Manipulator Dynamics (2)

Read Chapter 6
Dynamic equations for each link

- For each link there are two equations to describe the effects of force and torque to the motion:
  - Newton equation
    \[ iF = m \dot{v}_c \]
  - Euler’s equation
    \[ iN = c_{\dot{i}} I \dot{\omega} + i \omega \times c_i I \omega \]

\{C_i\} has its origin at the center of mass of the link and has the same orientation as the link frame \{i\}
Dynamic equations for all the links

- Use the iterative Newton-Euler algorithm

- Between links there are action and reaction forces (torques)

- For link \( n \), we have \( n+1f = 0 \) and \( n+1n = 0 \); therefore the equations can start from the last link and go inwards
Compute forces and torques

- Look the top figure in the previous slide, one has:

\[ iF = if - iR_{i+1}i^{+1}f \]

\[ in = in - iR_{i+1}i^{+1}n - iP_{Ci} \times IF - iP_{i+1} \times iR_{i+1}i^{+1}f \]

- From the above, one can obtain the following iterative equations:

\[ if = IF + iR_{i+1}i^{+1}f \]

\[ in = IN + iR_{i+1}i^{+1}n + iP_{Ci} \times IF + iP_{i+1} \times iR_{i+1}i^{+1}f \]

- Note that \( n^{+1}f = 0 \) and \( n^{+1}n = 0 \); so we can start the iteration from link \( n \).

- Since every joint has only one torque applied, we have

\[ \tau_i = (in)^T \widehat{IZ} \]

Question: how the above equation should be changed for a prismatic joint?
The iterative Newton-Euler dynamics algorithm (1)

- The iterative algorithm takes two steps
  - First step: perform *outward iterations* to compute velocities and accelerations
  - Second step: perform *inward iterations* to computer forces and torques

- First step:
  - Angular acceleration

\[ A_{\Omega_c} = A_{\Omega_B} + A_{R_B} B_{\Omega_c} \]

\[ i+1 \omega = i+1 R_i \ i \Omega + (\dot{\theta}_{i+1})^i+1 \hat{Z} \]

\[ i+1 \omega = i+1 R_i \ i \omega + (\dot{\theta}_{i+1})^i+1 \hat{Z} \]

\[ i+1 \dot{\omega} = i+1 R_i \ i \dot{\omega} + i+1 R_i \ i \omega \times \dot{\theta}_{i+1} i+1 \hat{Z} + \ddot{\theta}_{i+1} i+1 \hat{Z} \]

- Linear acceleration

\[ i+1 v = i+1 R_i ( i v + i \omega \times i P_{i+1} ) \]

\[ i+1 \dot{v} = i+1 R_i [ i \dot{\omega} \times i R_{i+1} i+1 P + i \omega \times ( i \omega \times i R_{i+1} i+1 P ) + i \dot{v} ] \]

- For the center of the link:

\[ i+1 \dot{v}_{C_{i+1}} = i+1 \dot{\omega} \times i+1 P_{C_{i+1}} + i+1 \omega \times ( i+1 \omega \times i+1 P_{C_{i+1}} ) + i+1 \dot{v} \]
The iterative Newton-Euler dynamics algorithm (2)

- Second step:

  - Draw the Newton-Euler equations

\[ i+1 F = m_{i+1} i+1 \dot{v}_{c_{i+1}} \]

\[ i+1 N = \dot{c}_{i+1} I i+1 \dot{\omega} + i+1 \omega \times c_{i+1} I i+1 \omega \]

- Inward iteration from link \( n \) to link \( l \):

\[ i f = i F + i R_{i+1} i+1 f \]

\[ i n = i N + i R_{i+1} i+1 n + i P_{C_i} \times i F + i P_{i+1} \times i R_{i+1} i+1 f \]

\[ \tau_i = (i n)^T i Z \]
Iterative algorithm example

- Use the OSU hexapod as an example which is similar to the manipulator shown below – to simplify the problem, let $m_1$ and $m_2$ be point mass and located at the tip of the link.
Use the two iterations

- Determine the values needed in the algorithm
  - The position vectors of the center of mass
    \[
    ^1P_{C_1} = D_1 \hat{X}_1
    \]
    \[
    ^2P_{C_2} = D_2 \hat{X}_2
    \]
  - The inertia tensor at the center of mass
    \[
    c_1 I_1 = 0
    \]
    \[
    c_2 I_2 = 0
    \]
  - Rotational matrices
    \[
    ^1R_0 = \begin{bmatrix}
    c_1 & s_1 & 0 \\
    -s_1 & c_1 & 0 \\
    0 & 0 & 1
    \end{bmatrix}
    \]
    \[
    ^2R_1 = \begin{bmatrix}
    c_2 & 0 & s_2 \\
    -s_2 & 0 & c_2 \\
    0 & -1 & 0
    \end{bmatrix}
    \]
• Outward iteration
  - For link 1

\[ \dot{\omega} = \dot{\theta}_1 \hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \quad \dot{\omega} = \ddot{\theta}_1 \hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \]

\[ \dot{v} = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \quad \dot{v}_c = \begin{bmatrix} 0 \\ D_1 \ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -D_1(\dot{\theta}_1)^2 \\ 0 \\ 0 \end{bmatrix} \quad \dot{v} = \begin{bmatrix} -D_1(\dot{\theta}_1)^2 \\ D_1 \ddot{\theta}_1 \\ g \end{bmatrix} \]

\[ F = m_1 \begin{bmatrix} -D_1(\dot{\theta}_1)^2 \\ D_1 \ddot{\theta}_1 \\ g \end{bmatrix} = \begin{bmatrix} -m_1 D_1(\dot{\theta}_1)^2 \\ m_1 D_1 \ddot{\theta}_1 \\ m_1 g \end{bmatrix} \quad N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
For link 2

\[
\begin{align*}
{\ddot{w}} &= {\ddot{R}R_1}\dot{\theta}_1\ddot{z}_1 + \dot{\theta}_2\ddot{z}_2 = 
\begin{bmatrix}
\frac{s_2\dot{\theta}_1}{c_2\dot{\theta}_1} \\
\frac{0}{c_2\dot{\theta}_1} \\
0
\end{bmatrix}
\end{align*}
\]

\[
{\ddot{\phi}} = \begin{bmatrix}
\frac{c_2}{-s_2} & 0 & s_2 \\
\frac{0}{c_2} & c_2 & 0 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
-D_1(\dot{\theta}_1)^2 \\
D_1\ddot{\theta}_1 \\
g
\end{bmatrix}
= \begin{bmatrix}
\frac{-c_2D_1(\dot{\theta}_1)^2}{s_2D_1(\dot{\theta}_1)^2} \\
\frac{s_2D_1(\dot{\theta}_1)^2}{-D_1\ddot{\theta}_1}
\end{bmatrix}
\]

\[
{\ddot{v}}_{c_2} = {\ddot{w}} \times 2P_{c_2} + {\ddot{w}} \times (2\omega \times \dot{2P}_{c_2}) + {\ddot{\phi}}
\]

\[
{\ddot{v}}_{c_2} = \begin{bmatrix}
\frac{s_2\dot{\theta}_1\dot{\theta}_2}{c_2\dot{\theta}_1} + s_2\ddot{\theta}_1 \\
-s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix}
\times \begin{bmatrix}
D_2 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
\frac{s_2\dot{\theta}_1}{c_2\dot{\theta}_1} \\
\frac{0}{c_2\dot{\theta}_1} \\
0
\end{bmatrix}
\times \left(\begin{bmatrix}
\frac{s_2\dot{\theta}_1}{c_2\dot{\theta}_1} \\
\frac{0}{c_2\dot{\theta}_1} \\
0
\end{bmatrix} \times \begin{bmatrix}
D_2 \\
0 \\
0
\end{bmatrix}\right)
+ \begin{bmatrix}
\frac{-c_2D_1(\dot{\theta}_1)^2}{s_2D_1(\dot{\theta}_1)^2} \\
\frac{s_2D_1(\dot{\theta}_1)^2}{-D_1\ddot{\theta}_1}
\end{bmatrix}
\begin{bmatrix}
s_2g \\
c_2g
\end{bmatrix}
\]

\[
{\ddot{F}} = m_2 \begin{bmatrix}
\frac{-c_2^2D_2(\dot{\theta}_1)^2 - D_2(\dot{\theta}_2)^2 - c_2D_1(\dot{\theta}_1)^2 + s_2g}{D_2\ddot{\theta}_2 + c_2s_2D_2(\dot{\theta}_1)^2 + s_2D_1(\dot{\theta}_1)^2 + c_2g} \\
\frac{2D_2s_2\dot{\theta}_1\dot{\theta}_2 - c_2D_2\ddot{\theta}_1 - D_1\ddot{\theta}_1}{2D_2s_2\dot{\theta}_1\dot{\theta}_2 - c_2D_2\ddot{\theta}_1 - D_1\ddot{\theta}_1}
\end{bmatrix}
\]

\[
{\ddot{N}} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
• Inward iteration
  - For link 2

\[ 2f = 2F \]

\[
\begin{bmatrix} D_2 \\ 0 \\ 0 \end{bmatrix} \times 2f = m_2 \begin{bmatrix} 0 \\ -D_2(2D_2s_2\dot{\theta}_1\dot{\theta}_2 - c_2 D_2\ddot{\theta}_1 - D_1\ddot{\theta}_1) \\ D_2(D_2\ddot{\theta}_2 + c_2s_2D_2(\dot{\theta}_1)^2 + s_2D_1(\dot{\theta}_1)^2 + c_2g) \end{bmatrix}
\]

- For link 1

\[ 1f = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & -1 \\ s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} -c_2D_2(\dot{\theta}_1)^2 - D_2(\dot{\theta}_2)^2 - c_2D_1(\dot{\theta}_1)^2 + s_2g \\ D_2\ddot{\theta}_2 + c_2s_2D_2(\dot{\theta}_1)^2 + s_2D_1(\dot{\theta}_1)^2 + c_2g \\ 2D_2s_2\dot{\theta}_1\dot{\theta}_2 - c_2 D_2\ddot{\theta}_1 - D_1\ddot{\theta}_1 \end{bmatrix} m_2 + \begin{bmatrix} -m_1D_1(\dot{\theta}_1)^2 \\ m_1D_1\ddot{\theta}_1 \\ m_1g \end{bmatrix} \]

\[ 1n = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & -1 \\ s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} -c_2D_2(\dot{\theta}_1)^2 - D_2(\dot{\theta}_2)^2 - c_2D_1(\dot{\theta}_1)^2 + s_2g \\ D_2\ddot{\theta}_2 + c_2s_2D_2(\dot{\theta}_1)^2 + s_2D_1(\dot{\theta}_1)^2 + c_2g \\ 2D_2s_2\dot{\theta}_1\dot{\theta}_2 - c_2 D_2\ddot{\theta}_1 - D_1\ddot{\theta}_1 \end{bmatrix} m_2 + \begin{bmatrix} -m_1D_1(\dot{\theta}_1)^2 \\ m_1D_1\ddot{\theta}_1 \\ m_1g \end{bmatrix} \]

\[ 1n = 1R_2 2n + 1P_{c_1} \times 1F + 1P_2 \times 1R_2 2f \quad \text{Figure out the details?} \]
Final dynamic equation

• Since every joint has only one degree of freedom, one has:

\[ \tau_1 = ( ^1n)^T ^1Z \]
\[ \tau_2 = ( ^2n)^T ^2Z \]

• Ultimately one can obtain:

\[ \tau = M(\Theta) \ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) \]

• where

\[ \Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \]
\[ \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \]
Lagrangian formulation

• Another approach is using the Lagrangian formulation
  - Step 1 obtain the Lagrangian of a manipulator

\[ L(\Theta, \dot{\Theta}) = k(\Theta, \dot{\Theta}) - u(\Theta) \]

where \( k(\Theta, \dot{\Theta}) = \sum_i^n k_i = \sum_i^n \frac{1}{2} m_i (v_{c_i})^T v_{c_i} + \frac{1}{2} (i \omega)^T c_i I i \omega \) is the kinetic energy of the manipulator, and

\[ u(\Theta) = \sum_i^n u_i = \sum_i^n [-m_i (0 g)^T 0 P_{c_i}] \]

is the potential energy of the manipulator

• The dynamic equation can be expressed as

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\Theta}} - \frac{\partial L}{\partial \Theta} = \tau \quad \rightarrow \quad \frac{d}{dt} \frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau \]

Becomes the same as the one shown earlier

\[ \tau = M(\Theta) \ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) \]