ECE7850 Lecture 10

Continuous Time Optimal Control of Switched Nonlinear Systems

With only one week left, we will only focus on basic concepts and key results.

• Finite-dimensional optimization problems

• Elementary calculus of variations

• Pure and relaxed switched optimal control problems

• Chattering lemma and embedding principle

• Solution to relaxed problems

• Examples: switching time optimizations and switched optimal control.
Finite-dimensional optimization problems

- Basic building block (1D case):
  \[
  \begin{cases}
  \min_x g(x) \\
  \text{subj. to } x \in [a, b]
  \end{cases}
  \]
  - Necessary local min condition for interior point \( x^* \in (a, b) \):

- Necessary local min condition for \( x^* = a \):
Consider a general constrained optimization problem in $\mathbb{R}^n$:

$$\begin{align*}
\min_x & \quad J(x) \\
\text{subj. to} & \quad x \in X
\end{align*}$$

$X$ is a closed subset in $\mathbb{R}^n$ with boundary $\partial X$

**Definition 1 (Local minimizer):** $x^* \in X$ is called a local min if there exists $r > 0$, such that

$$J(x^*) \leq J(x), \forall x \in N(x^*; r) \cap X$$ (1)

Necessary conditions: conditions weaker than (1) that can be used to eliminate non-optimal solutions
• We call \( d \in \mathbb{R}^n \) a **feasible direction** if \( \exists \alpha > 0 \) such that \( x + \alpha d \in X \) for all \( \alpha \in [0, \bar{\alpha}] \)

• Feasible cone at \( x \): \( F_X(x) = \{0\} \cup \{ \text{feasible directions} \} \)

• An obvious necessary condition: cost does not increase along any feasible direction

\[
\forall d \in F_X(x^*), \text{ we have } J(x^* + \alpha d) \geq J(x^*), \text{ for sufficiently small } \alpha
\]  

\[
(2)
\]

• **Directional derivative**: \( DJ(x^*; d) = \lim_{\alpha \to 0} \frac{J(x^* + \alpha d) - J(x^*)}{\alpha} \).

• 1-sided directional derivative: \( DJ^+(x^*; d) = \lim_{\alpha \downarrow 0} \frac{J(x^* + \alpha d) - J(x^*)}{\alpha} \).
• Optimality conditions in terms of directional derivatives:

  – Whenever the 1-sided directional derivative exists, (2) requires

    \[ DJ^+(x^*; d) \geq 0, \forall d \in F_X(x^*) \]

  – If \( DJ \) also exists, then we need \( DJ(x^*; d) = 0 \).
Example 1 \[
\begin{aligned}
\min_x \quad & J(x) \\
\text{subj. to} \quad & h(x) \leq 0
\end{aligned},
\]
where \( J : \mathbb{R}^n \to \mathbb{R} \) and \( h : \mathbb{R}^n \to \mathbb{R} \) are differentiable.
• Feasible direction can be restrictive (especially for nonlinear equality constraints)

• Need to replace $F_X(x)$ with tangent cone $T_X(x)$ in general: including all limiting tangent directions when approaching $x$ inside $X$

• Note $cl(F_X(x)) \subseteq T_X(x)$
• A large class of optimal control problems can be viewed as optimization problem in infinite-dimensional space

  – $X$ becomes a space of control input signals (function of time)

  – $J$ becomes function of control signal (functional)

  – But the results are still based on the same key concepts: necessary conditions, feasible direction, and directional derivatives

  – We just need slight generalizations.
Elementary Calculus of Variations

• Let $\mathcal{V}$ be a general normed vector space (may be infinite dimensional)

• Consider constrained optimization: $\min_{x \in \mathcal{X}} J(x)$, where $\mathcal{X} \subseteq \mathcal{V}$ and $J : \mathcal{X} \rightarrow \mathbb{R}$ is a functional.

• Generalization of directional derivative concept to function space: Gateaux derivative

$$\delta J(x; \eta) = \lim_{\alpha \to 0} \frac{J(x + \alpha \eta) - J(x)}{\alpha}$$

• It is also called the first variation of $J$ along $\eta$

• Gateaux derivative (or first variation) $\delta J(x; \eta)$ (according to above definition) may not be linear in $\eta$. Definitions in the literature often directly requires it to be linear in $\eta$. 
• one-sided **Gateaux derivative**:

\[
\delta J^+(x; \eta) = \lim_{\alpha \downarrow 0} \frac{J(x + \alpha \eta) - J(x)}{\alpha}
\]

• Optimality conditions:

- \( \delta J^+(x; \eta) \geq 0 \) for all feasible/admissible directions \( \eta \)

- \( \delta J(x; \eta) = 0 \), for all feasible/admissible directions \( \eta \)
• **Note:** $\delta J(x; \eta) = g'(0)$, where $g(\alpha) = J(x + \alpha \eta)$.

• **Basic calculus of variation problem:** Find a $C^1$ function $x : [a, b] \to \mathbb{R}$ with given $x(a) = x_0$ and $x(b) = x_f$ to minimize $J(x) = \int_a^b l(x, x', t) dt$.

  – this curve optimization problem can be viewed as a control problem: find control $u \in C^0$ to minimize $J(u) = \int_a^b l(x, u, t) dt$ subject to dynamic constraint $\dot{x} = u$.

  – Derive $\delta J(x; \eta)$ for admissible $\eta$ that satisfies $\eta(a) = \eta(b) = 0$.

\[
J(x + \alpha \eta) = \int_a^b l(x + \alpha \eta, x' + \alpha \eta', t) dt = \int_a^b (l(x, x', t) + 2l_x(x, x', t) \alpha \eta + l_{x'}(x, x', t) \alpha \eta') dt + o(\alpha)
\]
- **Fundamental lemma of calculus of variation**: If a continuous function \( \xi : [a, b] \to \mathbb{R} \) satisfies
\[
\int_a^b \xi(t)\eta(t)\,dt = 0
\]
for all \( C^1 \) function \( \eta : [a, b] \to \mathbb{R} \) with \( \eta(a) = \eta(b) = 0 \), then \( \xi \equiv 0 \).

- **Euler-Lagrange Equation:**
Example: \( J(x) = \int_0^{\pi/2} [\ddot{x}^2(t) - x^2(t)] dt \) with boundary conditions \( x(0) = 0, x(\pi/2) = 1. \)
Switched Optimal Control Problems

• Switched nonlinear systems:

\[ \dot{x}(t) = f_{\sigma(t)}(t, x(t), u(t)), \quad \text{where } \sigma(t) \in \mathcal{Q} = \{1, \ldots, q\} \tag{3} \]

• Hybrid control: \( \xi(t) = (u(t), \sigma(t)) \) with constraints:

\[ u(t) \in \mathcal{U} \subset \mathbb{R}^m, \sigma(t) \in \mathcal{Q}, \quad \text{where } \mathcal{U} \text{ bounded and convex} \]

• State trajectory driven by \( \xi \): \( x(t; \xi) \), or simply \( x(t) \).

• A finite time horizon, w.l.g., assume \( \mathcal{T} = [0, 1] \).

• State trajectory constraint: \( h_j(x(t)) \leq 0, \forall j \in \mathcal{J} = \{1, 2, \ldots n_s\}, \forall t \in \mathcal{T} \).
• Cost function: $J(x(1; \xi))$
  
  – only penalize terminal state.
  
  – problems with running cost can be reduced to this form by introducing additional state.

• Notations to emphasize dependence on $\xi$:
  
  – $\phi_t(\xi) \triangleq x(t; \xi)$, $\psi_{j,t}(\xi) \triangleq h_j(x(t; \xi))$, $J(\xi) \triangleq J(x(1; \xi))$
  
  – Overall constraint functional: $\Psi(\xi) \triangleq \max_{j,t} \psi_{j,t}(\xi)$.

• Assumptions [$A1$]:
  
  – $f_i(t, x, u), h_j(x), J(x)$ are Lipchitz continuous w.r.t all arguments
  
  – $\frac{\partial f_i}{\partial x}(t, x, u), \frac{\partial f_i}{\partial u}(t, x, u), \frac{\partial h_i}{\partial x}(x), \frac{\partial J}{\partial x}(x)$ exist and are Lipchitz continuous w.r.t. all arguments
• An equivalent way to write system dynamics:

\[ \dot{x} = f(t, x, u, d) \triangleq \sum_{i=1}^{q} d_i(t) f_i(t, x, u) \]

where \( d(t) = [d_1(t), \ldots, d_q(t)]^T \) is a corner of the \( q \)-simplex:

\[ \Sigma_q^p = \{(d_1, \ldots, d_q) \in \{0, 1\}^q | \sum_{i=1}^{q} d_i = 1\} \]

• Control Spaces:

  – We say \( f : [0, 1] \rightarrow F \) belongs to \( L_2([0, 1], F) \) if \( \|f\|_{L_2} = \left( \int_{0}^{1} \|f(t)\|_2^2 dt \right)^{1/2} < \infty \)

  – Continuous input space: \( U = L_2([0, 1], U) \)
– Discrete input space: $\mathcal{D}_p = L_2([0, 1], \Sigma_p^q)$

– Overall optimization space: $\mathcal{X} = L_2([0, 1], \mathbb{R}^m) \times L_2([0, 1], \mathbb{R}^q)$

– Pure optimization space: $\mathcal{X}_p = \mathcal{U} \times \mathcal{D}_p$

• (Pure) Switched Optimal Control Problem:

$$\mathcal{P}_p : \quad J^*_p = \begin{cases} \inf_{\xi} J(\xi) \\ \text{subj. to} \quad \Psi(\xi) \leq 0, \quad \xi \in \mathcal{X}_p \end{cases}$$

• Challenges: space $\mathcal{X}_p = \mathcal{U} \times \mathcal{D}$ is not a vector space due to $\mathcal{D}$, on which gradient of $J$ and $\Psi$ are not well defined.
Embedding Principle and Chattering Lemma

• Key idea for solving $\mathcal{P}_p$ is to “embed” the switched systems into a larger class of nonlinear systems for which $d$ takes values inside the entire $q$-simplex (not just the corner points).

• $q$-simplex: $\Sigma^q = \{(d_1, \ldots, d_q) \in [0, 1]^q | \sum_{i=1}^{q} d_i = 1\}$.

• Relaxed System:

$$\dot{x}(t) = \sum_{i \in Q} d_i(t) f_i(t, x(t), u(t)), \text{ with } x(0) = x_0. \quad (5)$$

  – $d(t) \in \Sigma_p^q \Rightarrow$ original switched systems
  – $d(t) \in \Sigma_i^q \Rightarrow$ relaxed switched systems
  – The set of all trajectories of the switched system is contained in that of the relaxed system.
• Relaxed control spaces:

  – Related discrete input space: \( D_r = L^2([0, 1], \Sigma_q) \)

  – Relaxed optimization space: \( X_r = U \times D_r \)

• Relaxed Switched Optimal Control Problem \( \mathcal{P}_r \):

\[
\mathcal{P}_r : \quad J^*_r = \begin{cases} 
\inf_{\xi} J(\xi) \\
\text{subj. to} \\
\Psi(\xi) \leq 0, \quad \xi \in X_r
\end{cases}
\]

\[ (6) \]

• Obviously: \( J^*_r \leq J^*_p \)

• Problem \( \mathcal{P}_r \) can be solved using classical optimal control methods
Main solution idea:

- solve $P_r$, resulting in $\xi^*_r \in \mathcal{X}_r$
- project back to pure space: $\Gamma(\xi^*_r) \rightarrow \xi^*_p \in \mathcal{X}_p$

Question: can we find a good projection without losing much on performance?

- Answer: Yes. The cost of any relaxed control input $\xi_r$ can be approximated arbitrarily well by a pure control input $\xi_p$. This is known as the Chattering Lemma.

Lemma 1 (Chattering Lemma) $\forall \epsilon > 0, \forall \xi \in \mathcal{X}_r, \exists \xi_p \in \mathcal{X}_p$ s.t. $\|\phi_t(\xi_r) - \phi_t(\xi_p)\|_2 \leq \epsilon$
Proof of chattering lemma:

– We show the case with \( M = 2 \) with no continuous control. The result can be easily extended to the general case.

– Given an arbitrary \( \alpha(t) \in [0, 1] \). Let \( \phi_t \) be the solution to

\[
\dot{x} = \alpha(t) f_0(t, x(t)) + (1 - \alpha(t)) f_1(t, x(t)).
\]

– We want to construct another \( \tilde{\alpha}(t) \in \{0, 1\} \) so that the corresponding solution \( \tilde{\phi}_t \) to

\[
\dot{x}(t) = \tilde{f}(t, x(t)) = \tilde{\alpha}(t) f_0(t, x(t)) + (1 - \tilde{\alpha}(t)) f_1(t, x(t))
\]

satisfies the desired inequality.

– Given partition \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) with \( t_{k+1} - t_k = \Delta t \). Choose \( t'_k \in (t_k, t_{k+1}) \) such that \( \int_{t_k}^{t_{k+1}} (1 - \alpha(\tau)) d\tau = \int_{t_k}^{t_{k+1}} \alpha(\tau) d\tau \). We propose to construct

\[
\tilde{\alpha}(t) = \begin{cases} 
0 & \text{if } t \in [t_k, t'_k) \\
1 & \text{if } t \in [t'_k, t_{k+1})
\end{cases}
\]
– Now let’s derive a bound for \( \| \phi_t - \tilde{\phi}_t \| \). Note that
\[
\phi_t - \tilde{\phi}_t = \int_0^t f(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) d\tau = \int_0^t [f(\tau, \phi_\tau) - \tilde{f}(\tau, \phi_\tau)] d\tau + \int_0^t [\tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau)] d\tau
\] (7)
Define \( f^\Delta(t) = f_1(t, \phi_t) - f_0(t, \phi_t) \), \( \forall t \in [0, 1] \).

– first term of (7)
\[
\begin{align*}
&= \sum_k \left( \int_{t_k}^{t_{k+1}} (1 - \alpha(\tau)) f^\Delta(\tau) d\tau - \int_{t_k}^{t_{k+1}} \alpha(\tau) f^\Delta(\tau) d\tau \right) \\
&= \sum_k \left( f^\Delta(t_k) \left[ \int_{t_k}^{t_{k+1}} (1 - \alpha(\tau)) d\tau \right] - f^\Delta(t_k) \left[ \int_{t_k}^{t_{k+1}} \alpha(\tau) d\tau \right] + e_k \right) = \sum_k e_k
\end{align*}
\]
where
\[
\| e_k \| = \left\| \int_{t_k}^{t_{k+1}} (1 - \alpha(\tau))(f^\Delta(\tau) - f^\Delta(t_k)) d\tau - \int_{t_k}^{t_{k+1}} \alpha(\tau)(f^\Delta(\tau) - f^\Delta(t_k)) d\tau \right\| \leq \tilde{L} \Delta t^2
\]
Hence, choose \( \Delta t \) small enough: first term \( \leq \frac{\epsilon}{2} \)

– second term of (7):
\[
\left\| \int_0^t [\tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau)] d\tau \right\| \leq \int_0^t L \| \phi_\tau - \tilde{\phi}_\tau \| d\tau \\
\Rightarrow \| \phi_t - \tilde{\phi}_t \| \leq \epsilon' + L \int_0^t \| \phi_\tau - \tilde{\phi}_\tau \| d\tau
\]
by Gronwall inequality \( \leq \epsilon \)

Embedding Principle and Chattering Lemma 22
• The proof is constructive, but is not the best way to construct pure control input.

• A more effective projection strategy based on wavelet can be found in [VGBS13].
Solving Relaxed Switched Optimal Control Problem

• Now the question comes back to how to solve the relaxed optimal control $\xi^*_r$?

• This is a classical optimal control problem. Analytical solution usually does not exist.

• $\xi^*_r$ can be found using gradient type of algorithms in functional space.

• The key is to compute the directional derivative: $DJ(\xi; \eta)$ and $D\psi_{j,t}(\xi; \eta)$

  – If $DJ(\xi; \eta) < 0$, we can decrease cost by moving in $\eta$ direction

  – If $\max_{j,t} D\psi_{j,t}(\xi, \eta) < 0$, we can reduce infeasibility by moving in $\eta$ direction

  – Once we have $DJ$ and $D\psi_{j,t}$, numerous algorithms are available to find local min (See [Polak97])
• Since $d$ can be varied continuously for problem $P_r$. We move $d$ into $u$ and deal with typical nonlinear system: $\dot{x} = f(t, x, u)$. In this case, $\xi = u$.

• Directional derivative of state trajectory $D\phi_t(u; \eta)$ is given by

$$D\phi_t(u; \eta) = \int_0^t \Phi(t, \tau; u) \left( \frac{\partial f}{\partial u}(\tau, \phi_\tau(u), u(\tau)) \cdot \eta(\tau) \right) d\tau$$

(8)

where $\Phi(t, \tau; u)$ is the unique solution to

$$\frac{\partial \Phi}{\partial t}(t, \tau) = \frac{\partial f}{\partial x}(t, \phi_t(u), u(t)) \Phi(t, \tau)$$

(9)
With $D\phi_t(u; \eta)$, we can easily find

$$DJ(u; \eta) = \frac{\partial J}{\partial x}(\phi_t(u))D\phi_t(u; \eta), \quad D\psi_{j,t}(u; \eta) = \frac{\partial h_j}{\partial x}(\phi_t(u))D\phi_t(u; \eta)$$  \hspace{1cm} (10)

• Example 2 (Switching Time Optimization Problem): Consider switched linear system

$$\dot{x} = A_{\sigma(t)}x(t), \quad \sigma(t) = \begin{cases} 
1 & t \in [0, u_1) \\
2 & t \in [u_1, u_2) \\
1 & t \in [u_2, 10]
\end{cases}, \quad A_1 = \frac{1}{10} \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \frac{1}{10} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where mode sequence is known and the transition time $u = [u_1, u_2]$ needs to be optimized with cost function $J(u) = \frac{1}{20} \int_0^{10} \|x(t)\|^2 dt$. 
solution to Example 2:
solution to Example 2 (cont.):
• Example 3 (Switched optimal control problem): Quadrotor Model

\[ \dot{x} = f_i(x, u) \text{ where} \]

\[
f_1(x, u) = \begin{bmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} \sin x_3(t) (u(t) + Mg) \\ \cos x_3(t) (u(t) + Mg) - g \\ 0 \end{bmatrix}, \quad f_2(x, u) = \begin{bmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} \sin x_3(t) \\ \cos x_3(t) - g \\ -Lu(t) \end{bmatrix} \]

\[ f_2(x, u) = \begin{bmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} \sin x_3(t) \\ \cos x_3(t) - g \\ \frac{Lu(t)}{I} \end{bmatrix} \]

Cost function: \[ \int_0^{t_f} 5u^2(t)dt + 5(x_1(t_f) - 6)^2 + 5(x_2(t_f) - 1)^2 + \sin \left( \frac{x_3(t_f)}{2} \right) \]

constraint: \[ u(t) \in [0, 10^{-3}] \text{ and } x_2(t) \geq 0 \]