Continuous Time Optimal Control of Switched Nonlinear Systems

With only one week left, we will only focus on basic concepts and key results.

- Finite-dimensional optimization problems
- Elementary calculus of variations
- Pure and relaxed switched optimal control problems
- Chattering lemma and embedding principle
- Solution to relaxed problems
- Examples: switching time optimizations and switched optimal control.
Finite-dimensional optimization problems

- Basic building block (1D case):
  \[
  \min_x g(x) \\
  \text{subj. to } x \in [a, b]
  \]

- Necessary local min condition for interior point \( x^* \in (a, b) \)
  \[
  \text{near } x^*, \quad g(x) = g(x^*) + g'(x^*)(x-x^*) + o(|x-x^|)
  \]

  \( x^* \in (a,b), \quad x \in (-\varepsilon + x^*, \varepsilon + x^*) \)

  \[
  g(x) \geq g(x^*) \implies g'(x^*) = 0
  \]

- Necessary local min condition for \( x^* = a \):
  \[
  g(x) = g(x^*) + g'(x^*)(x-x^*) + o(|x-x^|) \quad x > x^* \]

  \[
  g(x) \geq g(x^*), \quad x \in [a, a+\varepsilon)
  \]

  \[
  \implies g'(x^*) > 0
  \]

feasible direction
Consider a general constrained optimization problem in $\mathbb{R}^n$:

$$\begin{cases}
\min_x J(x) \\
\text{subj. to } x \in X
\end{cases}$$

$X$ is a closed subset in $\mathbb{R}^n$ with boundary $\partial X$

Definition 1 (Local minimizer): $x^* \in X$ is called a local min if there exists $r > 0$, such that

$$J(x^*) \leq J(x), \forall x \in \mathcal{N}(x^*; r) \cap X$$

Necessary conditions: conditions weaker than (1) that can be used to eliminate non-optimal solutions
• We call \( d \in \mathbb{R}^n \) a **feasible direction** if \( \exists \bar{\alpha} > 0 \) such that \( x + \alpha d \in X \) for all \( \alpha \in [0, \bar{\alpha}] \).

• Feasible cone at \( x \): \( F_X(x) = \{0\} \cup \{ \text{feasible directions} \} \)

• An obvious necessary condition: cost does not increase along any feasible direction.

\[
\forall d \in F_X(x^*), \quad \text{we have} \quad J(x^* + \alpha d) \geq J(x^*), \quad \text{for sufficiently small } \alpha
\]  

\( \text{(2)} \)

• **Directional derivative:** \( DJ(x^*; d) = \lim_{\alpha \to 0} \frac{J(x^* + \alpha d) - J(x^*)}{\alpha} = g'(0) \)

• **1-sided directional derivative:** \( DJ^+(x^*; d) = \lim_{\alpha \downarrow 0} \frac{J(x^* + \alpha d) - J(x^*)}{\alpha} = g'(0^+) \)
• Optimality conditions in terms of directional derivatives:

- Whenever the 1-sided directional derivative exists, (2) requires

\[ D^+ J(x^*; d) \geq 0, \forall d \in F_X(x^*) \]

\[ g'(a^+) \geq 0 \]

\[ g(a) = g(b) + g'(a^+) (a - c) + o(a), \quad a \to c \]

\[ \frac{\partial J}{\partial x^*} (d) \]

- If \( D^+ J \) also exists, then we need \( D^+ J(x^*; d) = 0 \).

\[ g'(a^+) = g'(a^-) = g'(b) < DJ^+ \]

\[ \begin{cases} D^+ J(x^*; d) \geq 0 \\ D^+ J(x^*; -d) \geq 0 \\ D^+ J(x^*; d) = 0 \end{cases} \Rightarrow D^+ J(x^*; d) = 0 \]
Example 1 \[
\begin{aligned}
\min_x \quad & J(x) \\
\text{subj. to} \quad & h(x) \leq 0
\end{aligned},
\text{where } J : \mathbb{R}^n \to \mathbb{R} \text{ and } h : \mathbb{R}^n \to \mathbb{R} \text{ are differentiable.}
\]

\[
x = \{ x \in \mathbb{R}^n : h(x) \leq 0 \}
\]

1. Necessary condition for interior points \( x^* \in X^0, h(x^*) = 0 \):
   \[
   \nabla J(x^*) \cdot d \geq 0, \quad \forall \ d \in \mathbb{R}^n
   \]
   \[
   \Rightarrow (\nabla J(x^*))^T \cdot d \geq 0, \quad \forall \ d \in \mathbb{R}^n
   \Rightarrow \nabla J(x^*) = 0
   \]

2. What about \( x^* \in \partial X, h(x^*) = 0 \) we need:
   \[
   \nabla h(x^*) = 0, \quad \forall \ d \in \mathbb{R}^n
   \]

   Suppose:
   \[
   \nabla h(x^*) \quad \text{and} \quad F_X(x^*) = \{ d \in \mathbb{R}^n : \nabla h(x^*)^T \cdot d \leq 0 \}
   \]
   \[
   (\nabla J(x^*))^T \cdot d \geq 0, \quad \forall \ d \in \mathbb{R}^n
   \]

   \[\nabla J \quad \text{and} \quad \nabla h\]

   \[\nabla J \cdot d < 0 \quad \text{but} \quad \nabla h \cdot d \leq 0 \quad \Rightarrow \quad \nabla J + \mu \nabla h = 0, \quad \mu \geq 0
   \]

   KKT condition

Finite-dimensional optimization problems
• Feasible direction can be restrictive (especially for nonlinear equality constraints)

\[ X = \{ x \in \mathbb{R}^n : x_1^2 + x_2^2 = 1 \} , \quad F_x(x^*) = \{ 1 \} \]

\[ \text{tangent vector} \]

• Need to replace \( F_x(x) \) with tangent cone \( T_x(x) \) in general: including all limiting tangent directions when approaching \( x \) inside \( X \)

\[ T_x(x^*) = \{ 1 \}^n , \quad 0 \leq D^T(x^*, d) \leq 0 , \quad \forall d \in T_x(x^*) \]

• Note \( cl(F_x(x)) \subseteq T_x(x) \)

Finite-dimensional optimization problems
A large class of optimal control problems can be viewed as optimization problem in infinite-dimensional space

- \( X \) becomes a space of control input signals (function of time)

- \( J \) becomes function of control signal (functional)

- But the results are still based on the same key concepts: necessary conditions, feasible direction, and directional derivatives

- We just need slight generalizations.
Elementary Calculus of Variations

- Let $\mathcal{V}$ be a general normed vector space (may be infinite dimensional)

- Consider constrained optimization: $\min_{x \in \mathcal{X}} J(x)$, where $\mathcal{X} \subseteq \mathcal{V}$ and $J : \mathcal{X} \to \mathbb{R}$ is a functional.

- Generalization of directional derivative concept to function space: **Gateaux derivative**

\[
\delta J(x; \eta) = \lim_{\alpha \to 0} \frac{J(x + \alpha \eta) - J(x)}{\alpha}
\]

- It is also called the **first variation** of $J$ along $\eta$

- Gateaux derivative (or first variation) $\delta J(x; \eta)$ (according to above definition) may not be linear in $\eta$. Definitions in the literature often directly requires it to be linear in $\eta$. 
• one-sided **Gateaux derivative**:

\[
\delta J^+(x; \eta) = \lim_{\alpha \downarrow 0} \frac{J(x + \alpha \eta) - J(x)}{\alpha}
\]

\[
q'(s^*)
\]

• Optimality conditions:

- \( \delta J^+(x; \eta) \geq 0 \) for all feasible/admissible directions \( \eta \)

- \( \delta J(x; \eta) = 0 \), for all feasible/admissible directions \( \eta \)
• Note: \( \delta J(x; \eta) = g'(0) \), where \( g(\alpha) = J(x + \alpha \eta) \).

• Basic calculus of variation problem: Find a \( C^1 \) function \( x : [a, b] \rightarrow \mathbb{R} \) with given \( x(a) = x_0 \) and \( x(b) = x_f \) to minimize \( J(x) = \int_a^b l(x, x', t) \, dt \).

  – this curve optimization problem can be viewed as a control problem: find control \( u \in C^0 \) to minimize \( J(u) = \int_a^b l(x, u, t) \, dt \) subject to dynamic constraint \( \dot{x} = u \).

  – Derive \( \delta J(x; \eta) \) for admissible \( \eta \) that satisfies \( \eta(a) = \eta(b) = 0 \).

\[
\begin{align*}
\delta J(x; \eta) &= J(x + \alpha \eta) - J(x) = \int_a^b \left( l(x + \alpha \eta, x', t) - l(x, x', t) \right) \, dt \\
&= \int_a^b \left[ l(x, x', t) \eta' + o(\alpha) \right] \, dt + o(\alpha)
\end{align*}
\]

\[
\Rightarrow \frac{\partial J}{\partial \eta} = \frac{\delta J(x; \eta)}{\delta \eta} = J(x) + \int_a^b \left[ l_x(x, x', t) \eta' + l_{x'}(x, x', t) \eta'' \right] \\
\Rightarrow \frac{\partial J}{\partial \eta} = \int_a^b \left[ l_x(x, x', t) \eta - \frac{d}{dt} \left( l_{x'}(x, x', t) \eta \right) \right] \, dt \\
\Rightarrow \frac{\partial J}{\partial \eta} = \int_a^b \left[ l_x(x, x', t) \eta - l_{x'}(x, x', t) \eta \right] \, dt
\]

for optimal \( x \), we need \( \frac{\partial J}{\partial \eta} = 0 \), \( \forall \eta \) admissible.
- **Fundamental lemma of calculus of variation**: If a continuous function $\xi : [a, b] \rightarrow \mathbb{R}$ satisfies $\int_a^b \xi(t) \eta(t) dt = 0$ for all $C^1$ function $\eta : [a, b] \rightarrow \mathbb{R}$ with $\eta(a) = \eta(b) = 0$, then $\xi \equiv 0$.

**Why?** If not, $\exists t_1 \in [a, b]$ s.t. $\xi(t_1) > 0 \Rightarrow \xi(t) > 0, \forall t \in (-e_{t_1}, e_{t_1})$

Choose $\eta$

- **Euler-Lagrange Equation**: By above lemma, we know $DJ(x; \eta) = 0$

  $$\Rightarrow L_x(x, x', t) = \frac{d}{dt} L_{x'}(x, x', t), \forall t \in [a, b]$$
Example: \( J(x) = \int_0^{\pi/2} [\dot{x}^2(t) - x^2(t)] \, dt \) with boundary conditions \( x(0) = 0, \, x(\pi/2) = 1. \)

Find \( x(t) \in C^1 \) to minimize \( J(x) \)

\[
(\dot{x}^2(t) - x^2(t)) = \dot{x}(t) \cdot \dot{x}(t) - x(t) \cdot x(t) = 2 \dot{x}^2(t) - 2 x(t) \frac{\partial \dot{x}(t)}{\partial t} + 2 x(t) \frac{\partial x(t)}{\partial t}
\]

By E.L. equation

\[
-2x(t) = \frac{d}{dt} \left[ 2 \dot{x}(t) \right] \Rightarrow \dot{x}(t) + x(t) = 0
\]

\[
\begin{align*}
\alpha_1 &= x(t) \\
\alpha_2 &= \dot{x}(t)
\end{align*}
\]

\[
\frac{d}{dt} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = e^{At} \begin{bmatrix} \alpha_1^0 \\ \alpha_2^0 \end{bmatrix} = \begin{bmatrix} \cos t + \sin t \\ -\sin t + \cos t \end{bmatrix} \begin{bmatrix} \alpha_1^0 \\ \alpha_2^0 \end{bmatrix}
\]

\[
\begin{align*}
\alpha_1(0) &= 1 & \alpha_1(\pi/2) &= \alpha_2(\pi/2) \\
\alpha_2(0) &= 0 & \alpha_2(\pi/2) &= \sin(\pi/2) = 1
\end{align*}
\]

\[
\begin{align*}
\alpha_1(\pi/2) &= 1 \Rightarrow \alpha_2(\pi/2) = 0 + 0 \Rightarrow x(\pi/2) = 1
\end{align*}
\]
Switched Optimal Control Problems

• Switched nonlinear systems:

\[ \dot{x}(t) = f_{\sigma(t)}(t, x(t), u(t)), \text{ where } \sigma(t) \in \mathcal{Q} = \{1, \ldots, q\} \]  \hspace{1cm} (3)

• Hybrid control: \( \xi(t) = (u(t), \sigma(t)) \) with constraints:

\[ u(t) \in \underbrace{\mathcal{U}} \subset \mathbb{R}^m, \sigma(t) \in \mathcal{Q}, \text{ where } \mathcal{U} \text{ bounded and convex} \]

• State trajectory driven by \( \xi \): \( x(t; \xi) \), or simply \( x(t) \).

• A finite time horizon, w.l.g., assume \( T = [0, 1] \).

• State trajectory constraint: \( h_j(x(t)) \leq 0, \forall j \in \mathcal{J} = \{1, 2, \ldots n_s\}, \forall t \in T \).
• Cost function: \( J(x(1; \xi)) \). In general, \( J(\xi) = \int_0^1 ((x_0, y_0), t) \, dt + \mathcal{J}_f(x(1)) \)

  – only penalize terminal state. Introduce new state \( \dot{z} = (x_0, y_0, t) \Rightarrow z(t) = \int_0^t ((x_0, y_0), s) \, ds \)

  \( \Rightarrow \mathcal{J}(\xi) = \mathcal{J}(\xi) \equiv \mathcal{J}_f(\mathcal{N}(x_0), z(1)) \)

  – problems with running cost can be reduced to this form by introducing additional state.

• Notations to emphasize dependence on \( \xi \):

  – \( \phi_t(\xi) \triangleq x(t; \xi), \psi_{j,t}(\xi) \triangleq h_j(x(t; \xi)), J(\xi) \triangleq J(x(1; \xi)) \)

  – Overall constraint functional: \( \Psi(\xi) \triangleq \max_{j,t} \psi_{j,t}(\xi). = \max_{j,t} \frac{\partial h_j}{\partial \xi}(x(t; \xi)) \) (constraints \( \Leftrightarrow \Psi(\xi) \leq 0 \))

• Assumptions [A1]:

  – \( f_i(t, x, u), h_j(x), J(x) \) are Lipchitz continuous w.r.t all arguments \( \frac{|\int_{t_1}^{t_2} (t \xi_1, u) - \int_{t_1}^{t_2} (t \xi_2, u)|}{|t_2 - t_1|} \)

  – \( \frac{\partial f_i}{\partial x}(t, x, u), \frac{\partial f_i}{\partial u}(t, x, u), \frac{\partial h_j}{\partial x}(x), \frac{\partial J}{\partial x}(x) \) exist and are Lipchitz continuous w.r.t. all arguments
• An equivalent way to write system dynamics:

\[ \dot{x} = f(t, x, u, d) \triangleq \sum_{i=1}^{q} d_i(t) f_i(t, x, u) \]

where \( d(t) = [d_1(t), \ldots, d_q(t)]^T \) is a corner of the \( q \)-simplex:

\[ \sum_{i=1}^{q} d_i = 1 \]

• Control Spaces:

– We say \( f : [0, 1] \rightarrow F \) belongs to \( L_2([0, 1], F) \) if \( \|f\|_{L_2} = \left( \int_0^1 \|f(t)\|_2^2 dt \right)^{1/2} < \infty \)

– Continuous input space: \( \mathcal{U} = L_2([0, 1], U_t) \)

  e.g., \( \mathcal{U} = \{ u_1 \in [0, 1], u_2 \leq [2, 3] \} \), s.t. \( u(t) \in \mathcal{U} \)
- Discrete input space: \( \mathcal{D}_p = L_2([0, 1], \Sigma_p^q) \) \( d(\cdot) \in \mathcal{D}_p \implies d(t) \in \Sigma_p^q \), 
  \( d_1(t) = \{0, 1\} \), 
  \( d_2(t) = \{0, 1\} \), 
  \( \sum t \leq d_1 + d_2 = 1 \)

- Overall optimization space: \( \mathcal{X} = L_2([0, 1], \mathbb{R}^m) \times L_2([0, 1], \mathbb{R}^q) \)

- Pure optimization space: \( \mathcal{X}_p = \mathcal{U} \times \mathcal{D}_p \)

- (Pure) Switched Optimal Control Problem:

  \[
  \mathcal{P}_p : \quad J^*_p = \begin{cases} 
  \inf_{\xi} & J(\xi) \\
  \text{subj. to} & \Psi(\xi) \leq 0, \quad \xi \in \mathcal{X}_p 
  \end{cases}
  \]

- Challenges: space \( \mathcal{X}_p = \mathcal{U} \times \mathcal{D} \) is not a vector space due to \( \mathcal{D} \), on which gradient of \( J \) and \( \Psi \) are not well defined.
Embedding Principle and Chattering Lemma

- Key idea for solving $\mathcal{P}_p$ is to “embed” the switched systems into a larger class of nonlinear systems for which $d$ takes values inside the entire $q$-simplex (not just the corner points).

- $q$-simplex: $\Sigma_p^q = \{ (d_1, \ldots, d_q) \in [0,1]^q \mid \sum_{i=1}^q d_i = 1 \}$.

- Relaxed System:

\[
\dot{x}(t) = \sum_{i=Q} d_i(t) f_i(t, x(t), u(t)), \quad \text{with} \quad x(0) = x_0. \tag{5}
\]

- $d(t) \in \Sigma_p^q \Rightarrow$ original switched systems

- $d(t) \in \Sigma_q^q \Rightarrow$ relaxed switched systems

- The set of all trajectories of the switched system is contained in that of the relaxed system.
• Relaxed control spaces:

  – Related discrete input space: \( D_r = L^2([0, 1], \Sigma^q) \)

  – Relaxed optimization space: \( X_r = \mathcal{U} \times D_r \)

• Relaxed Switched Optimal Control Problem \( P_r \):

\[
P_r : \quad J^*_r = \inf_{\xi} \begin{cases} J(\xi) \\ \text{subj. to} \quad \Psi(\xi) \leq 0, \quad \xi \in X_r \end{cases}
\]

• Obviously: \( J^*_r \leq J^*_p \)

• Problem \( P_r \) can be solved using classical optimal control methods
• Main solution idea:
  
  – solve $P_r$, resulting in $\xi^*_r \in \mathcal{X}_r$

  – project back to pure space: $\Gamma(\xi^*_r) \rightarrow \xi^*_p \in \mathcal{X}_p$

• Question: can we find a good projection without losing much on performance?

  – Answer: Yes. The cost of any relaxed control input $\xi_r$ can be approximated arbitrarily well by a pure control input $\xi_p$. This is known as the Chattering Lemma.

• **Lemma 1 (Chattering Lemma)** $\forall \epsilon > 0, \forall \xi \in \mathcal{X}_r, \exists \xi_p \in \mathcal{X}_p \text{ s.t. } \|\phi_t(\xi_r) - \phi_t(\xi_p)\|_2 \leq \epsilon$
Proof of chattering lemma:

- We show the case with $M = 2$ with no continuous control. The result can be easily extended to the general case.

- Given an arbitrary $\alpha(t) \in [0, 1]$. Let $\phi_t$ be the solution to

$$
\dot{x} = f(t, x(t)) = \alpha(t)f_0(t, x(t)) + (1 - \alpha(t))f_1(t, x(t)).
$$

- We want to construct another $\bar{\alpha}(t) \in \{0, 1\}$ so that the corresponding solution $\bar{\phi}_t$ to

$$
\dot{x}(t) = \bar{f}(t, x(t)) = \bar{\alpha}(t)f_0(t, x(t)) + (1 - \bar{\alpha}(t))f_1(t, x(t))
$$

satisfies the desired inequality.

- Given partition $0 = t_0 < t_1 < \cdots < t_n = 1$ with $t_{k+1} - t_k = \Delta t$. Choose $t'_k \in (t_k, t_{k+1})$ such that

$$
\int_{t_k}^{t_{k+1}} (1 - \alpha(\tau))d\tau = \int_{t_k}^{t'_k} \alpha(\tau)d\tau.
$$

We propose to construct

$$
\bar{\alpha}(t) = \begin{cases} 
0 & \text{if } t \in [t_k, t'_k) \\
1 & \text{if } t \in [t'_k, t_{k+1})
\end{cases}
$$

Embedding Principle and Chattering Lemma
– Now let’s derive a bound for $\|\phi_t - \tilde{\phi}_t\|$. Note that
\[
\phi_t - \tilde{\phi}_t = \int_0^t f(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) \, d\tau = \int_0^t [f(\tau, \phi_\tau) - \tilde{f}(\tau, \phi_\tau)] \, d\tau + \int_0^t [\tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau)] \, d\tau
\] (7)
Define $f^\Delta(t) = f_1(t, \phi_t) - f_0(t, \phi_t)$, $\forall t \in [0, 1]$.

– first term of (7)
\[
= \sum_k \left( \int_{t_k}^{t_{k+1}} (1 - \alpha(\tau)) f^\Delta(\tau) \, d\tau - \int_{t_k'}^{t_{k+1}'} \alpha(\tau) f^\Delta(\tau) \, d\tau \right)
= \sum_k \left( f^\Delta(t_k) \left[ \int_{t_k}^{t_{k+1}} (1 - \alpha(\tau)) \, d\tau \right] - f^\Delta(t_k) \left[ \int_{t_k'}^{t_{k+1}'} \alpha(\tau) \, d\tau \right] + e_k \right) = \sum_k e_k
\]
where
\[
\|e_k\| = \| \int_{t_k}^{t_{k+1}} (1 - \alpha(\tau)) \left( f^\Delta(\tau) - f^\Delta(t_k) \right) \, d\tau - \int_{t_k'}^{t_{k+1}'} \alpha(\tau) \left( f^\Delta(\tau) - f^\Delta(t_k) \right) \, d\tau \| \leq \tilde{L} \Delta t^2
\]
Hence, choose $\Delta t$ small enough: first term $\leq \frac{\epsilon}{2}$

– second term of (7):
\[
\| \int_0^t \left[ \tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) \right] \, d\tau \| \leq \int_0^t L \|\phi_\tau - \tilde{\phi}_\tau\| \, d\tau
\]
\[
\Rightarrow \|\phi_t - \tilde{\phi}_t\| \leq \epsilon + L \int_0^t \|\phi_\tau - \tilde{\phi}_\tau\| \, d\tau
\]
by Gronwall inequality $\leq \epsilon$
The proof is constructive, but is not the best way to construct pure control input.

A more effective projection strategy based on wavelet can be found in [VGBS13].
Solving Relaxed Switched Optimal Control Problem

• Now the question comes back to how to solve the relaxed optimal control $\xi^*_r$?

• This is a classical optimal control problem. Analytical solution usually does not exist.

• $\xi^*_r$ can be found using gradient type of algorithms in functional space.

• The key is to compute the directional derivative: $D J(\xi; \eta)$ and $D \psi_{j,t}(\xi; \eta)$
  
  – If $D J(\xi; \eta) < 0$, we can decrease cost by moving in $\eta$ direction
  
  – If $\max_{j,t} D \psi_{j,t}(\xi, \eta) < 0$, we can reduce infeasibility by moving in $\eta$ direction
  
  – Once we have $D J$ and $D \psi_{j,t}$, numerous algorithms are available to find local min (See [Polak97])
• Since $d$ can be varied continuously for problem $P_r$. We move $d$ into $u$ and deal with typical nonlinear system: $\dot{x} = f(t, x, u)$. In this case, $\xi = u$.

• Directional derivative of state trajectory $D\phi_t(u; \eta)$ is given by

$$D\phi_t(u; \eta) = \int_0^t \Phi(t, \tau; u) \left( \frac{\partial f}{\partial u}(\tau, \phi_t(u), u(\tau)) \eta(\tau) \right) d\tau$$  \hspace{1cm} (8)$$

where $\Phi(t, \tau; u)$ is the unique solution to

$$\frac{\partial \Phi}{\partial t}(t, \tau) = \frac{\partial f}{\partial x}(t, \phi_t(u), u(t)) \Phi(t, \tau)$$  \hspace{1cm} (9)$$
• With $D\phi_t(u; \eta)$, we can easily find $J(x(\eta; \eta))$

$$DJ(u; \eta) = \frac{\partial J}{\partial x}(\phi_t(u))D\phi_t(u; \eta), \quad D\psi_{i,t}(u; \eta) = \frac{\partial \psi_{i,t}}{\partial x}(\phi_t(u))D\phi_t(u; \eta)$$  \hspace{1cm} (10)

chain rule:

Egested Magnus

• Example 2 (Switching Time Optimization Problem): Consider switched linear system

$$\dot{x} = A_{\sigma(t)}x(t), \quad \sigma(t) = \begin{cases} 
1 & t \in [0, u_1) \\
2 & t \in [u_1, u_2), \\
1 & t \in [u_2, u] 
\end{cases} \quad A_1 = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where mode sequence is known and the transition time $u = [u_1, u_2]$ needs to be optimized with cost function $J(u) = \frac{1}{2} \int_0^u \|x(t)\|^2 dt$. 

\[ \begin{array}{ccc} 1 & \rightarrow & 2 & \rightarrow & 1 \\
\u_1 & \u & \u_2 \end{array} \]
solution to Example 2:

\[ \dot{x} = f(t, x, u) = \begin{cases} [A_1 x \quad 1] & t \in [0, u_1) \\ [A_2 x \quad 1] & t \in [u_1, u_2) \\ [A_1 x \quad 1] & t \in [u_2, 1] \end{cases} \]

\[ \dot{x}(0) = \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \dot{z}(t) = \begin{bmatrix} x^2(t) + x_2^2(t) \end{bmatrix} \in \mathbb{R}^3 \]

\[ z(1) = \int_0^1 \|x(t)\|^2 dt \]

\[ \dot{x}(t) = [A_i x(t)] \quad i \in \mathbb{Q} \]

First: compute \( D \phi_t(u; 2) = \lim_{\alpha \to 0} \frac{\phi_t(u+\alpha 2) - \phi_t(u)}{\alpha} \)

\[ \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f}{\partial u_1} \\ \frac{\partial f}{\partial u_2} \end{bmatrix} \]

\[ f(t, x, u) = f_1(x) + \bigcup_{u \in \mathbb{Z}} (t-u, t][f_2(x) - f_1(x)] \quad \forall t \in (u_1 - \varepsilon, u_1 + \varepsilon) \]

\[ \frac{\partial f}{\partial u_1} = \delta(t-u_1)(f_1(x) - f_2(x)) \ldots (1) \]

similarly, \[ \frac{\partial f}{\partial u_2} = \delta(t-u_2)(f_2(x) - f_1(x)) \ldots (2) \]
solution to Example 2 (cont.):

\[ \eta = [\eta_1, \eta_2] \]

\[ \Phi_t(u, \eta) = \int_0^t \Phi(t, \tau) \frac{\partial f}{\partial u} \cdot \eta \ d\tau = \int_0^t \Phi(t, \tau) \left( \frac{\partial f_1}{\partial u_1} \eta_1 + \frac{\partial f_2}{\partial u_2} \eta_2 \right) d\tau \]

\[ = \Phi_t(u_1) \left[ f_1(x(u_1)) - f_2(x(u_1)) \right] + \Phi_t(u_2) \left[ f_2(x(u_2)) - f_1(x(u_2)) \right] \]

\[ I_{t \geq u_1} \]

\[ I_{t > u_1} = \begin{cases} 0 & \text{if } t < u_1 \\ 1 & \text{if } t \geq u_2 \end{cases} \]

\[ \Phi(t, u) \text{ solves: } \]

\[ \frac{\partial}{\partial t} \Phi(t, \tau) = \frac{\partial f}{\partial x} \left( t, x, u \right) \Phi(t, \tau) \]

\[ \Phi(t, t) = 1 \]

\[ \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} \\ \vdots \\ \frac{\partial f_2}{\partial x_1} \\ \vdots \end{bmatrix} = \begin{cases} \text{form 1} & \tau \in [0, u_1) \\ \text{form 2} & \tau \in [u_1, u_2) \\ \text{form 1} & \tau \in [u_2, \infty) \end{cases} \]
When \( u_1 = 0.3 \), \( u_2 = 0.6 \)

\[
J(\alpha) = \sqrt{4 \eta_1 - 5.7 \eta_2} \quad \text{to compute this}
\]

\[
D_J(u, \eta) = \frac{2J}{\partial x^i} \frac{\partial D\phi_t(u, \eta)}{\partial x^i}
\]

you need to solve this

\[
\begin{bmatrix} 0 & 0 & \beta \end{bmatrix} \left( \begin{array} \end{array} \right) = 0
\]

\[\text{ODE}(\tau)\]
Example 3 (Switched optimal control problem): Quadrotor Model

\[ \dot{x} = f_i(x, u) \]

where

\[
f_1(x, u) = \begin{bmatrix}
x_4(t) \\
x_5(t) \\
x_6(t) \\
\frac{\sin x_3(t)}{M} (u(t) + Mg) \\
\frac{\cos x_3(t)}{M} (u(t) + Mg) - g \\
0
\end{bmatrix},
\]

\[
f_2(x, u) = \begin{bmatrix}
x_4(t) \\
x_5(t) \\
x_6(t) \\
g \sin x_3(t) \\
g \cos x_3(t) - g \\
\frac{-Lu(t)}{I}
\end{bmatrix},
\]

\[
f_2(x, u) = \begin{bmatrix}
x_4(t) \\
x_5(t) \\
x_6(t) \\
g \sin x_3(t) \\
g \cos x_3(t) - g \\
\frac{Lu(t)}{I}
\end{bmatrix}
\]

Cost function:

\[
\int_0^t 5u^2(t)dt + 5(x_1(t_f) - 6)^2 + 5(x_2(t_f) - 1)^2 + \sin \left( \frac{x_3(t_f)}{2} \right)
\]

Constraint: \( u(t) \in [0, 10^{-3}] \) and \( x_2(t) \geq 0 \)