This note summarizes and further clarifies the three major uses of a rotation matrix discussed in Lecture note 3:

1. to represent an orientation;
2. to change the reference frame in which a vector or a frame is represented;
3. to rotate a vector or a frame;

Consider three frames \{a\}, \{b\}, and \{c\} with unit axes \{\hat{x}_a, \hat{y}_a, \hat{z}_a\}, \{\hat{x}_b, \hat{y}_b, \hat{z}_b\}, and \{\hat{x}_c, \hat{y}_c, \hat{z}_c\}, respectively.

1 Representation of Orientation:

Let \( R \) be a rotation matrix:

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]

where \( r_1, r_2, r_3 \) denote the three column vectors and \( r_{ij} \) denotes its \((i,j)\) entry. We call \( R \) a representation of orientation of \{b\} relative to \{a\} if the columns of \( R \) are the coordinates of unit axes \{\hat{x}_b, \hat{y}_b, \hat{z}_b\} in \{a\}, i.e.,

\[
\begin{align*}
\hat{x}_b &= r_{11} \hat{x}_a + r_{21} \hat{y}_a + r_{31} \hat{z}_a = \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} r_1 \\
\hat{y}_b &= r_{12} \hat{x}_a + r_{22} \hat{y}_a + r_{32} \hat{z}_a = \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} r_2 \\
\hat{z}_b &= r_{13} \hat{x}_a + r_{23} \hat{y}_a + r_{33} \hat{z}_a = \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} r_3
\end{align*}
\]

(1)

If \( R \) is defined as above, then each of its columns describes how the corresponding \{b\}-frame unit axis is oriented relative to frame \{a\}. Therefore, \( R \) can be used to represent the orientation of one frame relative to another. Note that in matrix form, equation (1) can be equivalently written as

\[
\begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} = \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} R
\]

(2)
2 Change Reference Frame

Let $R_{ab}$ be a rotation matrix that represents the orientation of {b} relative to {a}. Let $p_a$ and $p_b$ be the coordinates of the same physical point p in frames {a} and {b}, respectively. Since they represent the same physical point, we must have that

$$\begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} p_a = \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} p_b = \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} R_{ab} p_b$$

where the second equality is due to equation (2). Thus, we can conclude that

$$p_a = R_{ab} p_b$$

Therefore, $R_{ab}$ not only represents the orientation of {b} relative to {a}, but also can be used to change the coordinates of the same point p from reference frame {b} to frame {a}.

Following a similar argument, we can see that $R_{ab}$ can be used to change the reference frame in which another frame is represented. To see this, let $R_{bc}$ represent the orientation of {c} relative to {b}. Using equation (2) twice, we can get

$$\begin{bmatrix} \hat{x}_c & \hat{y}_c & \hat{z}_c \end{bmatrix} = \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} R_{bc} = \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} R_{ab} R_{bc}$$

(3)

According to the definition of a rotation matrix, equation (3) implies that $R_{ab} R_{bc}$ represents the orientation of frame {c} relative to frame {a}. By our subscript convention, we can write this as

$$R_{ac} = R_{ab} R_{bc}$$

In the above equation, $R_{bc}$ is still interpreted as the orientation of frame {c} relative to {b}. The matrix $R_{ab}$ here is viewed as changing reference frame for the representation of the orientation of {c}. In other words, multiplying $R_{ab}$ changes the orientation of {c} from “relative to {b}” to “relative to {a}”.

3 Rotation Operator

A rotation matrix can also be viewed as an operator that rotates a frame or a point. Let $R$ be the matrix satisfies equation (1). Then $R$ represents the orientation of {b} relative to {a}. Further assume that the frame {b} is obtained by rotating {a} about an axis $\hat{\omega}$ by an angle $\theta$.

**Rotating a frame:** $R$ can be equivalently viewed as an operator that rotates {a} to obtain {b}. It can be seen from (2) that such an operation is accomplished by postmultiplying the axes {$\hat{x}_a, \hat{y}_a, \hat{z}_a$} by $R$. Here, the rotation axis $\hat{\omega}$ is expressed in {a} (i.e. the coordinates of the $\hat{\omega}$ vector is specified with respect to {a}).

Given the above set up, what does $RR$ mean? Using (2), we have

$$\begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} RR = \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} R$$

Therefore, $RR$ can be viewed as rotating {a} about $\hat{\omega}$ by $\theta$ and then rotating {b} about $\hat{\omega}$ by $\theta$. It is important to know that the rotation axes for the two rotations have the same coordinates but correspond to different physical vectors in general. For the first operation, $\hat{\omega}$ is expressed in {a}, while for the second rotation operation, $\hat{\omega}$ is expressed in {b}.
In general, for an arbitrary frame \( \{c\} \) with orientation represented by \( R_c \) relative to some fixed frame, the multiplication \( R_c R \) means rotating \( \{c\} \) about the axis \( \hat{\omega} \) by an angle \( \theta \), where \( \hat{\omega} \) is expressed in frame \( \{c\} \). In other words, \( R_c R \) is the orientation of the new frame obtained by the rotation operation.

**Rotating a vector:** Now consider the matrix vector multiplication \( Rp \), where \( R \) is defined in the same way as above. Previously, we have viewed \( R \) as a change of reference frames for the representations of the same physical vector in different frames. Now we want to view this as rotating \( p \) about \( \hat{\omega} \) by an angle \( \theta \). The rotated vector is given by \( p' = Rp \). Note that only one frame is involved in this interpretation, namely, \( p \) and \( p' \) are the coordinates of two different points (before and after rotation) in the same frame. The rotation axis \( \hat{\omega} \) must also be expressed in the same frame.

In order for the above interpretation to work, the vectors \( p \), \( p' \) and \( \hat{\omega} \) have to be expressed in the same frame. If there are more than 1 frame involved, we need to do a change of coordinate system. To see this, let \( p_c \) and \( p_d \) be the coordinates of a physical point \( p \) in frames \( \{c\} \) and \( \{d\} \), respectively.

Define

\[
p'_c = Rp_c, \quad p''_d = Rp_d
\]

According to our previous interpretation, \( Rp_c \) means rotating \( p \) about \( \hat{\omega} \) (expressed in \( \{c\} \)) by \( \theta \). Let the rotated vector be \( p' \) with coordinates \( p'_c \) and \( p'_d \) in frames \( \{c\} \) and \( \{d\} \), respectively. Similarly, \( Rp_d \) means rotating \( p \) about \( \hat{\omega} \) (expressed in \( \{d\} \)) by \( \theta \). Let the resulting rotated vector be \( p'' \) with coordinates \( p''_c \) and \( p''_d \) in frames \( \{c\} \) and \( \{d\} \), respectively. Note that the same \( R \) matrix corresponds to different rotation operations (i.e. \( p' \neq p'' \)) depending on the frames we are working with.

Now let’s focus on the first rotation operation that rotates \( p \) to \( p' \). Let \( R_{cd} \) be the orientation of \( \{d\} \) relative to \( \{c\} \). Then we know \( p'_d = R_{cd}^{-1} p'_c \). Therefore, we have the following

\[
p'_d = R_{cd}^{-1} p'_c = R_{cd}^{-1} Rp_c = (R_{cd}^{-1} RR_{cd})p_d
\]  \( \text{(4)} \)

Recall that \( p' \) is obtained by rotating \( p \) about \( \hat{\omega} \) (expressed in \( \{c\} \)) by \( \theta \). This operation can be represented by \( R \) if we are working with frame \( \{c\} \) (i.e. the same frame in which \( \hat{\omega} \) is expressed). Equation (4) indicates that this operation can be equivalently represented by \( (R_{cd}^{-1} RR_{cd}) \) if we work with frame \( \{d\} \).