ECE7850 Lecture 4:

Basics of Stability Analysis

- Basic Stability Concepts
- Lyapunov Stability Theorems
- Converse Lyapunov Functions
- Semidefinite Programming (SDP)
- Basic Polynomial Optimization
- Computational Techniques for Stability Analysis

Basic Stability Concepts

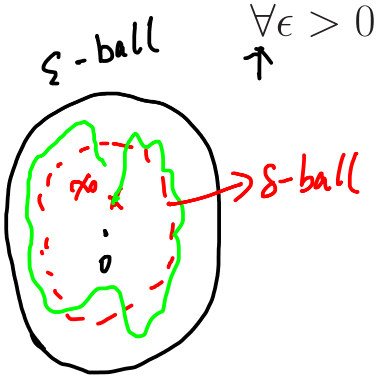
- Consider a time-invariant nonlinear system:

$$\dot{x} = f(x) \text{ with IC } x(0) = x_0 \tag{1}$$

- Assume: f Lipschitz continuous; origin is an isolated equilibrium $f(0) = 0$

- $x = 0$ *stable* in the sense of Lyapunov, if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \forall t \geq 0$$



trajectories starting from δ-ball remains in ε-ball in the sense of Lyapunov

- $x = 0$ *asymptotically stable* if it is stable and δ can be chosen so that

$$\|x(0)\| \leq \delta \Rightarrow x(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

$\Leftarrow$ origin is attractive

if the above condition holds for all δ, then *globally asymptotically stable*

$\delta \in \mathbb{R}$

otherwise, it is only locally asymp. stable.

- *Region of Attraction*: $R_A = \{x \in \mathbb{R}^n :$ whenever $x(0) = x,$ then $x(t) \rightarrow 0\}$

- $x = 0$ *exponential stable* if there exist positive constants δ, λ, c such that

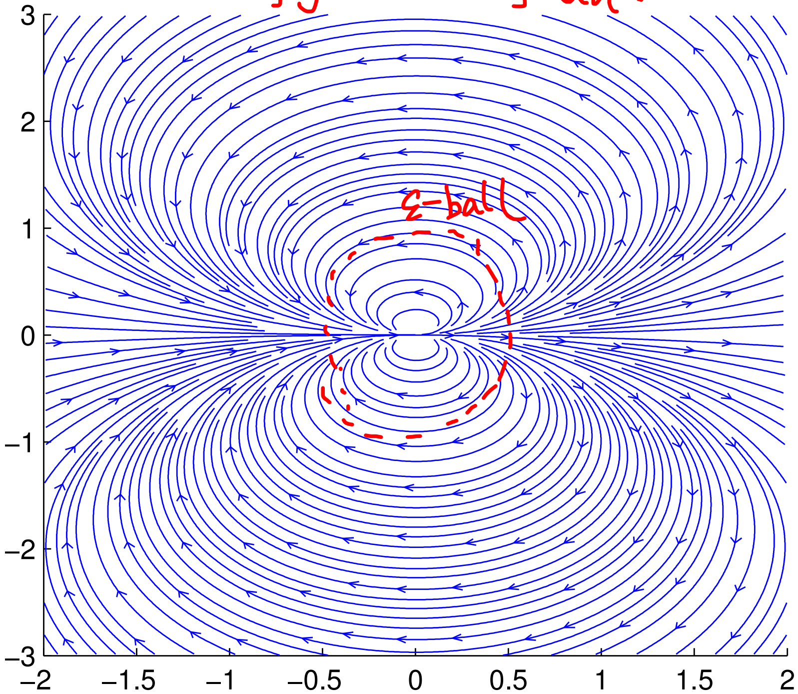
$$\|x(t)\| \leq c\|x(0)\|e^{-\lambda t}$$

norm of state trajectory decays exponentially.

- Does attractive implies stable in Lyapunov sense?
  - Answer is NO. e.g.: $\begin{cases} \dot{x}_1 = x_1^2 - x_2^2 \\ \dot{x}_2 = 2x_1 x_2 \end{cases}$

by inspection: $\forall x_0$, $x(t) \to 0$, as $t \to \infty$

butterfly vector field.



However, it is not stable in the Lyapunov sense.

why?: there is no δ-ball that guarantees trajectory remains inside a given ε-ball.

Stability Analysis Using Lyapunov Functions

How to verify stability of a system:

- Trivial answer: explicit solution of ODE $x(t)$ and check stability definitions

- Need to determine stability without explicitly solving the ODE

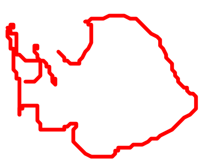
- Preferably, analysis only depends on the vector field

idea: Find a simple scalar-valued function

- The most powerful tool is: Lyapunov function

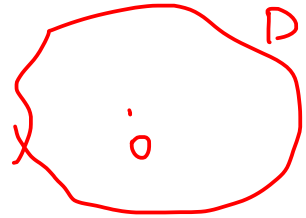
complex domain in high-dim space $\Rightarrow$

$\dot{x}(t) = f(x(t))$



we want: $V(x(t)) \to 0 \Rightarrow x(t) \to 0$

requirement!



- Classes of functions: $\boxed{(\text{Assuming } 0 \in D \subseteq \mathbb{R}^n)}$
  - $g : D \to \mathbb{R}$ is called positive semidefinite (PSD) on D if $g(0) = 0$ and $g(x) \geq 0, \forall x \in D$
  - $g : D \to \mathbb{R}$ is called positive definite (PD) on D if $g(0) = 0$ and $g(x) > 0, \forall x \in D \setminus \{0\}$
  - g is negative semidefinite (NSD) if $-g$ is PSD ; g is negative definite (ND) if $-g$ is PD
  - $g : \mathbb{R}^n \to \mathbb{R}$ is radically unbounded if $V(x) \to \infty$ as $\|x\| \to \infty$ eg. $V(x) = \|x\|$ radially unbounded ($V(x) = \cos x$) $\Leftrightarrow$ X
  - $\mathcal{C}^n$: n-times continuously differential functions $g : \mathbb{R}^n \to \mathbb{R}^m$

- Lie derivative of a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ along vector field g is:

scalar-valued function

$V : \mathbb{R}^n \to \mathbb{R}$

we want tell whether $V(x(t))$ will go to 0 or not along system trajectory.

$$\boxed{\mathcal{L}_g V(x)} \triangleq \left(\frac{\partial V}{\partial x}(x)\right)^T g(x) \triangleq \frac{\partial V}{\partial x} g(x)$$ ← notation some people use

If we view $V(x(t))$ as a function of t ($x(t) = [x_1(t) \cdots x_n(t)]^T$)

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \cdot \frac{\partial x_i}{\partial t} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \cdot f_i(x) = \left(\frac{\partial V}{\partial x}\right)^T f$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \end{bmatrix} = f(x) \qquad = (\nabla V)^T f$$

C^1

- **Theorem 1 (Lyapunov Theorem)** *Let $D \subset \mathbb{R}^n$ be a set containing an open neighborhood of the origin. If there exists a PD function $V : D \to \mathbb{R}$ such that*

$$\mathcal{L}_f V \text{ is NSD} \tag{2}$$

the value of V along system trajectory is non increasing

then the origin is stable. If in addition,

$$\mathcal{L}_f V \text{ is ND} \tag{3}$$

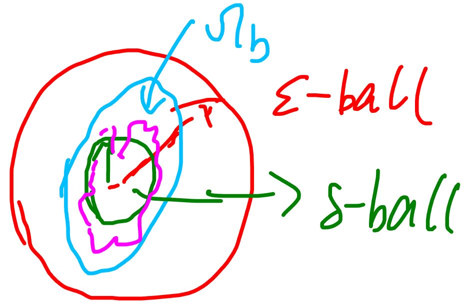
decrease along sys traj

then the origin is asymptotically stable.

- Remarks:
  - A PD $\mathcal{C}^1$ function satisfying (2) or (3) will be called a Lyapunov function
  - For the latter case, if V is also radially unbounded $\Rightarrow$ globally asymptotically stable

$D = \mathbb{R}^n$

- *Proof of Lyapunov Theorem*:



① show stability:

If we define sublevel set $\Omega_b = \{x \in \mathbb{R}^n : V(x) \le b\}$

V nonincreasing $\Rightarrow$ If $x(0) \in \Omega_b$ then, $x(t) \in \Omega_b$ $\forall t$

Therefore, if we can choose $b > 0$ s.t. $\Omega_b \subseteq B(0, \varepsilon)$ ...<i>

and choose $\delta > 0$, s.t. $B(0, \delta) \subseteq \Omega_b$ ...<ii>

then we are done.

Q: Can we find b and δ to ~~se~~ satisfy <i> and <ii>

1°: because V is continuous $\Rightarrow$ $m = \min_{\|x\| = \varepsilon} V(x)$ exists due to

Weierstrass thm

V is P.D. $\Rightarrow$ $m > 0$

$\Rightarrow$ we ~~ca~~ can choose $b \in (0, m)$

2°: $\because$ $V(x)$ is continuous at origin, so for any $b>0$

$\exists\ \delta>0$ s.t. $\forall\, x\in B(0,\delta)$ we have $|V(x)-V(0)| = V(x) < b$

$$\Rightarrow B(0,\delta)\subseteq \Omega_b$$

Q.E.D.

(2): show asymptotically stability: suppose $x(0)\in B(0,\delta)$

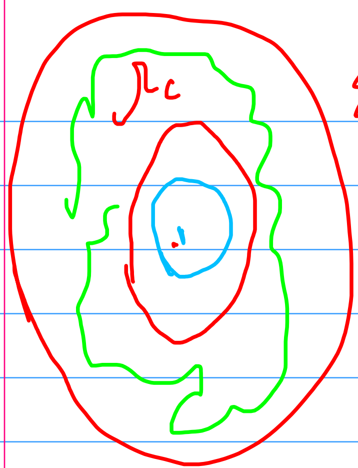
we know $V(x(t))$ decreasing monotonically and $V(x(t))\geq 0$

$\Rightarrow c=\lim_{t\to\infty} V(x(t))$ exists. and $c\geq 0$ (due to monotonic convergence theorem)

We just need to show $c=0$

show! If NOT: i.e. $c>0$ $\Rightarrow$ $x(t)\notin \Omega_c=\{x\in\mathbb{R}^n \mid V(x)\leq c\}$

choose $B(0,\beta)\subseteq\Omega_c$ (due to continuity of V at 0)



$\dot V$ is ND

Let $a = -\max\limits_{\beta \le \|x\| \le \varepsilon} \dot V(x) \Rightarrow a > 0$

$$\Rightarrow V(x(t)) = V(x(0)) + \int_0^t \dot V(x(s))\, ds$$

$$\le V(x(0)) - a \cdot t$$

$\Rightarrow$ for some large t $\quad V(x(t)) < 0$

$\Rightarrow$ contradiction !

$\Rightarrow c = 0$

- Definition: $V : D \to \mathbb{R}$ is called an Exponential Lyapunov Function (ELF) on $D \subset \mathbb{R}^n$ if $\exists k_1, k_2, k_3 > 0$ and $\alpha > 0$ such that

$$\begin{cases} k_1\|x\|^{\alpha} \le V(x) \le k_2\|x\|^{\alpha} & \cdots \text{(1)} \\ \mathcal{L}_f V(x) \le -k_3\|x\|^{\alpha} & \cdots \text{(2)} \end{cases}$$

- **Theorem 2 (ELF Theorem)** *If system (1) has an ELF, then it is exponentially stable.*

- Proof:

$$(2) \Rightarrow \dot V(x(t)) \le -k_3\|x(t)\|^{\alpha} \le -\frac{k_3}{k_1} V(x(t)) \Rightarrow \dot V \le -\frac{k_3}{k_1} V$$

1. show $V(x(t)) \le V(x(0))e^{-(k_3/k_1)t}$

$$\Rightarrow V(x(t)) \le e^{-ct} V(x(0)), \quad \text{where } c = \frac{k_3}{k_1}$$

2. show $x(t) \le ce^{-\lambda t}\|x(0)\|$

$$\Downarrow$$

$$\Rightarrow \|x(t)\|^{\alpha} \le \frac{1}{k_1} V(x(t)) \le \frac{1}{k_1} e^{-ct} V(x(0))$$

$$\le \frac{1}{k_1} e^{-ct} k_2 \|x(0)\|^{\alpha}$$

$$= \frac{k_2}{k_1} e^{-ct} \|x(0)\|^{\alpha}$$

$$\Rightarrow \|x(t)\| \le \left(\frac{k_2}{k_1}\right)^{\frac{1}{\alpha}} e^{-\frac{c}{\alpha}t} \|x(0)\|$$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- **Example 1** $\begin{cases} \dot{x}_1 = -x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 = x_1 - x_2 - x_1^2 - x_2^3 \end{cases}$ *Try* $V(x) = \|x\|^2$ $= x_1^2 + x_2^2$

$\begin{bmatrix} \frac{\partial v}{\partial x_1} \\ \frac{\partial v}{\partial x_2} \end{bmatrix}$

Q: whether system stable or NOT ?

<1>: $V(x) = \|x\|^2$ is P.D. and is smooth

<2>: $L_f V = \left(\frac{\partial V}{\partial x}(x)\right)^T f(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \cdot \begin{bmatrix} -x_1 + x_2 + x_1 x_2 \\ x_1 - x_2 - x_1^2 - x_2^3 \end{bmatrix}$

$= 2\left[-x_1^2 + x_1 x_2 + \cancel{x_1^2 x_2} + x_1 x_2 - x_2^2 - \cancel{x_1^2 x_2} - x_2^4 \right]$

$= 2\left[-(x_1 - x_2)^2 - x_2^4 \right]$

is ND $\Rightarrow$ asym stable.

Remark: fail to find L.F. does not mean instability

- **Example 2** $\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = -x_2 \end{cases}$

  – *Fact: The system is GAS (Homework: try $V(x) = \ln(1 + x_1^2) + x_2^2$)*

  – *Can we find a simple quadratic Lyapunov function? First try: $V(x) = x_1^2 + x_2^2$*

$$\dot{V} = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{bmatrix} = 2\left[-x_1^2 + x_1^2 x_2 - x_2^2\right] = -2\left[x_1^2(1 - x_2) + x_2^2\right]$$

Is $\dot{V}$ ND on $\mathbb{R}^2$?

try Let $x_1 = \sqrt{8}$

$$\Rightarrow \dot{V} = -2\left[8(1 - x_2) + x_2^2\right]$$

$$x_2^2 - 8x_2 + 8 = (x_2 - 4)^2 - 8$$

  – *In fact, the system does not have any (global) polynomial Lyapunov function*

When there is a Lyapunov Function?

- Converse Lyapunov Theorem for Asymptotic Stability

$$\begin{cases} \text{origin asymptotically stable}; \\ f \text{ is locally Lipschitz on } D \\ \text{with region of attraction } R_A \end{cases} \Rightarrow \exists V \text{ s.t.} \begin{cases} \text{V is continuuos and PD on } R_A \\ \mathcal{L}_f V \text{ is ND on } R_A \\ V(x) \to \infty \text{ as } x \to \partial R_A \end{cases}$$

hard to prove

- Converse Lyapunov Theorem for Exponential Stability

$$\begin{cases} \text{origin exponentially stable on } D; \\ f \text{ is } \mathcal{C}^1 \end{cases} \Rightarrow \exists \text{ an ELF } V \text{ on } D$$

→ sys simple

- Proofs are involved especially for the converse theorem for asymptotic stability

- **IMPORTANT**: proofs of converse theorems often assume the knowledge of system solution.

Semi-definite Programming

- Converse Lyapunov function theorems are not constructive

$$= [x_1, x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Basic idea for Lyapunov function synthesis

$V(x) = x^T P x \qquad V(x) = a x_1^2 + c x_2^3 + d x_3^5$

  - Select Lyapunov function structure (e.g. quadratic, polynomial, piecewise quadratic, ...)
  - Parameterize Lyapunov function candidates

$$V(x) = \begin{cases} x^T P_1 x, & x \in \text{Region1} \\ x^T P_2 x, & x \in \text{Region2} \end{cases}$$

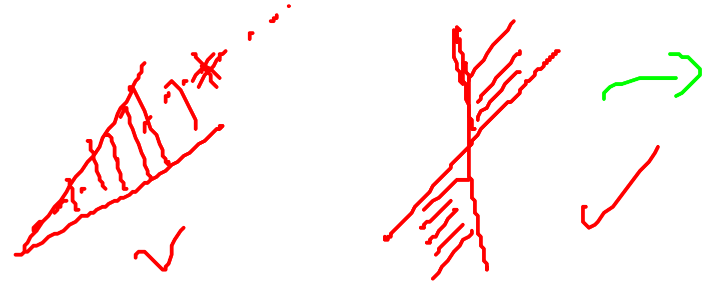
  - Find values of parameters to satisfy Lyapunov conditions

- Many Lyapunov synthesis problems can be formulated as Semidefinite programming (SDP) problems.

Convex Cone

- Recall: A set S is convex if $x_1, x_2 \in S$ implies $\lambda x_1 + (1-\lambda)x_2 \in S$, $\forall \alpha \in [0,1]$.

- A set S is a *cone* if $\lambda > 0$, $x \in S \Rightarrow \lambda x \in S$.

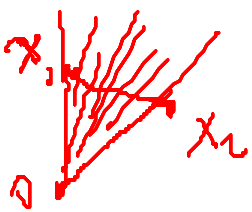


non convex

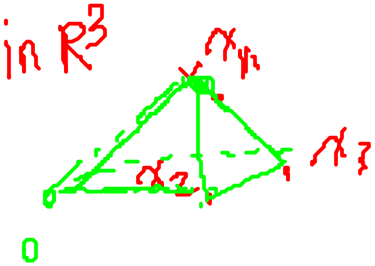
cone

- Conic combination of x_1 and x_2:

  $x = \alpha_1 x_1 + \alpha_2 x_2$ with $\alpha_1, \alpha_2 \geq 0$



in R^3



called polyhedral cone

- *convex cone:* (i) a cone that is convex; (ii) equivalently, a set that contains all the conic combinations of points in the set

Real Symmetric Matrices:

- $\mathcal{S}^n$: set of real symmetric matrices $A^T = A$

- All eigenvalues are real

symmetric matrix is diagonalizable

- There exists a full set of orthogonal eigenvectors

- *Spectral decomposition*: If $A \in \mathcal{S}^n$, then $\boxed{A = Q\Lambda Q^T}$, where Λ diagonal and Q is unitary.

Q is unitary if $Q^T Q = QQ^T = I$ $\Longleftrightarrow$ let $Q = [q_1, q_2, \ldots, q_n]$

$q_i^T q_j = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$

Positive Semidefinite Matrices

- $A \in \mathcal{S}^n$ is called *positive semidefinite (p.s.d.)*, denoted by $A \succeq 0$, if $x^T A x \geq 0, \forall x \in \mathbb{R}^n$

it defines a PSD quadratic form $V(x) = x^T A x$

- $A \in \mathcal{S}^n$ is called *positive definite (p.d.)*, denoted by $A \succ 0$, if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$

$V(x) = x^T P x$ is PD

- $\mathcal{S}_+^n$: set of all p.s.d. (symmetric) matrices

- $\mathcal{S}_{++}^n$: set of all p.d. (symmetric) matrices

- p.s.d. or p.d. matrices can also be defined for non-symmetric matrices. But we focus on symmetric ones. but nonsymmetric

e.g.: $\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{A}$ is P.D $V(x) = [x_1 \ x_2] \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$

- Other equivalent definitions for symmetric p.s.d. matrices:

  $A \in S_+^n$

  – All $2^n - 1$ principal minors of A are nonnegative

  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$: we have $2^3 - 1 = 7$: $|A|$, $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $|a_{11}|$, $|a_{22}|$, $|a_{33}|$

  are leading principal minors

  – All eigs of A are nonnegative

  ✗ – There exists a factorization $A = B^T B$

- Other equivalent definitions for p.d. matrices:

  – All n leading principal minors of A are positive

  – All eigs of A are strictly positive

  ✗ – There exists a factorization $A = B^T B$ with B square and nonsingular.

- Useful facts: TAT^{-1} : similarity transformation.

  – If T nonsingular, $A \succ 0 \Leftrightarrow T^T AT \succ 0$; and $A \succeq 0 \Leftrightarrow T^T AT \succeq 0$;

  S_+^n, S_{++}^n are invariant under congruent transformation.

  – $\mathcal{S}_+^n$ is a convex cone: *positive semidefinite cone*

  PSD cone

  $A, B \in S_+^n$, $\alpha A + \beta B \succeq 0$, $\forall \alpha \geq 0, \beta \geq 0$

  – Inner product on $\mathbb{R}^{m\times n}$: $< A, B > \triangleq tr(A^T B) \triangleq A \bullet B$.

  $\Rightarrow$ for $A, B \in S^n$ $\Rightarrow \langle A, B\rangle = tr(AB)$

  $\forall A, B \in \mathbb{R}^{m\times n}$: $tr(A^T B) = \sum_i \sum_j A_{ij} B_{ij}$ (similar to "dot" product in $\mathbb{R}$ Euclidean space)

  defines an inner product in $\mathbb{R}^{m\times n}$

  angle between $A, B \in S^n$: $\cos\theta = \dfrac{\langle A, B\rangle}{\sqrt{\langle A, A\rangle}\sqrt{\langle B, B\rangle}}$

X – For $A, B \in \mathcal{S}_+^n$, $tr(AB) \geq 0$ (the cone $\mathcal{S}_+^n$ is acute)

property, show in HW2

– Schur complement lemma: Define $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ has to be symmetric

1. $M \succ 0 \Leftrightarrow \begin{cases} A \succ 0 \\ C - B^T A^{-1} B \succ 0 \end{cases} \Leftrightarrow \begin{cases} C \succ 0 \\ A - BC^{-1}B^T \succ 0 \end{cases}$
2. If $A \succ 0$, then $M \succeq 0 \Leftrightarrow C - B^T A^{-1} B \succeq 0$
3. If $C \succ 0$, then $M \succeq 0 \Leftrightarrow A - BC^{-1}B^T \succeq 0$

e.g.: suppose: $AA^T \preceq I \Rightarrow I - A \cdot I^{-1} \cdot A^T \succeq 0 \Rightarrow \begin{bmatrix} I & A \\ A^T & I \end{bmatrix} \succeq 0$

– Proof of Schur complement lemma:

e.g.: we pick (2) to prove:

The assumption is: $A \succ 0$, $A \in S^n_{++}$

~~S~~ ~~assume~~ assume $M \succeq 0 \iff \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq 0, \forall x, y \in \mathbb{R}^n$

$= \overbrace{x^TAx + 2x^TBy + y^TCy}^{g(x,y)} \geq 0$

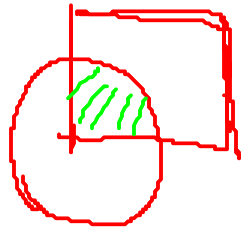
$\iff \underbrace{\min_x \left(x^TAx + 2x^TBy + y^TCy \right)}_{} \geq 0, \forall y$

$\iff \frac{\partial g}{\partial x} = 0 \iff x^* = -A^{-1}By \xRightarrow{\text{plug in } g(x,y)} y^T(C - B^TA^{-1}B)y \geq 0, \forall y$

\# (QED)

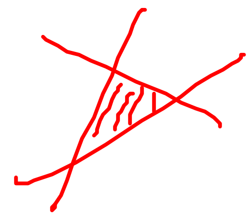
Operations that preserve convexity

- intersection of possibly infinite number of convex sets:



  - e.g.: polyhedron:

polytope is bounded polyhedron



  - e.g.: PSD cone:

S_+^n: define sets $G(z) = \{P \in S^n : z^T P z \geq 0\}$, for a particular $z \in \mathbb{R}^n$

$$S_+^n = \bigcap_{z \in \mathbb{R}^n} G(z)$$

- affine mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ (i.e. $f(x) = Ax + b$) $\longleftrightarrow$
  - $f(X) = \{f(x) : x \in X\}$ is convex whenever $X \subseteq \mathbb{R}^n$ is convex

    center of ellipsoid

    e.g.: Ellipsoid: $E_1 = \{\|x \in \mathbb{R}^n : (x - x_c)^T P(x - x_c) \leq 1\}$ or equivalently $E_2 = \{x_c + Au : \|u\|_2 \leq 1\}$

    $B = \{\|x\|^2 \leq r^2\}$ ball

    E_2

    $\Rightarrow$ $E_2 = f_2(B)$, $f_2(x) = x_c + Ax$

    $E_1 = f_1(B)$

    $f_1(x) = P^{\frac{1}{2}}(x - x_c)$

  - $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\}$ is convex whenever $Y \subseteq \mathbb{R}^m$ is convex

    e.g.: $\{Ax \leq b\} = f^{-1}(\mathbb{R}^n_+)$, where $\mathbb{R}^n_+$ is nonnegative orthant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, \forall i\}$

    pointwise

    $f^{-1}(x) = b - Ax$

Linear Matrix Inequality

- Given symmetric matrices $F_0, \ldots, F_m \in \mathcal{S}^n$, $x \in \mathbb{R}^n$ is variable

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \succeq 0$$

$$\begin{bmatrix} x_1 & x_2 \\ x_1 - x_3 & x_2 \end{bmatrix} \succeq 0$$

is called a *Linear Matrix Inequality* in $x \in \mathbb{R}^n$

- The function $F(x)$ is affine in x : VFY: $F(x) = F_0 + G(x)$

where $G(x)$ is linear

$G(\alpha x) = \alpha G(x)$

- The constraint set $\overset{E}{=}\{x \in \mathbb{R}^n : F(x) \succeq 0\}$ is nonlinear but convex

pick $x, y \in E$: $F(\alpha x + (1-\alpha) y) \in E$, $\forall \alpha \in [0, 1]$

VFY :

- **Example 3** *Characterize the constraint set:* $F(x) = \begin{bmatrix} x_1 + x_2 & x_2 + 1 \\ x_2 + 1 & x_3 \end{bmatrix} \succeq 0$

① is a LMI

$$F(x) = x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

② $F(x) \succeq 0 \iff \begin{cases} x_1 + x_2 \geq 0 \\ x_3 \geq 0 \\ (x_1+x_2)x_3 - (x_2+1)^2 \geq 0 \end{cases}$

set of nonlinear inequalities

- **Example 4** *Find a Lyapunov function* $V(x) = x^T P x$ *for a linear system* $\dot{x} = Ax$

requirements: ①: V P.D. ② $\dot{V}$ is ND. | Here, our unknown variable is $P \in S^n$

① $\Leftrightarrow$ $P \succ 0$ · ③

③ and ④ are LMIs of
(P is the unknown)

e.g.: represent $P = \sum_{j=1}^{n} \sum_{i=1}^{j} x_{ij} E^{ij}$

where $E^{ij} \in S^n$: is 0 except (i,j), (j,i) entries are 1.

$E^{(1,2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

②: $\dot{V} = \left(\frac{\partial V}{\partial x}\right)^T \cdot f(x) = (2Px)^T \cdot Ax = 2x^T PAx$ - - - (a)

$\dot{V}(x) = (x^T P x)' = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T PAx$

$= x^T \underbrace{(A^T P + PA)}_{} x$ - (b)

(a) = (b) for all x:

② $\Leftrightarrow$ $-(A^T P + PA) \succ 0$ ··· ④

$\left.\begin{matrix} ① & ③ \\ 2 & ④ \end{matrix}\right\} \Leftrightarrow \left\{ \begin{matrix} \sum_{j=1}^{n} \sum_{i=1}^{j} x_{ij} E^{ij} \succ 0 \\ \sum_{j=1}^{n} \sum_{i=1}^{j} x_{ij} (-A^T E^{ij} - E^{ij} A) \succ 0 \end{matrix} \right\} \Rightarrow$ LMI

Semidefinite Programming (SDP)

- SDP: optimization problem with linear objective, and LMI and linear equality constraints:

The standard LMI form of SDP

$$\begin{cases} \text{minimize:} & c^T x \\ \text{subject to:} & F_0 + x_1 F_1 + \cdots + x_n F_n \succeq 0 \\ & Ax = b \end{cases} \tag{4}$$

variable $x \in \mathbb{R}^n$

- Global optimal solution of $\boxed{\text{SDP}}$ can be found efficiently.

- Equivalent SDP (*Standard Prime Form*):

Standard conic form:

$$\begin{cases} \text{minimize:} & f_p(X) = C \bullet X \\ \text{subject to:} & A_i \bullet X = b_i, i = 1, \ldots, m \\ & X \succeq 0 \end{cases} \tag{5}$$

variable $X \in S^n$

$C \bullet X = \sum_{i,j} C_{ij} X_{ij}$

One can show form (4) $\Longleftrightarrow$ form (5)

Example: ~~form~~ (5): Let $C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$

$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $b_1 = 2$, $b_2 = 3$

(5) $\Rightarrow$ $\min \; x_{11} + 2x_{12} + x_{22}$

subj. to: $x_{11} + x_{21} = 2$

$x_{12} + x_{22} = 3$

$x_{12} = x_{21}$

$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$

- Dual form: for form (5)

derive dual form HW#3

$$\begin{cases} \text{maximize:} & f_d(y) = b^T y \\ \text{subject to:} & \Sigma_{i=1}^m y_i A_i \preceq C \end{cases}$$

variable $y \in \mathbb{R}^m$

- Weak duality: $f_p(X) \geq f_d(y)$ for any primal and dual feasible X and y

X feasible for (5), y is feasible for

$$f_p(X) - f_d(y) = C \bullet X - b^T y = C \bullet X - \sum_{i=1}^m y_i (A_i \bullet X) = \Big(C - \sum_{i=1}^m y_i A_i\Big) \bullet X$$

≥ 0 (PSD cone is acute) $B \succeq 0$

- Strong duality holds under Slater's condition: $f_p(X^*) = f_d(y^*)$

Slater condition: prime is strictly feasible, i.e. $\exists X$ s.t. $A_i \cdot X = b_i$

$X \succ 0$

- **Example 5** *LMIs* $A^TP + PA + I \preceq 0$, $P \succeq 0$ *indicate that the Lyapunov function* $V(x) = x^TPx$ *for linear system* $\dot{x} = Ax$ *proves the bound:* $\int_0^\infty \|x(\tau)\|^2 d\tau \leq x(0)^TPx(0)$. *Suppose* $x(0)$ *is fixed. How to find the best possible such bound?*

→ total traj energy

① show bound: $\dot{V}(x(t)) = x(t)^T[A^TP+PA]x(t) \leq -\|x(t)\|^2$

$0 \leq V(t) = V(0) + \int_0^t \dot{V}(x(t))dt \leq V(0) + \int_0^t \|x(t)\|^2 dt$

$V(0) = x(0)^TPx(0)$

⇒ bound.

② given $x(0)$:

$$\begin{cases} \min_P \; x(0)^TPx(0) \\ \text{subj. to: } -(A^TP+PA+I) \succeq 0 \\ P \succeq 0 \end{cases}$$

⇒ SDP

Basic Polynomial Optimization

Motivation: we want to consider general polynomial ~~typ~~-type of LF.

- $R_{n,d}$: Set of polynomials (with real coefficients) in n variables of degree d:

$$f(x) = \sum_{\alpha \in \mathcal{I}} c_\alpha x_1^{\alpha 1} \cdots x_n^{\alpha n}$$

  where $d = \max_{\alpha \in \mathcal{I}} \Sigma_i^n \alpha_i$.

  e.g. $= x_1 x_2 + x_3^3 + x_1 x_3^2$

- $P_{n,d} = \{f \in R_{n,d} : f(x) \geq 0, \forall x \in \mathbb{R}^n\}$: set of p.s.d. polynomials

- $\Sigma_{n,d} = \{f \in R_{n,d} : f = \Sigma_i g_i^2$, for some $g_i \in R_{n,d}\}$: Sum of Squares (SOS)

  e.g. $f(x) = x_1^2 + (x_1 - x_2 - 2x_3 x_6)^2$

- $\Sigma_{n,d} \subset P_{n,d}$

- checking $f \in P_{n,d}$ is NP-hard

  Nondeterministic Polynomial-time Hard

- checking $f \in \Sigma_{n,d}$ is a SDP problem

Representation of Polynomials

- monomial bases $Z_d(x)$: $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $\alpha_i \in \{0, \cdots, n\}$ $\sum \alpha_i \leq d$

e.g. $x \in \mathbb{R}^2$: $Z_2(x) = [1 \;\; x_1 \;\; x_2 \;\; x_1x_2 \;\; x_1^2 \;\; x_2^2]$

$x \in \mathbb{R}^1$ $Z_3(x) = [1 \;\; x_1 \;\; x_1^2 \;\; x_1^3]$

If in $\mathbb{R}^n$ with degree d:

$\dim(Z_d(x)) = \binom{n+d}{d} = \frac{(n+d)!}{d!\,n!}$

e.g. $\binom{4}{2} = \frac{4!}{2!\,2!} = 6$

- Linear representation:

e.g. $f = 3x_1^2 + 4x_1x_2 + 6x_2 = \overbrace{\underbrace{[0 \;\; 0 \;\; 6 \;\; 4 \;\; 3 \;\; 0]}_{\text{coordinate of } f \text{ w.r.t. } Z_2(x)}}^{c} Z_2(x)$

$\forall f \in \mathbb{R}_{n,d}$ has a unique representation $c \in \mathbb{R}^N$, where $N = \binom{n+d}{d}$

- Quadratic representation (Gram matrix representation): $f = 4x_1^4 + 4x_1^3 x_2 - 7x_1^2 x_2^2 - 2x_1 x_2^3 + 10x_2^4$ (*x1)

<1> ~~given~~ $f \in \mathbb{R}_{n,2d}$ can be represented as $z_d(x)^T Q z_d(x)$ with $Q \in S^n$

<2> e.g. * can be written as

$$f = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}^T \begin{bmatrix} q_1 & q_2 & q_3 \\ q_2 & q_4 & q_5 \\ q_3 & q_5 & q_6 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix} = q_1x_1^4 + 2q_2 x_1^3x_2 + (q_4+2q_3)x_1^2x_2^2 + 2q_5 x_1x_2^3 + q_6 x_2^4$$

match coefficients

$$\Rightarrow q_1 = 4,\ q_2 = 2,\ q_4 + 2q_3 = -7,\ q_5 = -1,\ q_6 = 10$$

(**x1)

$\Downarrow$

- Quadratic representation is not unique:

$\forall$ ~~Q~~ $\forall Q \in S^n$ and with entries satisfying (**x1) is a valid representation of (*x1)

<3> Quadratic representation allows us to check $f \in \Sigma_{n,d}$

- Hilbert showed in 1888: $P_{n,d} = \Sigma_{n,d}$ iff
  - $d = 2$ quadratic polynomials for all n
  - $n = 1$ univariate polynomials for all d
  - $n = 2, d = 4$, quartic polynomials in two variables.
- SOS decomposition for $f \in P_{n,d}$: if $\exists g_1, \ldots, g_s$ such that $f = \Sigma_i g_i^2$

Thm: If $f \in P_{n,d}$ is PSD, then $\exists r > 0$ s.t. $f(x)\left(\sum_{i=1}^{n} x_i^2\right)^r \in \Sigma_{n,d+2r}$

- **Theorem 3** *Let $Z_d(x)$ be the monomial basis of degree $\leq d$. Then $f(x) \in \Sigma_{n,d}$ iff there exists Q such that*

$$\begin{cases} Q \succeq 0 & \leftarrow \text{LMI with variable } Q \\ f(x) = Z_d(x)^T Q Z_d(x) & \text{equality constraints w.r.t. } Q \end{cases}$$

  - this is SDP problem (LMI feasibility problem)
  - comparing terms gives affine constraints on the elements of Q

① If: such Q exists $\Rightarrow$ $Q = VV^T$, let $V = [v_1 \cdots v_r]$

$$\Rightarrow f(x) = Z_d(x)^T V V^T Z_d(x) = \|V^T Z_d(x)\|^2 = \sum_{i=1}^{r} \left(v_i^T Z_d(x)\right)^2$$

② $\Rightarrow$

$$\begin{cases} \min_{Q,\eta} \ \eta \\ \text{s.t. } Q + \eta \cdot I \succeq 0 \\ \quad f(x) = Z(x)^T Q Z(x) \end{cases}$$

- **Example 6** $f = \cancel{2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4}$ same f in (*)

$$z(x) = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -1+2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix}$$

$$2q_3 + q_4 = -1$$

In particular, we choose $\lambda = 6$

$$Q = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

$$\Rightarrow f(x) = (2x_1x_2 + x_2^2)^2 + (2x_1^2 + x_1x_2 - 3x_2^2)^2$$

Numerical Construction of Lyapunov Functions

- Important conditions for many stability problems:

$$g_0(x) \geq 0 \text{ on } \{x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_k(x) \geq 0\}$$

- Conservative but useful condition: $\exists$ SOS $s_i(x)$ s.t.

$$g_0(x) - \sum_i s_i(x) g_i(x) \geq 0, \forall x \in \mathbb{R}^n$$

This is *Generalized S-Procedure* simple to prove .

check ~~wsth~~ whethere $\exists\ s_i \in \Sigma_{n,d}$ s.t. $g_0(x) - \sum s_i g_i \in \Sigma_{n,?}$

- Important special case: $g_i(x) = x^T G_i x, i = 0, 1, ...$ are quadratic polynomials:
  - Original condition: $\forall x \in \mathbb{R}^n, x^T G_1 x \geq 0, \ldots, x^T G_k x \geq 0 \Rightarrow x^T G_0 x \geq 0$
  - Sufficient condition (S-procedure): $\exists \alpha_1, \ldots, \alpha_k \geq 0$ with

$$\Rightarrow \quad G_0 \succeq \alpha_1 G_1 + \cdots + \alpha_k G_k$$

  - S-Procedure is lossless if $k = 1$ and $\exists \hat{x}, \hat{x}^T G_1 \hat{x} > 0$ (constraint qualification)

Application to Stability Analysis

- **Example 7** $\dot{x} = Ax + g(x)$ *with* $g(x) \leq \beta\|x\|^2$ (e.g. $g(x)$ can be unknown disturbance or $g(x)$ can be nonlinear term)

(Goal: Find LF for exponential stability of form

$V(x) = x^T P x$) $\Rightarrow$ $P > 0$ and $\dot{V}(x) \leq -\alpha V(x)$ (assume $\alpha > 0$ is given)

Key: deal with $g(x)$ (Let's denote $z = g(x)$)

$\Rightarrow \dot{V} + \alpha V = 2x^T P(Ax + g(x)) + \alpha x^T P x = x^T(A^T P + PA + \alpha P)x + 2x^T P \overset{z}{g(x)}$

$\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}$

$$= \begin{bmatrix} x \\ z \end{bmatrix}^T \underbrace{\begin{bmatrix} A^T P + PA + \alpha P & P \\ P & 0 \end{bmatrix}}_{-G_0} \begin{bmatrix} x \\ z \end{bmatrix} \leq 0$$

So we want

$\tilde{x}^T G_0 \tilde{x} \geq 0$

whenever $\tilde{x}^T G_1 \tilde{x} \geq 0$

for $\begin{bmatrix} x \\ z \end{bmatrix} \in \{ \underbrace{z^T z \leq \beta x^T x}_{} \}$

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \underbrace{\begin{bmatrix} \beta I & 0 \\ 0 & -I \end{bmatrix}}_{G_1} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0$$

By S-Procedure: $\Rightarrow$ $\underbrace{G_0 \succeq \beta G_1}_{\text{LMI (variable } P)}$ for some $\beta > 0$

- **Example 8** *Find Lyapunov function for* $\begin{cases} \dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ \dot{x}_2 = 3x_1 - x_2 \end{cases}$

$\Rightarrow$ Look for polynomial Lyapunov function: we want 4^{th}-order poly

i.e. $V(x) = z_d(x)\, Q\, z_d(x) = \sum_{0 \le i+j \le 4} c_{i,j}\, x_1^i x_2^j$

$d=2$

① we want $V \in SOS \Rightarrow Q \succeq 0$

② we want $-\dot{V} \in SOS$

e.g.: $L_f V = \sum \frac{\partial V}{\partial x_i} \cdot f_i(x) = \sum_{0 \le i+j \le 4} c_{ij} \cdot \left(i\, x_1^{i-1} \cdot (-x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3) + j\, x_2^{j-1} \cdot (3x_1 - x_2) \right)$

We should see $\dot{V} = L_f V = z(x)^T \cdot P(Q)\, z(x)$, where $P(Q)$ is affine in Q

$\Rightarrow P(Q) \prec 0$

$\Rightarrow$ SDP. solve using SOSTOOLS.3