ECE5463: Introduction to Robotics

Lecture Note 4: General Rigid Body Motion

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Spring 2018
Outline

• Representation of General Rigid Body Motion

• Homogeneous Transformation Matrix

• Twist and \( se(3) \)

• Twist Representation of Rigid Motion

• Screw Motion and Exponential Coordinate
General Rigid Body Configuration

- General rigid body configuration includes both the orientation $R \in SO(3)$ and the position $p \in \mathbb{R}^3$ of the rigid body.
- Rigid body configuration can be represented by the pair $(R, p)$
- **Definition (Special Euclidean Group):**

$$SE(3) = \{(R, p) : R \in SO(3), p \in \mathbb{R}^3\} = SO(3) \times \mathbb{R}^3$$
Special Euclidean Group

• Let \((R, p) \in SE(3)\), where \(p\) is the coordinate of the origin of \(\{b\}\) in frame \(\{s\}\) and \(R\) is the orientation of \(\{b\}\) relative to \(\{s\}\). Let \(q_s, q_b\) be the coordinates of a point \(q\) relative to frames \(\{s\}\) and \(\{b\}\), respectively. Then

\[ q_s = R q_b + p \]
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Homogeneous Representation

• For any point $x \in \mathbb{R}^3$, its homogeneous coordinate is $\tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$

• Similar, homogeneous coordinate for the origin is $\tilde{o} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

• Homogeneous coordinate for a vector $v$ is:

• Some rules of syntax for homogeneous coordinates:
Homogeneous Transformation Matrix

- Associate each \((R, p) \in SE(3)\) with a \(4 \times 4\) matrix:
  \[
  T = \begin{bmatrix}
  R & p \\
  0 & 1
  \end{bmatrix}
  \text{ with } T^{-1} = \begin{bmatrix}
  R^T & -R^T p \\
  0 & 1
  \end{bmatrix}
  \]

- \(T\) defined above is called a homogeneous transformation matrix. Any rigid body configuration \((R, p) \in SE(3)\) corresponds to a homogeneous transformation matrix \(T\).

- Equivalently, \(SE(3)\) can be defined as the set of all homogeneous transformation matrices.

- Slight abuse of notation: \(T = (R, p) \in SE(3)\) and \(Tx = Rx + p\) for \(x \in \mathbb{R}^3\)
Uses of Transformation Matrices

• Representation of rigid body configuration (orientation and position)

• Change of reference frame in which a vector or a frame is represented
Uses of Transformation Matrices

- Rigid body motion operator that displaces a vector
Uses of Transformation Matrices

- Rigid body motion operator that displaces a frame
Example of Homogeneous Transformation Matrix

In terms of the coordinates of a fixed space frame \{s\}, frame \{a\} has its \(\hat{x}_a\)-axis pointing in the direction \((0, 0, 1)\) and its \(\hat{y}_a\)-axis pointing \((-1, 0, 0)\), and frame \{b\} has its \(\hat{x}_b\)-axis pointing \((1, 0, 0)\) and its \(\hat{y}_b\)-axis pointing \((0, 0, -1)\). The origin of \{a\} is at \((3, 0, 0)\) in \{s\} and the origin of \{b\} is at \((0, 2, 0)\) in \{s\}. 
Example of Homogeneous Transformation Matrix

Fixed frame \{a\}; end effector frame \{b\}, the camera frame \{c\}, and the workpiece frame \{d\}. Suppose \|p_c - p_b\| = 4
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Towards Exponential Coordinate

- Recall: rotation matrix $R \in SO(3)$ can be represented in exponential coordinate $\hat{\omega} \theta$
  - $q(\theta) = \text{Rot}(\hat{\omega}, \theta)q_0$ viewed as a solution to $\dot{q}(t) = [\hat{\omega}]q(t)$ with $q(0) = q_0$ at $t = \theta$.

- The above relation requires that the rotation axis passes through the origin.

- We can find exponential coordinate for $T \in SE(3)$ using a similar approach (i.e. via differential equation)

- We first need to introduce some important concepts.
Differential Equation for Rigid Body Motion

• Rotation about axis that may not pass through the origin

\[ \mathbf{p}(t) = e^{\hat{\mathbf{\Omega}}t} \mathbf{p}(0), \]

where \( e^{\hat{\mathbf{\Omega}}t} \) is the matrix exponential of the \( 4 \times 4 \) matrix \( \hat{\mathbf{\Omega}}t \), defined (as usual) by

\[ e^{\hat{\mathbf{\Omega}}t} = I + \hat{\mathbf{\Omega}}t + (\hat{\mathbf{\Omega}}t)^2 \frac{1}{2!} + (\hat{\mathbf{\Omega}}t)^3 \frac{1}{3!} + \cdots \]

The scalar \( t \) is the total amount of rotation (since we are rotating with unit velocity). \( \exp(\hat{\mathbf{\Omega}}t) \) is a mapping from the initial location of a point to its location after rotating \( t \) radians.

In a similar manner, we can represent the transformation due to translational motion as the exponential of a \( 4 \times 4 \) matrix. The velocity of a point attached to a prismatic joint moving with unit velocity (see Figure 2.5b) is

\[ \dot{\mathbf{p}}(t) = \mathbf{v}. \quad (2.27) \]

Again, the solution of equation (2.27) can be written as \( \exp(\hat{\mathbf{\Omega}}t)\mathbf{p}(0) \), where \( t \) is the total amount of translation and \( \hat{\mathbf{\Omega}} = \begin{bmatrix} 0 & v & 0 \\ 0 & 0 & 0 \end{bmatrix} \). \quad (2.28)

The \( 4 \times 4 \) matrix \( \hat{\mathbf{\Omega}} \) given in equations (2.26) and (2.28) is the generalization of the skew-symmetric matrix \( \hat{\mathbf{\Omega}} \in \mathfrak{so}(3) \). Analogous to the definition of \( \mathfrak{so}(3) \), we define \( \mathfrak{se}(3) := \{ (\mathbf{v}, \hat{\mathbf{\Omega}}) : \mathbf{v} \in \mathbb{R}^3, \hat{\mathbf{\Omega}} \in \mathfrak{so}(3) \} \). \quad (2.29)

In homogeneous coordinates, we write an element \( \hat{\mathbf{\Omega}} \in \mathfrak{se}(3) \) as

\[ \hat{\mathbf{\Omega}} = \begin{bmatrix} \hat{\mathbf{\Omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}. \]
Differential Equation for Rigid Body Motion

• Consider the following differential equation in homogeneous coordinates

\[
\dot{p}(t) = \omega \times p(t) + v \quad \Rightarrow \quad \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix}
\]

(1)

• The variable \( v \) contains all the constant terms (e.g. \(-\omega \times q\) in the rotation example); thus, it may NOT be equal to the linear velocity of the origin of the rigid body.

• Solution to (1) is

\[
\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp \left( \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}
\]

• Motion of this form is characterized by \((\omega, v)\) which is called spatial velocity or Twist.
Twist

- Angular velocity and “linear” velocity can be combined to form the Spatial Velocity or Twist
  \[ \mathbf{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6 \]

- Each twist \( \mathbf{V} \) corresponds to a motion equation (1).

- For each twist \( \mathbf{V} = (\omega, v) \), let \( [\mathbf{V}] \) be its matrix representation
  \[ [\mathbf{V}] = \begin{bmatrix} \omega \\ v \\ 0 \\ 0 \end{bmatrix} \]

- With these notations, solution to (1) is given by
  \[ \begin{bmatrix} p(t) \\ 1 \end{bmatrix} = e^{[\mathbf{V}]t} \begin{bmatrix} p_0 \\ 1 \end{bmatrix} \]
$se(3)$

- Similar to $so(3)$, we can define $se(3)$:

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist. We follow the textbook notation to call the spatial velocity $\mathcal{V} = (\omega, v)$ a twist.

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$. 
Example of Twist

• \( \mathbf{\nu} = \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \) can have multiple different physical interpretations
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Exponential Map of $se(3)$: From Twist to Rigid Motion

Theorem 1.

For any $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$, we have $e^{[\mathcal{V}]\theta} \in SE(3)$

- **Case 1** ($\omega = 0$): $e^{[\mathcal{V}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$

- **Case 2** ($\omega \neq 0$): without loss of generality assume $\|\omega\| = 1$. Then

  \[
  e^{[\mathcal{V}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \quad \text{with } G(\theta) = I\theta + (1 - \cos(\theta))[\omega] + (\theta - \sin(\theta)) [\omega]^2 \tag{2}
  \]
Theorem 2.

Given any \( T = (R, p) \in SE(3) \), one can always find twist \( \mathcal{V} = (\omega, v) \) and a scalar \( \theta \) such that

\[
e^{[\mathcal{V}]\theta} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}
\]

Matrix Logarithm Algorithm:

- If \( R = I \), then set \( \omega = 0 \), \( v = p/\|p\| \), and \( \theta = \|p\| \).
- Otherwise, use matrix logarithm on \( SO(3) \) to determine \( \omega \) and \( \theta \) from \( R \). Then \( v \) is calculated as \( v = G^{-1}(\theta)p \), where

\[
G^{-1}(\theta) = \frac{1}{\theta} I - \frac{1}{2} [\omega] + \left( \frac{1}{\theta} - \frac{1}{2} \cos \frac{\theta}{2} \right) [\omega]^2
\]
Example of Exponential/Log
Quick Summary

• Angular and linear velocity can be combined to form a spatial velocity or twist $\mathcal{V} = (\omega, v)$

• Each twist $\mathcal{V} = (\omega, v)$ defines a motion such that any point $p$ on the rigid body follows a trajectory generated by the following ODE:

$$\dot{p}(t) = \omega \times p(t) + v$$

• Solution to this ODE (in homogeneous coordinate): $\tilde{p}(t) = e^{[\mathcal{V}]t} \tilde{p}(0)$.

• For any twist $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$, its matrix exponential $e^{[\mathcal{V}]\theta} \in SE(3)$, i.e., it corresponds to a rigid body transformation. We have an analytical formula to compute the exponential (Theorem 1)

• For any $T \in SE(3)$, we also have analytical formula (Theorem 2) to find $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$ such at $e^{[\mathcal{V}]\theta} = T$. 
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Screw Interpretation of Twist

• Given a twist $\mathcal{V} = (\omega, v)$, the associated motion (1) may have different interpretations (different rotation axes, linear velocities).

• We want to impose some nominal interpretable structure on the motion.

• Recall: an angular velocity vector $\omega$ can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis.

• Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a screw axis $S$ and a velocity $\dot{\theta}$ about the screw axis.
Screw Motion: Definition

- Rotating about an axis while also translating along the axis

\[
\begin{align*}
\hat{s} \dot{\theta} \times q + h \hat{s} \dot{\theta} & = p \times h = \\
\hat{s} \dot{\theta} & = \text{pitch} = \\
& \text{linear speed/angular speed}
\end{align*}
\]

- Represented by screw axis \( \{ q, \hat{s}, h \} \) and rotation speed \( \dot{\theta} \)
  - \( \hat{s} \): unit vector in the direction of the rotation axis
  - \( q \): any point on the rotation axis
  - \( h \): screw pitch which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis
Consider a point $p$ on a rigid body under a screw motion with (rotation) speed $\dot{\theta}$. Let $p(t)$ be its coordinate at time $t$. The overall velocity is

$$\dot{p}(t) = \hat{s}\dot{\theta} \times (p(t) - q) + h\hat{s}\dot{\theta}$$

(3)

Thus, any screw axis $\{q, \hat{s}, h\}$ with rotation speed $\dot{\theta}$ can be represented by a particular twist $(\omega, v)$ with $\omega = \hat{s}\dot{\theta}$ and $v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$. 
From Twist to Screw Axis

- The converse is true as well: given any twist \( \mathcal{V} = (\omega, v) \) one can always find \( \{q, \hat{s}, h\} \) and \( \dot{\theta} \) such that the corresponding screw motion (eq. (3)) coincides with the motion generated by the twist (eq. (1)).
  - If \( \omega = 0 \), then it is a pure translation \( (h = \infty) \)
  
  - If \( \omega \neq 0 \):
Examples Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_b$?

- What is the screw axis for twist $\mathbf{V} = (0, 2, 2, 4, 0, 0)$?
Implicit Definition of Screw Axis for a Given a Twist

- For any twist $\mathcal{V} = (\omega, v)$, we can always view it as a "screw velocity" that consists an screw axis $\mathbf{S}$ and the velocity $\dot{\theta}$ about the screw axis.

- Instead of using $\{q, \hat{s}, h\}$ to represent $\mathbf{S}$, we adopt a more convenient representation defined below:

- **Screw axis (corresponding to a twist):** Given any twist $\mathcal{V} = (\omega, v)$, its screw axis is defined as
  - If $\omega \neq 0$, then $\mathbf{S} := \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$.
  - If $\omega = 0$, then $\mathbf{S} := \mathcal{V}/\|v\| = (0, v/\|v\|)$.
Unit Screw Axis

• (unit) screw axis $S$ can be represented by

$$S = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$

• We have used $(\omega, v)$ to represent both screw axis (where $\|\omega\|$ or $\|v\| = 1$ must be unity) and a twist (where there are no constrains on $\omega$ and $v$)

• $S = (w, v)$ is called a screw axis, but we typically do not bother to explicitly find the corresponding $\{q, \hat{s}, h\}$. We can find them whenever needed.
Exponential Coordinates of Rigid Transformation

- Screw axis $S = (\omega, v)$ is just a normalized twist; its matrix representation is

$$[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

- Therefore, a point started at $p(0)$ at time zero, travel along screw axis $S$ at unit speed for time $t$ will end up at $p(t) = e^{[S]t}p(0)$

- Given $S$ we can use Theorem 1 to compute $e^{[S]t} \in SE(3)$;

- Given $T \in SE(3)$, we can use Theorem 2 to find $S = (\omega, v)$ and $\theta$ such that $e^{[S]\theta} = T$. We call $S\theta$ the **Exponential Coordinate** of the homogeneous transformation $T \in SE(3)$
Example of Exponential Coordinates

Exercise 3.29
Two frames \{a\} and \{b\} are attached to a moving rigid body. Show that the twist of \{a\} in space-frame coordinates is the same as the twist of \{b\} in space-frame coordinates.

Exercise 3.30
A cube undergoes two different screw motions from frame \{1\} to frame \{2\} as shown in Figure 3.32. In both cases, (a) and (b), the initial configuration of the cube is

\[ T_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

(a) For each case, (a) and (b), find the exponential coordinates \( S = (\omega, v) \) such that

\[ T_{02} = e^{[S]} T_{01}, \]

where no constraints are placed on \( \omega \) or \( v \).

(b) Repeat (a), this time with the constraint that \( \|\omega\| \in [-\pi, \pi] \).

Exercise 3.31
In Example 3.19 and Figure 3.16, the block that the robot must pick up weighs 1 kg, which means that the robot must provide approximately
More Discussions
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