Variable Neural Adaptive Robust Output Feedback Control of Uncertain Systems

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Abstract—The design of an adaptive robust output feedback controller is presented for a class of multi-input multi-output uncertain systems. The proposed output feedback controller uses a variable-structure radial basis function (RBF) network to approximate unknown system dynamics. The output feedback implementation is realized by a high-gain observer. The structure variation of the RBF network is taken into account in the stability analysis of the closed-loop system through the piecewise continuous quadratic Lyapunov function. The performance of the proposed variable neural adaptive robust output feedback controller is illustrated with simulations.

I. INTRODUCTION

In order to deal with systems with unknown dynamics or disturbances, many adaptive control strategies such as adaptive feedback linearization [1], adaptive backstepping [2], nonlinear damping and swapping [3] and switching adaptive control [4] have been introduced. Especially, a number of adaptive controllers have been developed for a class of feedback linearizable nonlinear systems including both single-input single-output (SISO) systems [5]–[7] and multi-input multi-output (MIMO) systems [8]–[11]. However, the proposed controllers often require the full access of the system states, which are usually not available in practice. The more realistic way of accessing the system states is through the observer. For example, the high-gain observers have been employed in [7], [12] to design output feedback based adaptive controllers. The advantage of using high-gain observers is that the control problem can be formulated in a standard singular perturbation form and then the singular perturbation theory can be applied to analyze the closed-loop system stability. Moreover, when the speed of the high-gain observer is sufficiently high, the performance of the output feedback controller recovers the performance of the state feedback controller [12].

Many types of function approximators have been utilized by adaptive controllers for approximating unknown system dynamics. For example, fixed-structure radial basis function (RBF) networks have been employed in [5], [7], [11]. The disadvantage of the fixed-structure RBF network is that the set of basis functions needs to be selected off-line, it is still necessary to pre-determine the number of hidden neurons. On the other hand, the RBF network has its own advantages over the MLNN, which include simpler structure, faster computation time and superior adaptive performance. Therefore, variable-structure RBF network based adaptive controllers have been recently proposed for SISO feedback linearizable uncertain systems in [13], [14], where variable-structure RBF networks preserve the advantages of RBF networks and, at the same time, overcome the limitations of fixed-structure RBF networks. However, the effect of the structure variation was not considered in the stability analysis in [13], [14], where the system stability is only analyzed for each fixed structure.

In this paper, we consider the same problem of output tracking control for a class of MIMO feedback linearizable uncertain systems as in [15], which has also been considered in [8]–[11]. In [15], an adaptive robust state feedback control strategy was proposed, where a variable-structure RBF network is used to approximate unknown system dynamics. The employed variable-structure RBF network avoids selecting basis functions off-line by allocating RBFs on-line dynamically. It can add RBFs to improve the approximation accuracy and later remove RBFs to prevent the network redundancy. The dwell time requirement was also introduced to avoid fast switching among different structures. However, the implementation of the proposed controller in [15] requires the access of the system states, which may not be possible in practice. To overcome this restriction, we propose a variable neural adaptive robust output feedback controller. We incorporate a high-gain observer into the state feedback control strategy proposed in [15]. The structure variation of the RBF network is also considered in the stability analysis through the piecewise continuous quadratic Lyapunov function that has been used for the stability analysis of switched and hybrid systems [16], [17].

II. PROBLEM STATEMENT

A. System Description

We consider a class of uncertain systems consisting of $p$ coupled subsystems modeled by the following equations,

$$y_i^{(n)} = f_i(x) + \sum_{j=1}^{p} g_{ij}(x)u_j + d_i, \quad i = 1, \ldots, p,$$  \hspace{1cm} (1)

where $y_i$, $u_i$ and $d_i$ are the output, input and disturbance of the $i$-th subsystem, respectively, and $f_i(x)$ and $g_{ij}(x)$ are
unknown functions with
\[ x = \begin{bmatrix} y_1 \cdots y_1^{(n_1-1)} \cdots y_p \cdots y_p^{(n_p-1)} \end{bmatrix}^\top \]
being the state vector of the whole system. The disturbance \( d_i \) may depend on both time \( t \) and state vector \( x \). Let \((A_i, b_i)\) be the canonical controllable pair that represents a chain of \( n_i \) integrators, and let \( c_i = [1 \ 0 \ \cdots \ 0]_{1 \times n_i}, \ y = [y_1 \ \cdots \ y_p]^\top, \ u = [u_1 \ \cdots \ u_p]^\top \) and \( d = [d_1 \ \cdots \ d_p]^\top \). The system (1) can be written in a compact form as
\[ \dot{x} = Ax + B (f(x) + G(x)u + d) \]
\[ y = Cx, \]
where \( A = \text{diag}[A_1, \cdots, A_p], \ B = \text{diag}[b_1, \cdots, b_p], \ C = \text{diag}[c_1, \cdots, c_p], \ f(x) = [f_1(x) \ \cdots \ f_p(x)]^\top \) and \( G(x) = [g_{ij}(x)]_{p \times p} \). The above system is often referred to as a square system, because there are same numbers of inputs and outputs. In this paper, we assume that \( f(x) \) and \( G(x) \) are Lipschitz continuous in \( x \), respectively, and that \( d \) is Lipschitz continuous in \( x \) and piecewise continuous in \( t \) with \( |d| \leq d_o \). In addition, we consider the case where the input matrix \( G(x) \) is definite with eigenvalues bounded away from zero for all \( x \) of interest. Without loss of generality, we assume that \( G(x) \) is positive definite with \( \Omega \). Let \( \Delta \Omega = \text{diag}[\omega_1, \cdots, \omega_p] \) be the output tracking error as
\[ e = \dot{y} - y. \]

\section*{B. Problem Formulation}

The control objective is such that the \( i \)-th system output \( y_i, i = 1, \ldots, p \), tracks a reference signal \( y_{di} \) that has bounded derivatives up to the \( n_i \)-th order as close as possible. Towards this objective, we propose a variable-structure RBF network based adaptive robust output feedback controller in this paper. To proceed, we define the desired system state vector as
\[ x_d = [y_{d1} \cdots y_{d1}^{(n_1-1)} \cdots y_{dp} \cdots y_{dp}^{(n_p-1)}]^\top \]
and \( y_d = [y_{d1}^{(n_1)} \cdots y_{dp}^{(n_p)}]^\top \). Because the reference signal \( y_{di} \) has bounded derivatives up to the \( n_i \)-th order, we have \( x_d \in \Omega_{x_d} \) and \( y_d^{(n)} \in \Omega_{y_d} \), where \( \Omega_{x_d} \) and \( \Omega_{y_d} \) are compact subsets of \( \mathbb{R}^{n_1} \) and \( \mathbb{R}^{n_p} \), respectively, and \( n = \sum_{i=1}^{p} n_i \). Let \( c_{pi} = y_i - y_d \) denote the \( i \)-th output tracking error. Then we define the output tracking error as \( e_y = [e_{y1} \ \cdots \ e_{yp}]^\top \) and the state tracking error as \( e = x - x_d \). It follows from (2) that the state tracking error dynamics are modeled by
\[ \dot{e} = Ae + B \left( f(x) + G(x)u - y_d^{(n)} + d \right). \]

Let \( K = \text{diag}[k_1, \ldots, k_p] \) be selected such that \( A_{m1} = A_i - b_i k_i \) is Hurwitz. Thus, \( A_{m} = \text{diag}[A_{m1}, \cdots, A_{mp}] \) is Hurwitz. Let \( \Omega_{e} \) be a compact set that includes all possible initial state tracking errors and let \( c_{e0} = \max_{e \in \Omega_{e}} \frac{1}{2} e^\top P_{m} e \), where \( P_{m} = \text{diag}[P_{m1}, \cdots, P_{mp}] \) is the solution to the continuous Lyapunov matrix equation \( A_{m}^\top P_{m} + P_{m} A_{m} = -2Q_{m} \) for \( Q_{m} = \text{diag}[Q_{m1}, \cdots, Q_{mp}] \) with \( Q_{mi} = Q_{mi} > 0 \). Choose \( c_e > c_{e0} \) and let \( \Omega_{e} = \{ e : \frac{1}{2} e^\top P_{m} e \leq c_e \} \).

Then we define the compact set \( \Omega_{e} = \{ x : x = e + x_d, e \in \Omega_{e}, x_d \in \Omega_{x_d} \} \), over which the unknown system dynamics are approximated by a variable-structure RBF network.

\section*{III. VARIABLE-STRUCTURE RBF NETWORK}

The variable-structure RBF network that we use to approximate unknown functions \( f(x) \) over a compact set \( \Omega_{x} \) is an improved version of the one proposed in [15]. It has \( N \) different admissible structures, where \( N \) is determined by the design parameters. For each admissible structure, the RBF network consists of \( n \) input neurons, \( M_v \) hidden neurons, and \( v \) output neurons. The \( k \)-th output of the RBF network with the \( v \)-th admissible structure can be represented as
\[ f_{k,v}(x) = \sum_{j=1}^{M_v} \omega_{kj,v} \xi_{j,v}(x), \]
where \( \omega_{kj,v} \) is the weight from the \( j \)-th hidden neuron to the \( k \)-th output neuron and \( \xi_{j,v}(x) \) is the radial basis function for the \( j \)-th hidden neuron. Let \( W_v = [\omega_{1,v} \ \cdots \ \omega_{M,v}] \) with \( \omega_{i,v} = [\omega_{1,v} \ \cdots \ \omega_{iM,v}]^\top \) and \( \xi_{j,v}(x) = [\xi_{1,v}(x) \ \cdots \ \xi_{M,v}(x)]^\top \). We have
\[ f_v(x) = W_v^\top \xi_v(x), \]
where \( f_v(x) = [f_{1,v}(x) \ \cdots \ f_{p,v}(x)]^\top \).

In the following subsections, we provide a brief description of the improved variable-structure RBF network. The major improvement over [15] lies in the RBF adding and removing operations.

\subsection*{A. Center Grid}

Recall that the unknown function \( f(x) \) is approximated over a compact set \( \Omega_{x} \subset \mathbb{R}^n \). To locate the centers of RBFs inside the approximation region \( \Omega_{x} \), we utilize an \( n \)-dimensional center grid with layer hierarchy, where each grid node corresponds to the center of one RBF. The grid nodes of the center grid are located at \( S_1 \times \cdots \times S_n \), where \( S_i \) is the set of locations of the grid nodes in the \( i \)-th coordinate and \( \times \) denotes the Cartesian product. The center grid is initialized inside the approximation region \( \Omega_{x} \) with \( S_i = \{ x_i, x_{ui} \}, \ i = 1, \ldots, n, \) where \( x_i \) and \( x_{ui} \) denote the lower and upper bounds in the \( i \)-th coordinate. The \( 2^n \) grid nodes of the initial grid are referred to as the boundary grid nodes, and they cannot be removed.

\subsection*{B. Adding RBFs}

If the elapsed time since the last operation, adding or removing, is greater than the dwell time \( T_d \), and \( \|e_y\| > \epsilon_{max} \), where \( \epsilon_{max} \) is a prespecified design parameter, for a period of time greater than \( T_d \), then the network attempts to add new RBFs, that is, add new grid nodes. First, the nearest neighboring grid node in the center grid, denoted \( c_{i(near)} \), to the current input \( x \) is located among existing grid nodes. Then the “nearer” neighboring grid node in the center grid denoted \( c_{i(near)} \) is located, where \( c_{i(near)} \) is determined such that \( x_i \) is between \( c_{i(near)} \) and \( c_{i(near)} \). The adding operation is performed for each coordinate independently. In the \( i \)-th coordinate, if the following conditions are satisfied:
\[ \begin{align*}
&1) \ x_i - c_{i(near)} > \frac{1}{2} (c_{i(near)} - c_i),
&2) \ x_i - c_{i(near)} > d_i(\text{threshold}),
\end{align*} \]
where $d_{i \text{(threshold)}}$ is a design parameter that specifies the minimum grid distance in the $i$-th coordinate and thus determines the number of admissible structures denoted by $N$, then a new location at the half of the sum of $c_{i \text{(nearest)}}$ and $c_{i \text{(nearest)}}$ is added into $S_i$. Otherwise, no new location is added to $S_i$. The layer of the newly added location is one level higher than the highest layer of the two adjacent existing locations in the same coordinate.

C. Removing RBFs

If the elapsed time since the last operation, adding or removing, is greater than the dwell time $T_d$, and $\|e_y\| \leq \rho e_{\text{max}}$, where $\rho \in (0, 1)$, for a period of time greater than $T_d$, then the network attempts to remove some of the existing RBFs, that is, some of the existing grid nodes to prevent network redundancy. The RBF removing operation is also implemented for each coordinate independently. In the $i$-th coordinate, if the following conditions are satisfied:

1. $c_{i \text{(nearest)}} \notin \{x_{i}, x_{ui}\}$,
2. the location $c_{i \text{(nearest)}}$ is in the higher than or in the same layer as the highest layer of the two neighboring locations in the same coordinate, 
3. $|x_i - c_{i \text{(nearest)}}| < \theta |c_{i \text{(nearest)}} - c_{i \text{(nearest)}}|$, $\theta \in (0, 0.5)$,

then the location $c_{i \text{(nearest)}}$ is removed from $S_i$. Otherwise, no location is removed from $S_i$.

D. Uniform Grid Transformation

The determination of the radius of the RBF is achieved by uniform grid transformation, whose details can be found in [15]. When implementing the output feedback controller, the state vector estimate, $\hat{x}$, is used rather than the actual state vector, $x$. The extended transformation to deal with $\hat{x} \notin \Omega_x$ can be found in [14].

IV. HIGH-GAIN TRACKING ERROR OBSERVER

We assume that the system states are not available for the feedback implementation. Thus, we adopt the high-gain observer as used in [7, 12].

\[
\dot{\hat{x}}_i = A_i \hat{x}_i + I_i (c_{yi} - \hat{c}_{yi}), \quad i = 1, \ldots, p
\]

(5)

to estimate the tracking error $e_i$ of the $i$-th subsystem. The observer gain $I_i$ is chosen as $I_i = [\alpha_{i1}/\epsilon \cdots \alpha_{in_i}/\epsilon^{n_i}]^T$, where $\epsilon \in (0, 1)$ is a design parameter and $\alpha_{ij}, j = 1, \ldots, n_i$, are selected so that the roots of the characteristic polynomial equation, $s^{n_i} + \alpha_{i1}s^{n_i-1} + \cdots + \alpha_{i(n_i-1)}s + \alpha_{in_i} = 0$, have negative real parts. Let $\hat{e} = [\hat{e}_1 \cdots \hat{e}_p]^T$ and $L = \text{diag}(l_1 \cdots l_p)$. Then the high-gain observer (5) can be rewritten as

\[
\begin{align*}
\dot{\hat{e}} &= A\hat{e} + L (e_y - C\hat{e}) \quad (6) \\
\end{align*}
\]

To facilitate the following stability analysis of the closed-loop system, we represent the estimation error dynamics in the singularly perturbed form. Let $\zeta = [\zeta_1^T \cdots \zeta_p^T]^T$, where $\zeta_i = [\zeta_{i1} \cdots \zeta_{in_i}]^T$ with

\[
\zeta_{ij} = \frac{e_{yi}^{(j-1)} - \tilde{e}_{yi}^{(j-1)}}{e_{ni-j}}, \quad j = 1, \ldots, n_i.
\]

(7)

It follows from (7) that $\dot{\hat{e}} = e - DC$, where $D = \text{diag}[D_1 \cdots D_p]$ and $D_i = \text{diag}[e_{ni-1} \cdots 1]$. Note that the $L_2^\infty$-norm of $D$ is 1, that is, $\|D\| = 1$. Combining (3), (6) and (7) yields

\[
c\zeta = Ac\zeta + eB (f(x) + G(x)u - y_d^{(n)} + d),
\]

(8)

where $A_c = \text{diag}[A_{c1} \cdots A_{cp}]$ and $A_{ci} = eD^{-1}_i(A_i - l_i c_i)D_i$ is a Hurwitz matrix. Applying the method from [18], we can prove the following proposition.

**Proposition 1:** Suppose that the control input $u$ is globally bounded. There exists a constant $\epsilon_i^* \in (0, 1)$ such that if $e \in (0, \epsilon_i^*)$, then $|\dot{\zeta}(t)| \leq \beta e$ with some $\beta > 0$ for $t \in [t_0 + T_1(\epsilon), t_0 + T_3)$, where $T_1(\epsilon)$ is a finite time and $t_0 + T_1$ is the moment when the tracking error $e(t)$ leaves the compact set $\Omega_e$ for the first time. Moreover, we have $\lim_{t \to T_1} T_1(\epsilon) = 0$ and $c_{ei} = 1/2 e(t_0 + T_1(\epsilon))^T P_n e(t_0 + T_1(\epsilon)) < c_e$.

We can now define the estimate of the system state vector $\hat{x}$ as $\hat{x} = x_d + \hat{e}$. In the following, we proceed to present the design of the proposed variable neural adaptive robust output feedback controller.

V. OUTPUT FEEDBACK CONTROLLER

**A. Controller Design**

Consider the adaptive robust output feedback controller

\[
\hat{u} = \hat{u}_{a,v} + \hat{u}_{s,v}
\]

\[
= G_0^{-1} (-f_\hat{v}(\hat{x}) + y_d^{(n)} - K\hat{e}) + \hat{u}_{s,v},
\]

(9)

where $G_0$ is a chosen positive definite matrix, $\hat{f}_\hat{v}(\hat{x}) = W_\hat{v}^T \xi_\hat{v}(\hat{x})$, and $\hat{u}_{s,v}$ is the robustifying component to be designed later. It is obvious that the controller architecture varies as the structure of the RBF network changes. Hence, the controller has $N$ different architectures, which are referred to as modes in the following. We constrain the weight vectors $\omega_i \in \mathbb{R}^{M_i}$, where $W_v = [\omega_{1,v} \cdots \omega_{p,v}]$, to reside in the compact sets $\Omega_i = \{\omega_i \in \mathbb{R}^{M_i} : \omega_i \leq \omega_{j,v}, \omega_{i,v} \leq \omega_{i,j}, 1 \leq j \leq M_i\}$, where $\omega_{i,j}$ and $\omega_{i,j}$, $i = 1, \ldots, p$, are design parameters. Let $W_v^* = [\omega_{1,v}^* \cdots \omega_{p,v}^*]$ denote the “optimal” constant weight matrix corresponding to each admissible network structure, that is,

\[
W_v^* = \arg \max_{\omega_i \in \Omega_i} \max_{x \in \Omega_x} \left\| f(x) - W_v^T \xi_v(x) \right\|.
\]

Note that this $W_v^*$ is only used for analytical purpose. Let $\Phi_v = W_v - W_v^* = [\phi_{1,v} \cdots \phi_{p,v}]$ with $\phi_{i,v} = \omega_{i,v} - \omega_{i,v}^*$, and let $c = \sum_{i=1}^p c_i$ with

\[
c_i = \max_v \left( \max_{\omega_i \in \Omega_i} \frac{1}{2\kappa} \phi_{i,v}^T \phi_{i,v} \right),
\]

(10)

where $\kappa > 0$ is a design parameter, and $\max_v(\cdot)$ denotes the maximization taken over all the admissible structures of the RBF networks. It is obvious that $c_i$ decreases as $\kappa$ increases. Let $\hat{\sigma} = B^T P_n \hat{e}$. The following projection based weight matrix adaptation law is employed,

\[
\dot{W}_v = \text{Proj} \left( W_v, \kappa \xi_v(\hat{x}) \hat{\sigma}^T \right),
\]

(11)
where \( \text{Proj}(W_v, \Theta_v) \) denotes \( \text{Proj}(\omega_{ij,v}, \theta_{ij,v}) \) for \( i = 1, \ldots, p \) and \( j = 1, \ldots, M_k \) and \( \text{Proj}(\omega_{ij,v}, \theta_{ij,v}) \) is the discontinuous projection operator used in [14]. It is easy to verify that

\[
\frac{1}{\kappa} \text{trace} \left( \Phi_v^T \left( W_v - \kappa \xi_v(\hat{x})(\sigma^T) \right) \right) \leq 0. \quad (12)
\]

Choose two positive constants \( d_f \) and \( d_g \) such that

\[
d_f \geq d_f^* = \max_{x \in \Omega_v} \left\| f(x) - W_v \xi_v(x) \right\|
\]

and

\[
d_g \geq d_g^* = \max_{x \in \Omega_v} \left\| G(x) - G_0 \right\|
\]

Then we define the robustifying component \( \hat{u}_{s,v} \) in (9) as

\[
\hat{u}_{s,v} = \begin{cases} 
-\hat{k}_{s,v} \frac{\sigma}{\| \sigma \|} & \text{if } \| \sigma \| \geq \nu \\
-\frac{\hat{k}_{s,v}}{\| u \|} & \text{if } \| \sigma \| < \nu, \end{cases}
\]

where \( \hat{k}_{s,v} = d_f + d_g^* \| u_{a,v} \| + d_o \) and \( \nu > 0 \) is a design parameter.

Recall that there exist peaking phenomena [19] associated with the high-gain observer described by (6). Thus, we cannot directly apply the output feedback controller defined in (9). In order to eliminate the peaking phenomena, we introduce saturation to the control input (9). Define \( \Omega_v = \left\{ e : \frac{1}{2} e^T P_m e \leq \epsilon \right\} \), where \( \epsilon_c > \epsilon_e \). Let \( u \) denote the state feedback controller obtained by replacing \( \hat{x} \) with \( x \) in (9), and let \( U_i \geq \max_{x \in \Omega_v} \left\{ \max \left| u_i(e, x_d, y_d, \omega_{i,v}) \right| \right\} \), where \( u_i \) is the \( i \)-th element of \( u \) and the inner maximization is taken over \( e \in \Omega_v, x_d \in \Omega_{x_d}, y_d^{(n)} \in \Omega_{y_d} \), and \( \omega_{i,v} \in \Omega_{i,v} \). Then the proposed adaptive robust output feedback controller takes the form

\[
u^* = U_1 \text{sat} \left( \frac{\hat{u}_p}{U_1} \right) \cdots U_p \text{sat} \left( \frac{\hat{u}_p}{U_p} \right) \quad (13)
\]

where sat is the saturation function.

Remark 1: The existence of a unique solution to (2) with the proposed variable neural adaptive robust output feedback controller can be established using the same arguments as used in [15].

B. Stability Analysis

Note that the control input \( u^* \) is globally bounded by construction. It follows from Proposition 1 that if \( e \in (0, \epsilon_1) \), then \( e(t) \in \Omega_v \) and \( \| \xi(t) \| \leq \beta e \) for \( t \in [t_0 + T_1(e), t_0 + T_3] \). Thus, we have \( \| e(t) - \hat{e}(t) \| \leq \| D \| \| \xi(t) \| \leq \beta e \) for \( t \in [t_0 + T_1(e), t_0 + T_3] \). There exists a constant \( \epsilon_2 \) such that \( \| e(t) - \hat{e}(t) \| \leq \beta \epsilon_2 \), which implies \( \hat{e}(t) \in \Omega_e \). Let \( \epsilon_2^* = \min \{ \epsilon_1, \epsilon_2 \} \). If \( e \in (0, \epsilon_2^*) \), the saturation of the adaptive robust output feedback (13) is not effective, that is, \( u^* = \hat{u} = u_{a,v} + \hat{u}_{s,v} \) for \( t \in [t_0 + T_1(e), t_0 + T_3] \). Substituting \( u^* = \hat{u}_{a,v} + \hat{u}_{s,v} \) into the state tracking error dynamics (3) gives

\[
\hat{e} = A e + B \left( f(x) + G(x)(\hat{u}_{a,v} + \hat{u}_{s,v}) - y_d^{(n)} + d \right) = A_m e + BK(e - \hat{e}) + BG(x) \hat{u}_{a,v} + Bd + B \left( f(x) - \hat{f}(x) \right) + B \left( G(x) - G_0 \right) \hat{u}_{a,v}, \quad (14)
\]

Now we consider the piecewise continuous quadratic Lyapunov function candidate

\[
V_v = \frac{1}{2} e^T P_m e + \frac{1}{2 \kappa} \text{trace} \left( \Phi_v^T \Phi_v \right) \quad (15)
\]

for \( t \in [t_0 + T_1(e), t_0 + T_3] \) and \( e \in (0, \epsilon_2^*) \). Whenever the proposed adaptive robust controller (9) is in the \( v \)-th mode. It follows from (10) that

\[
\frac{1}{2 \kappa} \text{trace} \left( \Phi_v^T \Phi_v \right) = \sum_{i=1}^{p} \frac{1}{2 \kappa} \alpha_i^T \alpha_i \leq \sum_{i=1}^{p} \beta_i = c. \quad (16)
\]

This Lyapunov function has jump discontinuities when the proposed adaptive robust output feedback controller switches among different modes. The time derivative of \( V_v \) evaluated along the solutions to (14) is

\[
\dot{V}_v = e^T P_m e + \frac{1}{\kappa} \text{trace} \left( \Phi_v^T \Phi_v \right) = -e^T Q_m e + \sigma^T K(e - \hat{e}) + \sigma^T G(x) \hat{u}_{a,v} + \frac{1}{\kappa} \text{trace} \left( \Phi_v^T \left( W_v \xi_v(\hat{x})(\sigma^T) \right) \right) + \sigma d
\]

\[
+ \sigma^T \left( f(x) - W_v \xi_v(\hat{x}) \right) + (\sigma - \hat{\sigma})^T d
\]

\[
+ \sigma^T \left( G(x) - G_0 \right) \hat{u}_{a,v} + (\sigma - \hat{\sigma})^T G(x) \hat{u}_{a,v} + (\sigma - \hat{\sigma})^T \hat{f}(x) - \hat{f}_v(\hat{x})
\]

\[
+ (\sigma - \hat{\sigma})^T \left( G(x) - G_0 \right) \hat{u}_{a,v}. \quad (17)
\]

For \( t \in [t_0 + T_1(e), t_0 + T_3] \), if \( e \in (0, \epsilon_2^*) \), then \( \| \xi(t) \| \leq \beta e \), \( \| e(t) - \hat{e}(t) \| \leq \beta e \), \( e(t) \in \Omega_e \), \( \hat{e}(t) \in \Omega_x \), \( x_d(t) \in \Omega_{x_d} \) and \( y_d^{(n)}(t) \in \Omega_{y_d} \). Hence, \( \sigma(t), \hat{\sigma}(t), x(t), \hat{x}(t), \hat{u}_{a,v}(t) \) and \( \hat{u}_{s,v}(t) \) are all bounded for \( t \in [t_0 + T_1(e), t_0 + T_3] \). Thus, we have

\[
\| \sigma K(e - \hat{e}) \| \leq r_1 e \quad (18)
\]

and

\[
\| (\sigma - \hat{\sigma})^T \left( f(x) - \hat{f}_v(\hat{x}) \right) + (\sigma - \hat{\sigma})^T \left( G(x) - G_0 \right) \hat{u}_{a,v} + (\sigma - \hat{\sigma})^T (d + G(x) \hat{u}_{a,v}) \| \leq r_2 e \quad (19)
\]

for some \( r_1, r_2 > 0 \). On the other hand, it follows from the Lipschitz continuity of the raised-cosine RBF that

\[
\| W_v^T \xi_v(x) - W_v^T \xi_v(\hat{x}) \| \leq L_v \| x - \hat{x} \|
\]

for some \( L_v > 0 \). Let \( L = \max_v L_v \). Taking into account that \( \| x - \hat{x} \| = \| e - \hat{e} \| \) gives

\[
\| \hat{\sigma}^T \left( f(x) - W_v^T \xi_v(\hat{x}) \right) \| \leq \| f(x) - W_v^T \xi_v(\hat{x}) \| \| \hat{\sigma} \|
\]

\[
+ \| W_v^T \xi_v(x) - W_v^T \xi_v(\hat{x}) \| \| \hat{\sigma} \|
\]

\[
\leq d_f^* \| \hat{\sigma} \| + L_v \| e - \hat{e} \| \| \hat{\sigma} \|
\]

\[
\leq d_f^* \| \hat{\sigma} \| + r_3 e \quad (20)
\]

for some \( r_3 > 0 \). It follows from (12) and (17)–(20) that

\[
\dot{V}_v \leq -e^T Q_m e + \hat{k}_{s,v} \| \hat{\sigma} \| + \sigma^T \left( G(x) \hat{u}_{a,v} + e \right), \quad (21)
\]
where $r = r_1 + r_2 + r_3$ and \( \hat{k}_{s,v} = d_{j}^* + d_{v}^* \|u_{s,v}\| + d_{o} \leq \hat{k}_{s,v} \).

If $\|\sigma\| > \nu$, then
\[
\hat{k}_{s,v}^* \|\sigma\| + \sigma^T G(x) \hat{u}_{s,v} \leq \hat{k}_{s,v}^* \|\sigma\| \leq 0. \tag{22}
\]

On the other hand, if $\|\sigma\| \leq \nu$, then
\[
\hat{k}_{s,v}^* \|\sigma\| + \sigma^T G(x) \hat{u}_{s,v} \leq \hat{k}_{s,v}^* \|\sigma\| \leq 0. \tag{23}
\]

Combining (22) and (23), we obtain
\[
\hat{k}_{s,v}^* \|\sigma\| + \sigma^T G(x) \hat{u}_{s,v} \leq \hat{k}_{s,v}^* \|\sigma\| \leq 0. \tag{24}
\]

where $\hat{k}_{s,v}^* = d_{j}^* + d_{v}^* \max_v (\max \|u_{s,v}\|) + d_{o}$ and the inner maximization is taken over $x_d \in \Omega_{x_d}, y_d \in \Omega_{y_d}$, and $e \in \Omega_e$. It follows from (16), (21) and (24) that
\[
\hat{V}_v \leq -e^T Q_m e + \frac{\hat{k}_{s,v}^*}{4} \nu + r\epsilon \leq -2\mu_m V_v + 2\mu_m \epsilon + \frac{\hat{k}_{s,v}^*}{4} \nu + r\epsilon = -\mu_m V_v - \mu_m (V_v - 2\epsilon - r\epsilon), \tag{25}
\]

where $\hat{c} = c + \hat{k}_{s,v}^*/(8\mu_m)$. Let $t_{0,v}$ and $t_{f,v}$ denote the initial and the final time instant, respectively, of a continuous time period when the controller is in the $v$-th mode. It follows from (21) that if $V_v(t) \geq 2\epsilon + r\epsilon$ for $t \in [t_{0,v}, t_{f,v}] \cap [t_0 + T_1(e), t_0 + T_3]$, then $V_v(t) \leq -\mu_m V_v(t)$, which implies that
\[
V_v(t) \leq \exp(-\mu_m(t - t_{0,v})) V_v(t_{0,v}) \tag{26}
\]

for $t \in [t_{0,v}, t_{f,v}] \cap [t_0 + T_1(e), t_0 + T_3]$.

**Theorem 1:** Let $t_1, t_2$ and $t_3$ be three consecutive switching time instants in the interval $[t_0 + T_1(e), t_0 + T_3]$ so that $v = v_1$ for $t \in [t_1, t_2]$ and $v = v_2$ for $t \in [t_2, t_3]$. Suppose that $V_v(t)$ satisfies (26) and $V_v(t) \geq 2\epsilon + r\epsilon$ for $t \in [t_1, t_3]$. If the dwell time $T_d$ of the variable-structure RBF network is selected such that
\[
T_d \geq \frac{1}{\mu_m} \ln \left( \frac{3}{2} \right), \tag{27}
\]

then $V_{v_2}(t_2) < V_{v_1}(t_1)$ and $V_{v_2}(t_3) < V_{v_1}(t_2)$.

**Proof:** This can be proved by replacing $2\epsilon$ with $2\epsilon + r\epsilon$ in the proof of Theorem 1 in [15].

**Theorem 2:** For the system (2) driven by the proposed adaptive robust output controller (13) with the adaptation laws (11), suppose that $T_d$ satisfies the condition (27). If $c_e \geq c_{e_1} + c$ and $c_e \geq 2\epsilon + c$, there exists a constant $\epsilon^* \in (0, 1)$ so that if $\epsilon \in (0, \epsilon^*)$, then $e(t) \in \Omega_e$ and $x(t) \in \Omega_x$ for $t \geq t_0$. Moreover, there exists a finite time $T \geq t_0 + T_1(e)$ such that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon + r\epsilon + c \tag{28}
\]

with some $r > 0$ for $t \geq T$. In addition, suppose that there exists a finite time $T_s \geq t_0 + T_1(e)$ such that $v = v_s$ for $t \geq T_s$. Then a finite time $T \geq T_s$ can be found such that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon + r\epsilon \tag{29}
\]

for $t \geq T$.

**Proof:** Recall that if $\epsilon \in (0, \epsilon^*)$, then $\hat{V}_v(t)$ satisfies (25) for $t \in [t_0 + T_1(e), t_0 + T_3)$, where we have $V(t_0 + T_1(e)) \leq c_{e_1} + c$. If, at the same time, $T_d$ satisfies the condition (27), it follows from Theorem 1 that
\[
V_v(t) \leq \max \{V(t_0 + T_1(e)), 2\epsilon + r\epsilon + c\} \leq \max \{c_{e_1} + c, 2\epsilon + r\epsilon + c\}, \tag{30}
\]

for $t \in [t_0 + T_1(e), t_0 + T_3)$. If $c_e > 2\epsilon + c$, there exists a constant $\epsilon_3$ so that if $\epsilon \in (0, \epsilon_3)$, then $c_e \geq 2\epsilon + r\epsilon + c$. Let $\epsilon^* = \min \{\epsilon^*_1, \epsilon^*_3\}$. If $\epsilon \in (0, \epsilon^*)$ and $c_e \geq c_{e_1} + c$, then $V_v(t) \leq c_{e_1} + c$ for $t \in [t_0 + T_1(e), t_0 + T_3)$ and hence $\frac{1}{2} e(t)^T P_m e(t) \leq c_{e_1} + c$ for $t \geq t_0 + T_1(e)$, which implies that $T_d \to \infty$. Because $e(t) \in \Omega_e$, for $t \in [t_0, t_0 + T_1(e)]$, we have $e(t) \in \Omega_e$ and $x(t) \in \Omega_x$ for $t \geq t_0$.

It follows from Theorem 1 and the above analysis that $V_v(t)$ will visit the interval $[0, 2\epsilon + r\epsilon]$ infinitely often for $t \geq t_0 + T_1(e)$. Let $T \geq t_0 + T_1(e)$ be the first time such that $V_v(T) \leq 2\epsilon + r\epsilon$. The trajectory of $V_v(t)$ starting at $t = T$ will stay inside the interval $[0, 2\epsilon + r\epsilon]$ until it jumps outside when the controller switches the mode. However, the jump of $V_v(t)$ between different modes satisfies that $|\Delta| \leq c$. Hence, we have $V_v(t) \leq 2\epsilon + r\epsilon + c$, which implies that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon + r\epsilon + c \tag{28}
\]

for $t \geq T$. If, in addition, there exists a finite time $T_s \geq t_0 + T_1(e)$ such that $v = v_s$ for $t \geq T_s$, then it follows from (26) that $V_v(t) \leq -\mu_m V_v(t) - \mu_m (V_v(t) - 2\epsilon - r\epsilon)$ for $t \geq T_s$. Thus, there exists a finite time $T \geq T_s$ such that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon + r\epsilon \tag{29}
\]

for $t \geq T$. The proof of the theorem is complete.

**Remark 2:** It can be seen from (28) and (29) that the tracking performance is directly proportional to the magnitude of $c, \nu$ and $\epsilon$. Recall that the magnitude of $c$ is inversely proportional to $\kappa$. Therefore, larger $\kappa$ and smaller $\nu$ and $\epsilon$ imply better tracking performance.

**VI. Simulation**

We illustrate the effectiveness of the proposed variable neural adaptive robust output feedback controller with a planar articulated two-link manipulator used in [15]. The detailed modeling of this two-link manipulator can be found in [20, p. 394]. Let $q_1$ and $q_2$ denote the angular positions of joint 1 and 2, respectively, and $\tau_1$ and $\tau_2$ denote the applied torques. We will assume that there exist input disturbances $\eta_1$ and $\eta_2$ associated with the applied torques $\tau_1$ and $\tau_2$, respectively. Thus, we can represent the dynamics of this two-link manipulator as
\[
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
-h\dot{q}_1 \\
-h\dot{q}_2
\end{bmatrix}
= \begin{bmatrix}
\tau_1 + \eta_1 \\
\tau_2 + \eta_2
\end{bmatrix},
\]

where $H_{11} = a_1 + 2a_3 \cos(q_2) + 2a_4 \sin(q_2), H_{12} = H_{21} = a_2 + a_3 \cos(q_2) + a_4 \sin(q_2), H_{22} = a_2$ and $h = a_3 \sin(q_2) - \cdots$.
also use the band-limited white noise signals for the input $I$, in [20, p. 396], that is, $I = I_e + m_e I_{ce} + m_e I_{te}$. In our simulation, we use the same numerical values as

\[ q_4 \cos(q_2) \text{ with } a_1 = I_1 + m_1 I_{c1} + I_e + m_e I_{ce} + m_e I_{te}^2, \]

\[ a_2 = I_e + m_e I_{ce}, \quad a_3 = m_e l_1 I_{ce} \sin(\delta_e) \] and

\[ a_4 = m_e l_1 I_{ce} \sin(\delta_e). \]

In our simulation, we use the same numerical values as in [20, p. 396], that is, $m_1 = 1.0 \text{ (kg)}$, $m_e = 2.0 \text{ (kg)}$, $I_1 = 0.12 \text{ (kg-m^2)}$, $I_e = 0.25 \text{ (kg-m^2)}$, $l_{c1} = 0.5 \text{ (m)}$, $l_{ce} = 0.6 \text{ (m)}$, $l_1 = 1 \text{ (m)}$ and $\delta_e = \pi/6 \text{ (rad)}$. We also use the band-limited white noise signals for the input disturbances $q_1$ and $q_2$ as in [15]. The manipulator is initially at rest, that is, $q_1 = q_2 = 0$ and $\dot{q}_1 = \dot{q}_2 = 0$.

We consider the same reference signals as in [11], which are defined as $q_{a1}(t) = \frac{\pi}{4} \cos(2\pi t)$ and $q_{a2}(t) = \frac{\pi}{4} \cos(2\pi t)$. Thus, we choose the grid boundaries for $q_1$, $q_1$, $q_2$, and $q_2$ to be $[-1.0, 1.0]$, $[-4.0, 4.0]$, $[-1.0, 1.0]$ and $[-5.5, 5.5]$, respectively. For the controller, we choose $k_1 = [1 \ 2]$, $k_2 = [4 \ 4]$, $Q_1 = Q_2 = 0.5 I_2$, $d_f = d_g = d_o = 5$, $\nu = 0.255$, $\phi = 0.1$ and $G_0 = 2 I_2$. The rest of the design parameters of the variable-structure RCRBF network are selected as $e_{\text{max}} = 0.005$, $\rho = 0.8$, $\vartheta = 0.3$, $d_{\text{threshold}} = [0.1 \ 0.4 \ 0.1 \ 0.55]$, $\alpha_1 = \alpha_2 = -25$, $\alpha_3 = \alpha_2 = 25$, $k = 500$ and $T_d = 1.5$. For the high-gain observer, we select $\alpha_{11} = \alpha_{21} = 2$, $\alpha_{12} = \alpha_{22} = 1$ and $\epsilon = 10^{-4}$. In addition, we select $U_1 = 250$ and $U_2 = 150$. It can be seen from Fig. 1 that the trajectories of the output tracking errors enter a small neighborhood around the origin and then stay therein. The variation of the number of hidden neurons is shown in Fig. 2.

When the tracking performance is poor, the network adds more RBFs in order to improve the approximation accuracy. When the tracking performance is acceptable, the network removes RBFs in order to avoid network redundancy.

VII. CONCLUSIONS

A variable neural adaptive robust output feedback controller has been proposed for the output tracking control of a class of MIMO uncertain systems. The variable-structure RBF network, which is used to approximate unknown system dynamics, can grow or shrink on-line dynamically according to the tracking performance. The piecewise continuous Lyapunov function is utilized in the stability analysis to analyze the effects of the structure variation of the RBF network. Simulation results confirm the effectiveness of the proposed variable neural adaptive robust output feedback controller.

REFERENCES


