Stabilization of Hybrid Systems using a Min-Projection Strategy

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Abstract

This paper describes a method how to stabilize a system consisting of several subsystems. The subsystems are described by nonlinear models with different vector fields. The method is denoted min-projection strategy, since the vector field associated with the smallest (skew) projection is selected for each state. Conditions are given guaranteeing (exponential) stability. It is also shown how these conditions can be formulated as a nonlinear optimization problem, or, for a pre-determined projection matrix, a linear matrix inequality (LMI) problem. Sliding motions may occur in the basic form of the strategy. However, it is shown how this behavior can be avoided by introducing hysteresis around the switch surfaces, still preserving stability of the closed-loop hybrid system. Two examples are given to motivate and exemplify the strategy.

Keywords: Hybrid systems, Switched systems, Exponential stability, Stabilization, Linear matrix inequalities, LMIs.

1 Introduction

This paper addresses issues concerning the stabilization of systems that switches among different system structures. The motivation is the large number of systems controlled by different controller structures or control-laws, see [12, 18, 4, 9, 5] and the references therein.

The stability of the closed-loop controlled system depends in general of the switching strategy. Switching between stable system structures not necessarily imply a stable closed-loop behavior [3, 14]. Contrary, by designing a proper switch strategy, unstable system structures can be stabilized [15].

During the last decade, several stability results of switched and hybrid systems have been proposed in the literature [13, 6, 3, 15]. Unfortunately, these results are not constructive how to design the switch strategy, but merely give sufficient conditions to guarantee stability of a closed-loop system using a proper switching strategy.

There exist constructive approaches to design a switching strategy. The approach proposed by Wicks et al. [19] describes how to (asymptotically) stabilize a system consisting of two linear subsystems for which there exists a stable convex combination.

Another approach is given by Malmberg et al. [10]. In this approach, it is assumed that each controller (vector field) has an associated Lyapunov function for which the energy decreases in a certain region, and these regions together cover the state space. Furthermore, it is assumed that the corresponding Lyapunov functions are equal at every change of controller. Malmberg et al. propose a min-switch strategy to satisfy this assumption, which implies that the controller corresponding to the smallest Lyapunov function should be selected. With the assumptions made, the min-switch strategy implies a stable closed-loop system. A similar approach is suggested by Bishop [1] where the individual Lyapunov functions are normalized to implicitly generate the switching surfaces.

Another approach to design stabilizing control laws is given by Xu et al. [20] for second-order switched systems. Since they restrict themselves to two dimensions, they can give sufficient as well as necessary conditions for the stabilization of the switched system.

The ideas of the strategy proposed in this paper originally appeared in our work [16]. However, we have refined the method to avoid sliding modes, if desirable. Furthermore, conditions guaranteeing stability is proposed and formulated as a nonlinear optimization problem, which, for a specific design parameter, results in a linear matrix inequality (LMI) problem [2]. Two examples are proposed to illustrate the strategy. The details of the strategy together with the complete proofs of the stability results in this paper and a detailed comparison with (some of) the above approaches can be found in [14].

The outline of the paper is as follows: The min-projection strategy is proposed in the next section together with an illustrating example. Conditions guaranteeing stability applying the min-projection strategy are given in Section 3. The min-projection strategy with hysteresis is described in Section 4 followed by an example. Some conclusions end the paper.
2 Design strategy

The problem considered in this paper is as follows: Let the dynamic evolution of the switched system be described by a differential equation of the form

\[ \dot{x} = f_{i(t)}(x), \text{ where } i(t) \in \{1, \ldots, N\} = I_N, \]

and \( x \in \mathbb{R}^n \). The design task is to select the index function \( i(t) \) from the index set \( I_N \) at each time instant to stabilize (in the sense of Lyapunov) the overall system. Even though the index function might just be dependent on \( t \), we will suggest a feedback design strategy where \( i(t) \) is a function of the state \( x \), or a hybrid feedback with memory in the loop to avoid sliding motions described later on.

To complicate the situation, it is possible to restrict a specific vector field \( f_i(x) \) to be selected only in parts of the state space. Therefore, define \( F(x) \) by

\[ F(x) = \{ f_i(x) \in I_N \mid f_i(x) \text{ can be selected in } x \}, \]

which denotes the set of all vector fields that are allowed to be selected for the continuous state \( x \). For a well-formulated problem, it is assumed that the set \( F(x) \) is non-empty at every state in the region of validity of the design, so at least some vector field can be chosen at each state \( x \).

There may be several reasons for not allowing a vector field to be selected at certain states. For instance, there may perhaps be requirements on the closed-loop system that obviously are not fulfilled for certain vector fields at certain states. Furthermore, the selection of a certain vector field may indirectly lead to a closed-loop system that does not satisfy the requirements, which is prevented by considering it as non-allowable or forbidden at certain states. Physical restrictions may also imply that certain vector fields do not exist at certain continuous states. It is also possible to force a vector field to be chosen at a certain state by allowing only that one to be selected. This may be useful in case when it is known \textit{a priori} that a certain vector field best satisfies the requirements in a specific region.

The design problem in this paper can be formulated as follows: For each state \( x \), choose one vector field \( f_i(x) \) from \( F(x) \) such that the closed-loop system becomes (exponentially) stable (if possible).

2.1 Min-projection strategy

The proposed design strategy can be formulated as follows:

Let \( P \in \mathbb{R}^{n \times n} \) be a matrix. For a specific \( x \) (in the region of validity), choose the vector field according to the criterion

\[ f_i(x) = \arg \min_{f_j(x) \in F(x)} x^T Pf_j(x). \]

This strategy is denoted the \textit{min-projection strategy} and is very appealing from an engineering point of view, since it is easy to apply (as well as understand). The intuition behind the strategy is that the closed-loop system should become stable if it is always possible to select a vector field that points in a (projected) direction such that the trajectories approach the equilibrium point (see Theorem 1), cf. Figure 1. By selecting the vector field corresponding to the smallest such projection, there is therefore a good chance to obtain a stable closed-loop system.

Before giving conditions guaranteeing stability of the strategy, we illustrate the strategy in the following example.

2.2 Example 1

Consider the problem of switching between the following two linear vector fields (the same vector fields as in [10]):

\[ A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ -5 & -1 \end{bmatrix}. \]

Applying the \textit{min-projection strategy}, with \( P \) equal to the identity matrix, implies that

\begin{align*}
A_1 x & \text{ is selected for } x_1 x_2 \leq 0, \\
A_2 x & \text{ is selected for } x_1 x_2 > 0,
\end{align*}

which results in a globally exponentially stable closed-loop system (which can be verified by the stability results and optimization formulations described later on in this paper). In fact, the \textit{min-projection strategy} coincides in this example with the time-optimal control to reach the origin. Some typical trajectory simulations of the obtained closed-loop behavior are pictured in Figure 2.

3 Conditions for stability

There are examples for which the application of the \textit{min-projection strategy} results in unstable closed-loop systems. However, the intuition behind the \textit{min-projection strategy} is that the closed-loop system should result in a stable system if it is always possible to select a vector field that points in a (projected) direction such that the trajectories approach the equilibrium point. The following theorem verifies this fact.

\textbf{Theorem 1} If \( P > 0 \), and there exist a constant \( \gamma > 0 \) such that for all states \( x \) (in the region of validity) at least
some \( f_i(x) \in F(x) \) satisfies
\[
2x^TP f_i(x) \leq -\gamma x^T x,
\]
then the min-projection strategy results in closed-loop system where the origin 0 is exponentially stable in the sense of Lyapunov.

In this theorem, it is assumed that the possible sliding motion dynamics is given by Filippov’s convex combination definition [7]. The proof of this theorem follows then by using the Lyapunov function \( V(x) = x^TPx \), whose time derivative satisfies \( V(x) \leq -\gamma x^T x \) for all trajectories [14].

If the conditions in Theorem 1 are satisfied, the trajectory converges according to
\[
||x(0)|| \leq k_1 e^{-k_2 t} ||x_0||,
\]
where \( k_1 > 0 \) and \( k_2 > 0 \), and \( x_0 \) is the initial state. If \( \lambda_{min} \) and \( \lambda_{max} \) denote the smallest respectively the largest eigenvalue of \( P \), then it can be shown that the \( k \)-parameters may be taken as \( k_1 = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \) and \( k_2 = \frac{\gamma}{2 \lambda_{max}} \) [14].

### 3.1 All vector fields selectable

If all vector fields may be selected in the entire region of validity, then the following lemma guarantees that the min-projection strategy results in a stable closed-loop system.

**Lemma 1** If \( P > 0 \), and there exist constants \( \gamma > 0 \) and \( \delta_i \geq 0 \), \( i \in I_N \), such that
\[
\sum_{i=1}^{N} \delta_i > 0 \quad \text{and} \quad \sum_{i=1}^{N} \delta_i (2x^TP f_i(x) + \gamma x^T x) \leq 0, \quad (1)
\]
then the min-projection strategy results in a closed-loop system where the origin 0 is exponentially stable in the sense of Lyapunov.

The proof follows by showing (by contradiction) that the conditions in Theorem 1 are valid [14].

### 3.2 Linear vector fields

The conditions in Lemma 1 (Theorem 1) can be verified by the following optimization problem in case of linear vector fields (still assuming that all vector fields are selectable).

**Problem 1** If there is a solution to
\[
\max \gamma \text{ subject to }
\]
\[
0. \quad \gamma > 0, \delta_i \geq 0, \ i \in I_N, \sum_{i=1}^{N} \delta_i = 1
\]
\[
1. \quad 0 < P \leq I
\]
\[
2. \quad \sum_{i=1}^{N} \delta_i (A_i^T P + PA_i) \leq -\gamma I
\]
then the min-projection strategy results in a closed-loop system where the origin 0 is exponentially stable in the sense of Lyapunov.

The unknown variables in this optimization problem are the \( \gamma \) and the \( \delta_i \)'s. Furthermore, if the matrix \( P \) is not decided a priori, we have to search for this variable as well.

If \( P \) is decided a priori, there is no need to require \( P \leq I \) in the first condition. However, if \( P \) is unknown and it is a part of the design strategy to select a \( P \) such that stability is guaranteed (if possible), then the problem is scaled in such a way that a solution \( P \leq I \) is sought. Without such a scaling, the optimal solution would otherwise be obtained for \( \gamma \) and \( ||P|| \) infinitely large.

If \( P \) is decided a priori, Problem 1 is an LMI problem [2] and can easily be solved by existing software [8]. However, if \( P \) is not given a priori, the complication is that the unknown variables \( \delta_i \) and \( P \) are multiplied, implying that the inequality in the second condition is nonlinear. Hence, the problem in this case is not an LMI problem. However, it may be solved as a nonlinear optimization problem. Conditions 1 and 2 are equivalent to the nonlinear constraints
\[
1. \quad 0 < \text{eig}(P), \text{eig}(P - I) \leq 0
\]
\[
2. \quad \text{eig}(\sum_{i=1}^{N} \delta_i (A_i^T P + PA_i) + \gamma I) \leq 0.
\]

where \( \text{eig}(Q) \) denotes the eigenvalues of \( Q \), which are real numbers since the matrices are symmetric. Hence, the optimization problem of finding the largest value of the linear objective function \( \gamma \) satisfying nonlinear inequality constraints is a standard optimization problem which for instance may be solved by the routine constr or attgoal in the Optimization Toolbox by MATLAB [11].

In practice, the success of finding the optimal solution to a nonlinear optimization problem depends strongly on
whether the initial start variables are close to the optimal solution or not, since nonlinear optimization problems in general are not convex optimization problems (which LMI problems are). The practical consequence is that even if there is a solution to the optimization problem where all constraints are satisfied, the numerical routines may find a solution corresponding to a local optimum, which hopefully satisfies the constraints. Any solution satisfying the constraints in Problem 1 guarantees stability applying the min-projection strategy, which is of primary concern, so it is usually not crucial if a local optimum is found instead of the global one. However, it is worse if no solution is found at all.

### 3.3 Split optimization problem

In the case $P$ is not decided a priori, it is possible to split Problem 1 in two less complex optimization problems in such a way that the possibility to find a valid solution increases. If there exists a solution to Problem 1, then it is necessary that

$$\text{Re}(\text{eig}(\sum_{i=1}^{N} \delta_i A_i + \lambda I)) \leq 0 \quad (2)$$

is satisfied for some $\delta_i$, $i \in I_N$, and $\lambda > 0$, where Re denotes the real part of a complex number. The reason is that $P > 0$ is a Lyapunov function for the matrix given by

$$A = \sum_{i=1}^{N} \delta_i A_i,$$

implying that the real part of all eigenvalues of this matrix must be (strictly) negative. Hence, if no stable convex combination of matrices $A_i$ exists, then there is no solution $P > 0$ satisfying the conditions in Problem 1. This implies without loss of generality that it is possible to first search for a stable convex combination $A$ and, if one is found, then to find a quadratic Lyapunov function for the stable convex combination. The first optimization problem may formally be formulated as (unknown $\lambda$ and $\delta_i$:s):

**PROBLEM 2**

\[
\max \lambda \ \text{subject to} \quad 
\begin{align*}
0. & \quad \lambda \geq 0, \delta_i \geq 0, \ i \in I_N, \sum_{i=1}^{N} \delta_i = 1 \\
1. & \quad \text{Re}(\text{eig}(\sum_{i=1}^{N} \delta_i A_i + \lambda I)) \leq 0
\end{align*}
\]

This nonlinear optimization problem is in general easier to solve than the previous one, since it does not contain the unknown matrix $P$, implying that the number of unknown variables is considerably reduced in case of a high dimension $n$. Note that a solution to Problems 1 and 2 exists if some of the matrices $A_i$ are stable.

The problem of finding a Lyapunov function verifying stability is an LMI problem in the unknown matrix $P$, and can be formulated as (unknown $P$ and $\gamma$):

**LMI PROBLEM 1** If there is a solution to

\[
\max \gamma \ \text{subject to} \quad 
\begin{align*}
0. & \quad \gamma > 0 \\
1. & \quad 0 < P \leq I \\
2. & \quad \sum_{i=1}^{N} \delta_i (A_i^T P + PA_i) \leq -\gamma I
\end{align*}
\]

then the min-projection strategy results in a closed-loop system where the origin $0$ is exponentially stable in the sense of Lyapunov.

The obtained solution is a solution satisfying the conditions in Problem 1. If desirable, the found $P$ matrix guarantees stability using the min-projection strategy with an estimate of $k_2 = \frac{1}{\text{Re} \gamma}$ ($= \gamma/2$ if $P$ is unknown and restricted by $P \leq I$). Furthermore, it is always possible to use the obtained solution as initial values to the original optimization problem in Problem 1 to find a better value of $\gamma$, if feasible.

It should finally be pointed out that there are systems for which it does not exist a stable convex combination but still results in a stable closed-loop system applying the min-projection strategy [14]. In this case, stability has to be checked afterwards. It may for instance be verified by simulations or through a more formal analysis, for instance by searching for a piecewise (multiple) quadratic Lyapunov functions using linear matrix inequalities [15, 14].

### 4 Min-projection strategy with hysteresis

The direct application of the min-projection strategy may result in a closed-loop system where sliding motions occur, due to the fact that the selected vector field is only a function of the continuous state and not the used vector field. The infinitely fast switchings in finite time between the vector fields $f_i(x)$ and $f_j(x)$ (or possible further vector fields) occur at states satisfying $x^T Pf_i(x) = x^T Pf_j(x)$ as a result of applying the min-projection strategy if the smallest vector field projection on each side of this surface points towards it [7, 17]. If sliding motions are undesirable, the vector field changes cannot occur exactly at the coinciding switching surface but have to be adjusted such that the switch sets become separated. This can be achieved by introducing hysteresis around the surfaces where sliding motions occur, meaning that the selected vector field will be a function of both the continuous state and the vector field around the sliding motion surfaces. Hence, a hybrid feedback with memory in the loop is obtained.

If Lemma 1 is satisfied, it is always possible to adjust the switch surfaces such that sliding motions are avoided and still guarantee stability of the closed-loop system, at a cost of having a lower estimate of the convergence rate $k_2$. For
a value \( \tilde{\gamma} \) satisfying \( 0 < \tilde{\gamma} < \gamma \), the regions of states where \( 2x^TPf_j(x) + \gamma x^T x \leq 0 \) are satisfied for the different vector fields. A vector field to be selected is the one corresponding to the strategy that is satisfying \( 2x^TPf_j(x) + \tilde{\gamma} x^T x \leq 0 \). Hence, by instead switching vector fields at the boundary of \( 2x^TPf_j(x) + \tilde{\gamma} x^T x \leq 0 \), that is, at states satisfying \( 2x^TPf_j(x) + \tilde{\gamma} x^T x = 0 \), sliding motions are avoided with the disadvantage that \( k_2 = \frac{2\gamma}{\max g_j} \) instead of previously \( k_2 = \frac{\gamma}{\max g_j} \). The larger a value of \( \gamma \) that can be obtained and the smaller a value of \( \tilde{\gamma} \) that is accepted, the larger the hysteresis region becomes. A value of \( \tilde{\gamma} = 0 \) only guarantees stability. However, it is most often the case that \( \gamma \) as well as \( \tilde{\gamma} \) are quite conservative estimates, and the trajectories usually converges much faster.

If it is desirable to avoid sliding motions, the **min-projection strategy** is adjusted in the following way. Initially, the first vector field to be selected is the one corresponding to the smallest projection. Denote this vector field by \( f_j(x) \). The vector field does not change at the states where another projected vector field becomes smaller if sliding motions will be the result, but remains the same until some state \( x \) satisfying \( x^TPf_j(x) + \tilde{\gamma} x^T x = 0 \) is reached, in which case the vector field corresponding to the smallest projection is selected. This adjusted strategy will be referred to the **min-projection strategy with hysteresis**. Since it may be difficult to decide whether sliding motions will occur in higher dimensions, all changes of vector fields may occur at surfaces \( x^TPf_j(x) + \tilde{\gamma} x^T x = 0 \).

### 5 Examples

To illustrate the **min-projection strategy** and the **min-projection strategy with hysteresis**, consider the problem of stabilizing the closed-loop system by switching among the following three individually unstable linear vector fields:

\[
A_1 = \begin{bmatrix}
1.8631 & -0.0053 & 0.9129 \\
0.2681 & -6.4962 & 0.0370 \\
-2.4311 & -5.1032 & 1.6428
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
-0.8690 & 0.0869 & 0.0185 \\
0.0369 & -5.9869 & 0.8214 \\
0.0372 & -0.0821 & -2.7388
\end{bmatrix},
\]

\[
A_3 = \begin{bmatrix}
0.1941 & 0.2904 & -0.1110 \\
-1.0360 & 3.0486 & -4.9284
\end{bmatrix},
\]

\[
eig(A_1) = \{ 3.1176, 0.3816, -6.4895, -3.5084 \},
\]

\[
eig(A_2) = \{ 1.6976, 0.2880 \},
\]

\[
eig(A_3) = \{ 0.4192 \pm 0.5453i, -5.4391 \},
\]

There does not exist a stable convex combination for any two matrices so the system cannot be stabilized by switching only between two of the linear systems. However, there is a stable control-law switching between the three linear systems. By solving Problem 2 and LMI problem 1 and using the obtained values as initial data to Problem 1, the result is a solution \( \delta_1 = 0.2533, \delta_2 = 0.3377, \delta_3 = 0.4090 \),

\[
P = \begin{bmatrix}
0.7001 & -0.2550 & -0.3024 \\
-0.2550 & 0.7219 & -0.1128 \\
-0.3024 & -0.1128 & 0.3549
\end{bmatrix},
\]

and \( \gamma = 0.4214 \). This implies that \( k_2 = 0.2107 \). Hence, applying the **min-projection strategy** with this \( P \) matrix results in a globally exponentially stable closed-loop system. Sliding motions occur (defined according to Filippov’s convex combination definition) but can be avoided if desirable by applying the **min-projection strategy with hysteresis**. Trajectory simulations in both cases are shown in Figure 3. Note that all continuous states converge to the origin.

**Figure 3:** Trajectory simulations applying the **min-projection strategy** (above) without hysteresis and (below) with hysteresis for a value of \( \tilde{\gamma} = 0.01 \).

The **min-projection strategy** and the **min-projection strategy with hysteresis** have been inspired by the work of Peleties, DeCarlo and Wicks [19, 18], and there are similarities but also differences between the proposed design methods. Peleties, DeCarlo and Wicks restrict the problem to switching between two individually unstable linear vector fields, and propose methods to stabilize the closed-loop system when a stable convex combination of the linear vector fields exists. We have no theoretical restriction on the number of vector fields used. Our strategy can be applied to linear as well as nonlinear vector fields and we obtain estimates of the convergence rate in the approach.
6 Conclusions

This paper has suggested a method to stabilize a system consisting of several subsystems. The method is denoted min-projection strategy since the vector field associated with the smallest (skew) projection is selected in each state. Conditions guaranteeing stability of the approach has been given and it has been shown how to formulate these conditions as a nonlinear optimization problem, or, for a pre-determined projection matrix $P$ as a linear matrix inequality (LMI) problem. The nonlinear optimization problem has been split into two less complex optimization problems by first searching for a convex combination of the linear matrices, which is a nonlinear optimization problem, and then finding the unknown positive definite matrix $P$ by solving an linear matrix inequality (LMI) problem. It has been shown how sliding motions can be avoided by introducing hysteresis around the switch surfaces. Two examples have been given to motivate and exemplify the stabilization strategy.

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References


