d₁(S₁(a), M₁)) \]

for a ∈ A₁, where ρ is the metric on A₁, and D₁ ⊂ A₁ is the primitive set of M₁.

Then, V is a strongly stability preserving mapping.

Furthermore, if in (ii), ψ₁(τ) = a₁r₉, a₁ > 0, b₁ > 0, i = 1, 2, then V is exponential stability preserving, i.e., the exponential stability of (S₁, M₁) and (S₂, M₂) are equivalent.

In applications, the boundedness properties of a general dynamical system play an important role. The following results address mappings which preserve boundedness properties.

**Theorem 4.2.** Assume that hypotheses (i) and (ii) of Theorem 4.1 hold with ψ₁, ψ₂ ∈ K⁺ and that M₁ and M₂ are bounded. Then,

(a) the uniform boundedness of (S₁, M₁) and (S₂, M₂) are equivalent; and
Theorem 5.1. Let \( (A_i, X_i, S_i), \ i = 1, 2 \) be two general dynamical systems and let \( A_2 \subseteq X_2 \) with the metric on \( A_2 \) induced by the metric on \( X_2 \). Assume that \( V : X_1 \times T \rightarrow X_2 \), and \( V(x,t) \) is continuous in \( x \in X_1 \) for each fixed \( t \in T \), and \( V \) satisfies

\( (i) \) \( V(S_1) \subseteq S_2 \); and

\( (ii) \) there exist \( \psi_1, \psi_2 \in K \) such that

\[ \psi_1(d_1(x, M_1)) \leq d_2(V(x,t), M_2) \leq \psi_2(d_1(x, M_1)) \]

for \( x \in X_1 \) and \( t \in T \), where \( d_1, d_2 \) are metrics on \( X_1, X_2 \), respectively. Then, the following statements are true:

(a) If \( S_1(D_1) = M_1 \), where \( D_1 \) is the primitive set of \( S_1 \), then the invariance of \( (S_1, M_1) \) implies the invariance of \( (S_2, M_2) \).

(b) The stability, asymptotic stability, uniform stability, and uniform asymptotic stability of \( (S_2, M_2) \) imply the same corresponding types of stability for \( (S_1, M_1) \).

(c) If in \( (ii) \) \( \psi_2(r) = ar^a, a > 0, b > 0 \), then the exponential stability of \( (S_2, M_2) \) implies the exponential stability for \( (S_1, M_1) \).

(d) If \( M_1, M_2 \) are bounded and if in \( (ii) \) \( \psi_1, \psi_2 \in K_R \), then the uniform boundedness and uniform ultimate boundedness of \( S_2 \) imply the same corresponding type of boundedness of \( S_1 \).

The significance of the above results lies in the fact that, if applicable, they reduce the qualitative analysis of system \( S_1 \) (which may be complicated) to the qualitative analysis of system \( S_2 \) (which may be simple).

Next, we choose \( S_2 \) to be determined by a system of ordinary differential inequalities with nonnegative solutions, i.e., \( S_2 \) is determined by

\[ Dz \leq h(z, t) \]

where \( D \) denotes a fixed Dini derivative, \( z \in C[\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+] \).

Corollary 5.1. Let \( (A, X, S) \) be a general dynamical system with \( T = \mathbb{R}^+ \), let \( X, S \) be a metric space, and let \( M \subseteq X \). Assume that there exists a \( V : X \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( V(x,t) \) is continuous in \( x \in X \) for each fixed \( t \in \mathbb{R}^+ \), and where \( V \) satisfies the following conditions:

(i) there exists \( h \in C[\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+] \) such that

\[ Dz \leq h(V(z,t), t) \]

for all \( z \in \mathbb{R}^+ \), where

\[ Dz = \frac{dz}{dt} \]

and

\[ D(V(p(t,a,t_0), t)) \geq D(V(p(t,a,t_0), t)) \]

(5.1)

and \( D(V(p(t,a,t_0), t)) \) denotes a Dini derivative with respect to \( t \); and

(ii) there exist \( \psi_1, \psi_2 \in K \), such that

\[ \psi_1(d_1(x, M_1)) \leq D(V(x,t), M_2) \leq \psi_2(d_1(x, M_1)) \]

for \( x \in X \) and \( t \in \mathbb{R}^+ \), where \( | \cdot | \) is any norm on \( \mathbb{R}^n \).

Then the statements \( (a)-(d) \) in Theorem 5.1 are true with \( (S_1, M_1) = (S, M) \) and \( (S_2, M_2) = (S_2, (0)) \).

The proof of Corollary 5.1 is a direct consequence of Theorem 5.1.

We point out that typical results of the existing comparison theory (see, e.g., [7,17-18,22]), may be phrased as follows: under certain reasonable assumptions on \( h \in C[\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+] \) one of several types of (Lyapunov) stability of an equilibrium of \( S_2 \) (assumed to be the origin \( 0 \), without loss of generality) is guaranteed by the corresponding type of stability of an equilibrium of \( S_2 \) (assumed to be zero, without loss of generality) where \( S_2 \) is the dynamical system determined by \( x = h(x, t) \),

\[ h(0,t) = 0 \quad \text{for all} \ t \in \mathbb{R}^+. \]

In the present context, the qualitative analysis of \( S \) is reduced to a qualitative analysis of \( S_2 \). A typical assumption is that \( h(x,t) \) in \( (E) \) is quasimonotone (see Definition 3.14).

The above discussion is made more precise by means of a sample result, Theorem 5.2. To establish these, we require two preliminary results.

Lemma 5.1. Assume that \( h \in C[\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+] \) and \( h(0,t) = 0 \) for all \( t \in \mathbb{R}^+ \). For

\[ \dot{x} = h(x(t)), \]

the equilibrium \( x = 0 \) is stable, then there exists a \( \delta > 0 \) such that any noncontinuable solution \( x(t) \) of \( (E) \) with \( \|x(0)\| < \delta \) must be defined for all \( t \in \mathbb{R}^+ \). If the equilibrium \( x = 0 \) is uniformly asymptotically stable in the large, then any noncontinuable solution \( x(t) \) of \( (E) \) must be defined for all \( t \in \mathbb{R}^+ \).

Lemma 5.2. Assume that \( h \in C[\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+] \) is quasimonotone and \( h(0,t) = 0 \) for all \( t \in \mathbb{R}^+ \). Then the stability, asymptotic stability, uniform stability, uniform asymptotic stability, exponential stability in the large and exponential stability in the large of \( (S_2, (0)) \) imply the same corresponding types of stability of \( (S_2, (0)) \). Moreover, the uniform boundedness and uniform ultimate boundedness of \( S_2 \) imply the same corresponding types of boundedness of \( S_2 \).

Corollary 5.1 and Lemma 5.2 imply now the following comparison theorem.

Theorem 5.2. Let \( (A, X, S) \) be defined as in Corollary 5.1. If in addition to the hypotheses of Corollary 5.1 it is assumed that \( (A, x) \) is quasimonotone, then statements \( (a)-(d) \) in Theorem 5.1 are true with \( \langle S_1, M_1 \rangle = (S, M) \) and \( \langle S_2, M_2 \rangle = (S_2, (0)) \).

We conclude by noting that in the application of Theorem 5.1 to obtain comparison results (such as Theorem 5.2) it is not necessary to choose \( S_2 \) to be a dynamical system determined by ordinary differential equations or inequalities. For instance, we may choose \( S_2 \) to be a system determined by partial differential equations or inequalities, or by integro-differential equations (resp., inequalities). The choice of \( S_2 \) will, in general, be determined in a natural way by the given application on hand.

6. Interconnected Systems

Stability preserving mappings and qualitative equivalence can also be employed to establish a comparison theory for general interconnected dynamical systems. Such results, which include existing results as special cases ([7,18,22]), are omitted due to space limitations.

7. Application to Discrete Event Systems

Discrete event systems (DES) are dynamical systems which evolve in time by the occurrence of events at possibly irregular time intervals. In this section we will show how several results of the previous sections apply to a class of logical DES called ‘Petri nets’.

A. Petri Net Models

For our discussions on Petri nets we will adhere to the standard notation given in [27] where a Petri net

\[ P = (P, T, F, W, M_0) \]

where

(i) \( P = \{p_1, p_2, ..., p_n\} \) is a finite set of places (represented with circles),

(ii) \( T = \{t_1, t_2, ..., t_n\} \) is finite set of transitions (represented with line segments),
(iii) \( F \subset (P \times T) \cup (T \times P) \) is a set of arcs (represented with arrows).

(iv) \( W : F \rightarrow \{0,1,2,3,\ldots \} \) is an arc weight function (represented with numbers labeling arcs and assumed for convenience that if \((p,t) \notin F\) or if \((t',p) \notin F\) we will extend the arc weight function so that \(W(t',p') = W(p,t) = 0\) for those cases and the arrow is omitted), and

(v) \( M_0 : P \rightarrow \{0,1,2,3,\ldots \} \) is an initial marking (represented with dark dots, i.e., tokens, in places).

It is assumed that \( P \cap T = \emptyset \) and \( P \cup T \neq \emptyset \). The Petri net structure is \( N = (P,T,F,W) \) so that \( P(N, M_0) \). The Petri net PN is normally referred to as the "General Petri net" while if "inhibitor arcs" are added it is called an "Extended petri net" \([27,28]\) (also recall that "finite capacity nets" can be reduced to General Petri nets and that Marked Graphs and State Machines \([27]\) are special cases of General Petri nets).

If the initial marking is pre-specified then we will refer to the Petri net as \((N,M_0)\) or simply \(PN\), whereas, if the initial marking is not specified we will refer to the net as \(N\). Also note that if \(W(p,t) = 0\) or \(W(t,p) = 0\) then this is often represented graphically by \(a\) arcs from \(p\) to \(t\) (to \(p\)) each with no numeric label.

Let \(M_0(p)\) denote the marking (i.e., the number of tokens) at place \(p\) at time \(k\) and let \(M_k = (M_k(p_1), \ldots , M_k(p_n))\) denote the marking (state) of \(P\) at time \(k\) (the \(\times\) will be dropped when \(k\) is not needed). A transition \(t_j\) \(T\) is said to be enabled at time \(k\) if \(M_k(p_j) \geq W(t_j,p_j)\) for all \(p_j \in P\) such that \((t_j,p_j) \in E\). It is assumed that at each time \(k\) there exists at least one transition to fire. If a transition is enabled, then it can fire. If an enabled transition \(t_j\) \(T\) fires at time \(k\), then the next firing marking for place \(p\) \(P\) is given by

\[
M_{k+1}(p) = M_k(p) + W(t_j,p_j) - W(t_j,p_t),
\]

where \((t_j,p_j) \in E\) and \((p_t,t_j) \in F\). Let \(R(M_k)\) denote the set of markings of \(PN\) (states) that can be reached from \(M_0\) and let \(R_k(M)\) denote the set of markings that are reachable from \(M\) in one transition firing.

Let \(A = [a_{ij}]\) denote an \(n \times m\) matrix of integers (the incidence matrix) where \(a_{ij} = -a_{ij} < a_{ij} = W(p_i,p_j)\) and \(a_{ij} = W(t_j,p_i)\). Let \(u_k \in \{0,1\}^m\) denote a firing vector where if \(t_j \in T\) is fired then its corresponding firing vector is \(u_k = [0,0,1,0,0,\ldots,0]^T\) with the "1" in the \(j^{th}\) position in the vector and zeros everywhere else. The matrix equations (non-linear difference equations defined on \(N^m\) with non-unique solutions) describing the dynamical behavior represented by a Petri net are given by \([27,28]\)

\[
M_{k+1} = M_k + A^T u_k \tag{7.1}
\]

where if at step \(k\) \(a_{ij} \leq M_k(p_j)\) for all \(p_j \in P\), then \(t_j \in T\) is enabled and if this \(t_j \in T\) fires then its corresponding firing vector \(u_k = [0,0,1,0,0,\ldots,0]^T\) is utilized in \((7.1)\) to generate the next state. Notice that if \(M_k \in R(M_0)\), and if we fire some sequence of \(d\) transitions \(t_j\) \(T\) corresponding firing vectors \(u_k, u_1, \ldots, u_{d-1}\), we will obtain \(M_d = M_0 + A^T u\) with

\[
u = \sum_{k=0}^{d-1} u_k
\]

where \(u\) is called the firing count vector.

An Extended Petri net is obtained from a General Petri net by adding inhibitor arcs (sometimes called "not arcs"). Let \(F_a \subset (P \times T)\) denote the set of inhibitor arcs for the Extended Petri net \(EPN = (P,T,F_a,W,M_0)\) \((\forall F_a \in F)\). We use a line with a small circle on the end to graphically represent the inhibitor arc. The inhibitor arc does not change in any way what happens when a transition \(t \in T\) fires (i.e., equation \((7.1)\) remains unchanged for the Extended Petri net). The inhibitor arc does, however, change which transitions are enabled at each step. The set of transitions in EPN enabled at time \(k\) is given by

\[
\begin{align*}
\{t_j : M_k(p_j) \geq W(p_t,t_j), \forall p_t \in P \ s.t. (p_t,t_j) \in F_a \} - \\
\{t_j : (p_t,t_j) \in F_a \ and \ M_k(p_t) = 0\}.
\end{align*}
\]

Hence, the inhibitor arc tests if a place has a zero marking. It is important to study properties of Extended Petri nets due to the fact that the addition of the inhibitor arc greatly enhances the "modeling power" of the Petri net \([28]\). The characterization and analysis of the qualitative properties of systems represented by Petri nets is based on the fact that Petri net models are a special case of the general dynamical system of Definition 3.1.

B. Boundedness Analysis of Petri Nets

The fact that systems represented by Petri nets are amenable to Lyapunov stability analysis was first pointed out in \([29,26]\). In this section we show that the Petri net theoretic boundedness properties and analysis \([27,28]\) are actually special cases of the boundedness definitions in Section 3 and the Lyapunov approach to boundedness analysis. Let \(\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in R^m\) and \(\xi > 0\); \(i = 1, \ldots, m\). Throughout this section we will use the metric \(d : N^m \times N^m \rightarrow R^+\) where

\[
d(M, M') = \sum_{i=1}^{m} \xi(M(p_i) - M'(p_i)) \tag{7.2}
\]

and we use \(D \subset N^m\) to denote a bounded set. Next, we state the standard definitions of boundedness for Petri nets \([27,28]\).

Definition 7.1: A Petri net \((N,M_0)\) is said to be \(\gamma\)-bounded or simply bounded if \(M(p) \leq \gamma\) for all \(p \in P\) and \(M \in R(M_0)\).

Definition 7.2: A Petri net \(N\) is said to be structurally bounded if it is bounded for any finite initial marking \(M_0\).

For a Petri net \((N,M_0)\): (i) \((N,M_0)\) is \(\gamma\)-bounded for some \(\gamma > 0\) if and only if the motions of \((N,M_0)\) which begin at \(M_0\) are bounded, (ii) \(N\) is structurally bounded if and only if \(N\) possesses Lagrange stability, and (iii) \(N\) is structurally bounded if and only if the motions of \(N\) are uniformly bounded. Also notice that for the Petri net \(N\), if \(N\) is uniformly ultimately bounded with bound \(B\) then \(N\) is structurally bounded (but the converse is not necessarily true).

Next, we show how the Petri net-theoretic approach to the analysis of structural boundedness (uniform boundedness) is actually the Lyapunov stability-theoretic approach. Moreover, we introduce the characterization and analysis of uniform ultimate boundedness for Petri nets.

Theorem 7.1: For the Petri net \(N\) with \(D = \emptyset\):

(i) \(N\) is uniformly bounded if there exists an m-vector \(\phi > 0\) such that \(A\phi \leq 0\), and

(ii) \(N\) is uniformly ultimately bounded if there exists an m-vector \(\phi > 0\) and \(n\)-vector \(\pi > 0\) such that \(A\phi \leq -\pi\).

\[
\square
\]

Theorem 7.1 shows that the standard approach to boundedness analysis for General Petri nets is actually a special case of a Lyapunov approach to boundedness analysis. What is actually shown is that in the Petri net-theoretic approach to the analysis of structural boundedness \([27]\), in choosing \(\phi\), one is actually choosing a Lyapunov function \(V(M) = M^T\phi\). Once this is recognized, it may be easier to study boundedness properties by using the existing wealth of experience in choosing Lyapunov functions. Part (ii) of Theorem 7.1 and Remarks 7.1 provide what seems to be a first characterization and analysis of uniform ultimate boundedness and asymptotic stability in the large for Petri nets. It is important to note that the Lyapunov approach also applies to the many subclasses of Petri nets (e.g., Marked Graphs and State Machines). Finally, the Lyapunov approach to boundedness studies can also be utilized in the boundedness analysis of Extended Petri nets (both in the sense of Remarks 7.1 and by directly using the results of the previous sections for stability analysis).

Application 7.1: (Networks of Computers) Suppose that we are given a network of computers arranged in a "ring". The type of network that we consider is similar to the networks addressed in \([29]\). These are useful in mutual exclusion problems, for ensuring fairness, etc. (For a more detailed explanation of their utility, see \([30]\)). The Extended Petri net model EPN for the network of computers is shown in Figure 1.

Each place \(p_i \in P\) represents a computer node in the network and
the state of the computer (representing the possession of resources for carrying on a communication) is modeled via $M(p_i)$. The communications between the nodes are represented with the transitions (e.g., transitions $t_1$ and $t_2$ each represent different ways that node $i$ can communicate with node 2). Note that although we show only 5 nodes in the network, the analysis here is also valid for a general ring with $N$ nodes.

For the incidence matrix $A$ resulting from Figure 1, choosing $A_i = [1 1 1 1 1]^T$ results in $AB = 0$ (since for each $t_j \in T$, $W(n, t_j) \geq W(t_j, p_i)$) so that the Petri net is uniformly bounded by Theorem 7.1.

\[ V(M) = M^T \xi, \quad \xi = \phi, \quad \text{and the metric } d. \]

We must show that $V(M) \geq V(M')$ for all $M \in \mathcal{M}(M', D, \delta)$ and $M' \in \mathcal{R}(M)$ (i.e., it must be shown that for any state $M \in \mathcal{M}(M', D, \delta)$ it is true that $V(M) \geq V(M')$ no matter which transition $t \in T$ occurs). This is easily seen for the production network by simply noting that the number of input arcs and output arcs for each transition are identical. Notice that compared with the Petri net-theoretic approach, in the present case, (i) there is no need to write matrix equations, and (ii) there is a need for analysis for numerical calculations. Overall, both approaches do not lend themselves to automatic verification for boundedness properties; however, there exists much experience and intuitive insights into the choice of Lyapunov functions to aid in the stability analysis of a particular system.

C. Analysis of Interconnected Petri Nets

In the present subsection we conduct a qualitative analysis of a class of interconnected Petri nets. We assume that there are $l$ subsystems which are represented by different Petri net models $PN_i = (P_i, T_i, W_i, M_{0i})$, $i = 1, \ldots, l$. These models have places $p_{ij} \in P_i$ and transitions $t_{ij} \in T_i$ (representing the $j$th place or transition in subsystem $i$). They have states $M_i^j$; incidence matrices $A_i$; of dimension $n_i \times m_i$; and firing vectors $\phi^i$. In our development, we use the matrices $d_i(M', M') = \sum_{j=1}^{m_i} e_i^j(M[p]) - M[p]$ where $e_i^j = [e_i^1, \ldots, e_i^{m_i}]^T$ with $e_i^j > 0$, $i = 1, \ldots, l$. Also, we will employ bounded sets $D$, and Lyapunov functions $V(M) = V(M)^T \phi$ where $\phi = [\phi_1^0, \ldots, \phi_{l-1}^0]^T$ with $\phi_i > 0$, $i = 1, \ldots, l$. We assume that $\xi_i = \phi_i$, $i = 1, \ldots, l$.

Using the Lyapunov approach, choose the Lyapunov function $V(M) = M^T \xi, \xi = \phi$, and the metric $d$. We must show that $V(M) \geq V(M')$ for all $M \in \mathcal{M}(M', D, \delta)$ and $M' \in \mathcal{R}(M)$ (i.e., it must be shown that for any state $M \in \mathcal{M}(M', D, \delta)$ it is true that $V(M) \geq V(M')$ no matter which transition $t \in T$ occurs).

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For the composite system, which we denote by $S_1$, we view each of the $l$ subsystems as a node in $S_1$, and we specify the "interconnecting structure". We assume that the interconnecting structure is represented by a special Petri net $N = (P, T, F, W)$ where $P$ denotes a set of "macro places" $P = \{P_1, \ldots, P_l\}$ represented by circles (labeled $P_i$) which can contain (graphically) the entire Petri net $PN_i$. Let $T = \{t_1, \ldots, t_a\}$ denote a set of "interconnecting transitions" and let

\[ F \subseteq \{(U, P) \times T \} \cup \{T \times (U, P)\} \]

denote the arcs which allow for the interconnections of the $l$ subsystems.

For the composite system $S_1$, the enabling and firing of transitions works in the standard manner described above. In general, while the dimension of each of the $l$ subsystems is relatively low, the dimension of $S_1$ is high. We will use $M = [M^1, M^2, \ldots, (M^l)^T]^T$ to denote the state of $S_1$, $d_1$ to denote a metric for $S_1$, and $D = \{0\}$ to denote a bounded set of composite system states; For simplicity, we will assume that transitions do not occur simultaneously in $S_1$ (and hence, transitions in the subsystems, $E_i \in T_i$, and in the interconnecting structure, $t_i \in T$, do not occur at the same time). The following result illustrates how the comparison result, Theorem 7.1, applies in the qualitative analysis of interconnected discrete event systems.

**Theorem 7.4.** The interconnected Petri net set $S_1$ is uniformly bounded if:

(i) The $l$ subsystems of $S_1$ are uniformly bounded. In particular, there exist $\phi^i > 0$ such that $A_i \phi^i \leq 0$, $i = 1, \ldots, l$.

(ii) There exists an $l$-vector $\Phi > 0$ such that $\Delta \Phi \leq 0$ where $\Delta = [\delta_{ij}]$ is an $l \times l$ matrix with

\[ \delta_{ij} = \sum_{k=1}^{l} \phi_k [W(t_j, p_k) - W(p_k, t_j)]] \quad \text{and } \quad \Phi_k \in (0, t_k) \in T. \]

\[ \square \]

Theorem 7.2 which is in the spirit of the methodology advanced in [7] for the qualitative analysis of large scale dynamical systems, simplifies the analysis of interconnected systems represented by Petri nets. In this approach, we first establish the existence of the appropriate vectors $\phi^i$, $i = 1, \ldots, l$, to ensure that the subsystems are uniformly bounded. Provided that the interconnecting structure satisfies condition (ii) of Theorem 7.2, we conclude that the overall sys-


