ECE7850 Lecture 5

Stability of Switched and Hybrid Systems

- Stability concepts of switched and hybrid Systems
- Stability under arbitrary switching
- Stability under slow switching
- Stability under state-dependent switching
Stability Concepts of Switched and Hybrid Systems

• Representative class of HS:

\[
\begin{align*}
\dot{x} &= f(x(t), q(t), u(t)) \\
q(t^+) &= \Phi(x(t), q(t), u(t), \sigma(t))
\end{align*}
\] (1)

• Key concepts and main results are better presented using switched autonomous systems:

\[
\dot{x} = f_{\sigma(t)}(x(t)), \quad \sigma(t) \in Q
\] (2)

– \( f_q : \mathbb{R}^n \rightarrow \mathbb{R}^n \) locally Lipschitz with a common isolated equilibrium at \( x = 0 \)

– In general, \( Q \) can be finite or infinite (even uncountable)

• Conceptual extensions to general HS can be done, but constructive results are scarce.
Why care: dynamic behaviors much richer than subsystems

- **Example 1** *Destabilize stable subsystems*

\[ A_1 = \begin{bmatrix} -0.5 & -0.4 \\ 3 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & -3 \\ 0.4 & -0.5 \end{bmatrix}, \quad \sigma(t) = \begin{cases} 1 & x_1 x_2 \geq 0 \\ 2 & \text{otherwise} \end{cases} \] (3)
**Example 2** Stabilize unstable subsystems

\[ A_1 = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix}, \quad \sigma(t) = \begin{cases} 
1 & x_1 < 0 \& x_2 \in [0.5x_1, -0.25x_1] \\
1 & x_1 \geq 0 \& x_2 \in [-0.25x_1, 0.5x_1] \\
2 & \text{otherwise}
\end{cases} \]
Categories of Stability Problems:

- Fix switching signal $\sigma(t)$: switched system becomes a time-varying system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \triangleq f(t, x(t))$$

- Stability of $x(t)$ depends critically on our assumptions on $\sigma$

- Categorize problems based on assumptions on $\sigma$
  - *Stability under arbitrary switching:*
    - $\sigma(t)$ is viewed as external disturbance input
    - Typically, assume $\sigma(t)$ is piecewise constant
– **Stability under Slow Switching:**

* restrictions on dwell times between consecutive jumps of $\sigma(t)$

– **Stability under State-Dependent Switching:**

* $\sigma(t)$ determined through a given state-feedback law $\sigma(t) = \nu(x(t))$

* Variable structure systems (i.e., piecewise affine systems) can be viewed as switched systems under (pre-defined) state-dependent switching laws

– **Switching stabilization**

* $\sigma(t)$ is a control input to be designed
Stability Under Arbitrary Switching

**Definition 1** \( \dot{x} = f_\sigma(x) \) is \([\ast]\) stable under arbitrary switching if the corresponding time varying system \( \dot{x} = f(x, t) = f_{\sigma(t)}(x(t)) \) is \([\ast]\) stable for all piecewise constant switching signal \( \sigma \)

- \([\ast]\) can represent: \{locally, globally\}+\{asymptotically, exponentially\} stable
- Each of these notions corresponds to slightly different stability conditions, but conceptual ideas are the same
  - We primarily focus on glb. asym. stability (GAS)

- For switched linear systems: these notions are equivalent under arbitrary switching
• Main tool is *Common Lyapunov Function*

• Definition: A $C^1$ and PD function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a *Common Lyapunov Function* for \( \dot{x} = f_q(x), q \in Q \) if \( \exists \) a PD continuous ($C^0$) function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

\[
\left( \frac{\partial V}{\partial x}(x) \right)^T f_q(x) \leq -W(x), \quad \forall x, \forall q \in Q
\]

• **Theorem 1** A switched system is GAS under arbitrary switching if its subsystems share a radially unbounded common Lyapunov function

proof idea: (i) $(V$ is PD)& $(V$ continuous at $0) \Rightarrow \forall \alpha > 0, \exists r$ s.t. $B(0, r) \subseteq \Omega_\alpha = \{ V(x) \leq \alpha \}$

(ii) $\dot{V}(x(t)) = \frac{\partial V}{\partial x} f_{\sigma(t)}(x(t)) \leq -W(x(t)) \leq 0$, $\forall t$. Choose $x(0) \in B(0, r)$, then $x(t) \in \Omega_\alpha, \forall t$ (⇒ Lyapunov stable).

(iii) Let $c = \lim_{t \rightarrow \infty} W(x(t))$. Continuity + PD of $W$ ⇒ $c = 0$ (following the same argument as in the proof of the classical Lyapunov function theorem.
• Can we replace condition (4) with \( \frac{\partial V}{\partial x} f_q(x) < 0, \forall x, q \in Q \)?

  – Yes, when \( Q \) is compact

  – Not in general

  – Example: \( \dot{x} = -\sigma x, \sigma \in (0, 1] \)

    * not stable under arbitrary switching: \( x(t) = e^{-\int_0^t \sigma(\tau) d\tau} x(0) \)

  * But if choose \( V(x) = \frac{x^2}{2} \), we have \( \frac{\partial V}{\partial x} f_\sigma(x) = -\sigma x^2 \)
• Example 3 \( \dot{x} = A_{\sigma}x \), with \( A_1 = \begin{bmatrix} -1 & 0.25 \\ -1 & -1 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} -1 & -1 \\ 0.25 & -1 \end{bmatrix} \)

– choose \( V(x) = x_1^2 + x_2^2 \), we have

\[
\frac{\partial V}{\partial x} A_1 x = -x_1^2 - 1.4375 x_2^2 - (x_1 + 0.75 x_2)^2 < 0 \\
\frac{\partial V}{\partial x} A_2 x = -x_1^2 - 1.4375 x_2^2 - (x_1 + 0.75 x_2)^2 < 0
\]

– \( V \) common Lyapunov function \( \Rightarrow \) GAS under arbitrary switching

• Finding a quadratic common Lyapunov function \( V(x) = x^T P x \) for switched linear systems \( \{A_i\} \) can be formulated as a LMI feasibility problem:

\[
A_i^T P + PA_i < 0, \quad i \in \mathcal{Q}
\] (5)

• Does GAS \( \Rightarrow \) quadratic common Lyapunov function?

– Not even for switched linear systems
When does common Lyapunov function exists?

- **Theorem 2** If a switched nonlinear system \( \{f_q(x)\}_{q \in Q} \) is GAS under arbitrary switching and \( Q \) is finite. Then all subsystems share a radially unbounded common Lyapunov function
  
  – can be generalized to infinite \( Q \) under extra technical conditions (See Liberzon03)

– GAS switched linear systems **may not** have a quadratic common Lyapunov function

  * **Example 4** \( A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix} \)

  * The system is GAS with no quadratic common Lyapunov function

  * LMI (5) is not feasible
– **Theorem 3** *Switched linear system is GAS* $\Rightarrow$ *exists a piecewise quadratic common Lyapunov function of the following form:*

$$V(x) = \max_{i \in I} \left\{ (l_i^T x)^2 \right\}$$

* $I$: finite index set

* $l_i \in \mathbb{R}^n$: constant vector
Stability Under Slow Switching

- Requiring stability under arbitrary switching may be too strong

- Often times, switching rate is limited

- Intuition: subsystem stability $\oplus$ slow enough switching $\Rightarrow$ stability of switched systems

Questions:

- how slow do we need?

- need slow switching all the time or just on average
• Dwell time: Let $t_1, t_2, \ldots$ be the switching time instants of $\sigma(t)$

  − $\tau$ is called the dwell time of $\sigma$ if $|t_{i+1} - t_i| \geq \tau$ for all $i$

  − $\bar{\tau}$ is called average dwell time of $\sigma$ if

    $$N_\sigma(t, s) \leq N_0 + \frac{t - s}{\bar{\tau}}$$

    for all $t \geq s \geq 0$ and some scalar $N_0 \geq 0$.

    Here, $N_\sigma(t, s)$ denotes the number of switches over $[s, t]$.

• Let $\bar{S}(\alpha)$ denote the set of all switching signals with average dwell time larger than $\alpha \geq 0$
Theorem 4 If all subsystems of (2) are glob. exponentially stable, then \( \exists \bar{\tau}^* > 0 \) for which (2) is GAS for all \( \sigma \in \bar{S}(\bar{\tau}^*) \).

– Given the exponential convergence rate of each subsystems, one can derive an expression for \( \bar{\tau}^* \).

– The specific value of \( \bar{\tau}^* \) is not important

– Similar result holds for the worst case dwell time
Stability Under State-Dependent Switching

- Get bored with various Lyapunov conditions?

- There are too many of them for various kinds of switched/hybrid systems.

- The results for state-dependent switching truly departs from the classical Lyapunov approach.

- This leads to the celebrated Multiple-Lyapunov Function (MLF) approach.
• Recall the Key Idea of Lyapunov Function

  – $V$ should be scalar-valued PD function such that $V \to 0 \Rightarrow x(t) \to 0$.

  – amendable to show $V \to 0$ based on model instead of explicit solutions of the system

• How to generalize standard Lyapunov results:

  – $V$ needs not be differentiable everywhere

  – $V$ needs not always decrease along system trajectories

• Relax differentiability assumption + switched systems $\Rightarrow$ **Multiple Lyapunov functions**
Multiple Lyapunov Function Theorem

- Partition $\mathbb{R}^n$: $\bigcup_j \Omega_j = \mathbb{R}^n$; each mode is associated with one region;

- Associate a Lyapunov-like function for each region/mode:
  
  $V_j$ is PD and $\mathcal{L}_{f_j}V_j$ is NPD on $\Omega_j$  \hspace{1cm} (6)

- Patch multiple Lyapunov-like functions: $V(x) = \begin{cases} 
V_1(x), & \text{if } x \in \Omega_1 \\
V_2(x), & \text{if } x \in \Omega_2 \\
\vdots & \vdots
\end{cases}$
• **Theorem 5** *(MLF Theorem):* Assume that (i) \( \exists \) partitions and Lyapunov-like functions described on previous slide; (ii) \( \exists \) PD functions \( \{W_j\} \) such that

\[
V_j(x(t_{j,k+1})) - V_j(x(t_{j,k})) \leq -W_j(x(t_{j,k}))
\]

where \( t_{j,k} \) the \( k \)th time vector field \( f_j \) is switched in. Then system (2) is GAS.
• Proof of MLF Theorem
• Possible extensions:

  – Relax monotonically decreasing condition for $V_j$: e.g.:

  $V_j(x(t)) \leq h(V_j(x(t_{j,k}))), \forall t \in (t_{j,k}, t_{j,k+1})$

  $h : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is continuous with } h(0) = 0.$

  – Allowing multiple $(V_{ji}, \Omega_{ji})$ for each mode $i$: 
• Remarks:

- $V_j$ differentiable and decrease along system trajectory inside $\Omega$,

- $V(x)$: piecewise differentiable, may increase at switching instants

• Caveat:

  * The theorem requires knowledge of the state trajectory at switching times.

  * Checking the MLF conditions is not easy except for some special cases.
- MLF Application for piecewise linear systems

- Given partition \( \bigcup \Omega_i = \mathbb{R}^n \) with \( \Omega_i = \{ E_i x \geq 0 \} \)

- For simplicity, assume one partition for each mode: \( \dot{x} = A_i x \), for \( x \in \Omega_i \)

- Assume active switching boundary \( \Omega_{ij} \subseteq \Omega_i \cap \Omega_j \) given by \( \Omega_{ij} = \{ b_{ij}^T x = 0 \} \) for all feasible switchings \( i \to j \)
Sufficient MLF conditions:

1. Consider Lyapunov-like candidate: \( V_i(x) = x^T P_i x, \ i \in \Omega_i \)

2. Meet Lyapunov condition:
\[
\begin{align*}
\alpha \|x\|^2 & \leq x^T P_i x \leq \beta \|x\|^2, \quad \text{on } \Omega_i \\
x^T (A_i^T P_i + P_i A_i) x & < 0, \quad \text{on } \Omega_i
\end{align*}
\]

⇒ Sufficient LMI conditions
3. Nonincreasing during switching: $x^T P_i x \geq x^T P_j x$ on $\Omega_{ij}$

$\Rightarrow$ Sufficient LMI conditions
– It often suffices to assume continuity of MLF on switching boundaries

* In this case, no need to distinguish $i \rightarrow j$ and $j \rightarrow i$ boundaries

* Suppose $\exists \{C_i\}_i^N$ such that $\Omega_{ij} = \Omega_{ji} = \{x \in \mathbb{R}^n : C_i x = C_j x\}$ for all $i, j$. Then one way to guarantee continuity is to assume the following form for MLF:

$$V(x) = x^T P_i x \triangleq x^T (C_i^T P C_i) x, \quad x \in \Omega_i$$

* Similar LMIs (w.r.t $P$ instead of $P_i$) can be derived without worrying about the nonincreasing condition (i.e. condition 3 on previous slide)
– Summary of LMI conditions:

* (nonincreasing across $\Omega_{ij}$): \( \exists \alpha_1, \alpha_2, \alpha_3 > 0 \), vectors \( t_{ij} \), symmetric matrices \( P_i \), and elementwise nonnegative matrices \( W_i, U_i \) such that

\[
\begin{align*}
P_i - E_i^T W_i E_i & \succeq \alpha_1 I \\
A_i^T P_i + P_i A_i + E_i^T U_i E_i & \preceq -\alpha_2 I \\
P_i - P_j + b_{ij} t_{ij}^T + t_{ij} b_{ij}^T & \succeq \alpha_3 I
\end{align*}
\] (7)

* (continuous on $\Omega_{ij}$): \( \exists \alpha_1, \alpha_2 > 0 \), symmetric matrix \( T \), and elementwise nonnegative matrices \( W_i, U_i \) such that

\[
\begin{align*}
P_i = C_i^T T C_i \\
P_i - E_i^T W_i E_i & \succeq \alpha_1 I \\
A_i^T P_i + P_i A_i + E_i^T U_i E_i & \preceq -\alpha_2 I
\end{align*}
\] (8)
– Example 5 $A_1 = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix}, \quad \sigma(t) = \begin{cases} 1 & x_1 < 0 \& x_2 \in [0.5x_1, -0.25x_1] \\ 1 & x_1 \geq 0 \& x_2 \in [-0.25x_1, 0.5x_1] \\ 2 & \text{otherwise} \end{cases}$
Example Continued: