Outline

• Mathematics of Rigid Body Transformation

• Rotation Matrix and $SO(3)$

• Euler Angles and Euler-Like Parameterizations

• Exponential Coordinate of $SO(3)$

• Quaternion Representation of Rotation
Rigid Body Transformation

• Object (Body) in $\mathbb{R}^3$: a collection of points, represented by a subset $O \subset \mathbb{R}^3$.

• Transformation of a body: A single mapping $g: O \rightarrow \mathbb{R}^3$ which maps the coordinates of points in the body from their initial to final configurations.

• Transformation on points induce an action on vectors in a natural way. Given a transformation $g: O \rightarrow \mathbb{R}^3$, define

$$\hat{g}(v) \overset{\triangle}{=} g(q) - g(p), \quad \text{where } v = q - p$$

- Note: $\hat{g}$ has a different domain than $g$. 
Rigid Body Transformation

• **Rigid Body Transformation:** A mapping $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is called a *rigid body transformation* if it satisfies the following two properties

1. Length preserved: $\|g(q) - g(p)\| = \|q - p\|$, for all $p, q \in \mathbb{R}^3$

2. Cross product is preserved: $\hat{g}(v \times w) = \hat{g}(v) \times \hat{g}(w)$, for all $v, w \in \mathbb{R}^3$.

• Implications:
  - Inner product is preserved:
    \[ \hat{g}(v)^T \hat{g}(w) = v^T w, \text{ for all } v, w \in \mathbb{R}^3 \]
  - Angles between vectors are preserved
  - Orthogonal vectors are transformed to orthogonal vectors
  - Right-handed coordinate frames are transformed to right-handed coordinate frames
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Representation of Orientation

• Basic Reference Frames:
  - **Fixed (or Space) Frame**: \{s\} = \{\hat{x}_s, \hat{y}_s, \hat{z}_s\}
  - **Body Frame**: \{b\} = \{\hat{x}_b, \hat{y}_b, \hat{z}_b\}

• Let \(x_{sb}, y_{sb}\) and \(z_{sb}\) be the coordinate of \(\hat{x}_b, \hat{y}_b, \hat{z}_b\) in frame \{s\}
Rotation Matrix

- Let \( R_{sb} = \begin{bmatrix} x_{sb} & y_{sb} & z_{sb} \end{bmatrix} \)

- \( R_{sb} \) constructed above is called a rotation matrix. We know:
  - \( R_{sb}^TR_{sb} = I \)
  - \( \det(R_{sb}) = 1 \)
Special Orthogonal Group

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

  $$SO(n) = \{ R \in \mathbb{R}^{n \times n} : RR^T = I, \det(R) = 1 \}$$

- $SO(n)$ is a group. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set $G$, together with an operation $\bullet$, satisfying the following group axioms:
  - **Closure**: $a \in G, b \in G \Rightarrow a \bullet b \in G$
  
  - **Associativity**: $(a \bullet b) \bullet c = a \bullet (b \bullet c), \forall a, b, c \in G$

  - **Identity element**: $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.

  - **Inverse element**: For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where $e$ is the identity element.
Use of Rotation Matrix

• Representing an orientation: \( R_{ab} = \begin{bmatrix} x_{ab} & y_{ab} & z_{ab} \end{bmatrix} \)

• Changing the reference frame:
  - \( p_a = R_{ab} p_b \)
  - \( R_{ac} = R_{ab} R_{bc} \)

• Rotating a vector or a frame:
  - **Theorem (Euler):** Any orientation \( R \in SO(3) \) is equivalent to a rotation about a fixed axis \( \hat{\omega} \in \mathbb{R}^3 \) through an angle \( \theta \in [0, 2\pi) \)
    \[
    R = \text{Rot}(\hat{\omega}, \theta)
    \]
Pre-multiplication vs. Post-multiplication

- Given $R \in SO(3)$, we can always find $\hat{\omega}$ and $\theta$ such that $R = \text{Rot}(\hat{\omega}, \theta)$.
- Premultiplying by $R$ yields a rotation about an axis $\hat{\omega}$ considered in the fixed frame;
- Postmultiplying by $R$ yields a rotation about $\hat{\omega}$ considered in the body frame.

Figure 3.9: (Top) The rotation operator $R = \text{Rot}(\hat{z}, 90^\circ)$ gives the orientation of the right-hand frame in the left-hand frame. (Bottom) On the left are shown a fixed frame $\{s\}$ and a body frame $\{b\}$, which can be expressed as $R_{sb}$. The quantity $R R_{sb}$ rotates $\{b\}$ by $90^\circ$ about the fixed-frame axis $\hat{z}_s$ to $\{b\}'$. The quantity $R_{sb} R$ rotates $\{b\}$ by $90^\circ$ about the body-frame axis $\hat{z}_b$ to $\{b\}''$. In other words, premultiplying by $R = \text{Rot}(\hat{\omega}, \theta)$ yields a rotation about an axis $\hat{\omega}$ considered to be in the fixed frame, and postmultiplying by $R$ yields a rotation about $\hat{\omega}$ considered as being in the body frame.
Coordinate System for $SO(3)$

- How to parameterize the elements in $SO(3)$?

- The definition of $SO(3)$ corresponds to implicit representation:

$$R \in \mathbb{R}^{3 \times 3}, \quad RR^T = I, \quad \text{det}(R) = 1$$

- 6 independent equations with 9 unknowns

- Dimension of $SO(3)$ is 3.
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Euler Angle Representation of Rotation

- Euler angle representation:
  - Start with \( \{b\} \) coincident with \( \{s\} \)
  
  - Rotate \( \{b\} \) about \( \hat{z}_b \) by an angle \( \alpha \), then rotate about the (new) \( \hat{y}_b \) axis by \( \beta \), and then rotate about the (new) \( \hat{z}_b \) axis by \( \gamma \). This yields a net orientation \( R_{sb}(\alpha, \beta, \gamma) \) parameterized by the ZYZ angles \( (\alpha, \beta, \gamma) \)

  \[
  R_{sb}(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma)
  \]
Other Euler-Like Parameterizations

- Other types of Euler angle parameterization can be devised using different ordered sets of rotation axes

- Common choices include:
  - ZYX Euler angles: also called *Fick angles* or yaw, pitch and roll angles
  - YZX Euler angles (Helmholtz angles)
Examples of Euler-Like Representations
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Towards Exponential Coordinate of $SO(3)$

• Recall the polar coordinate system of the complex plane:
  - Every complex number $z = x + jy = \rho e^{j\phi}$

  - Cartesian coordinate $(x, y) \leftrightarrow$ polar coordinate $(\rho, \phi)$

  - For some applications, the polar coordinate is preferred due to its geometric meaning.

• For any rotation matrix $R \in SO(3)$, it turned out $R = e^{[\hat{\omega}]\theta}$
  - $\hat{\omega}$: unit vector representing the axis of rotation

  - $\theta$: the degree of rotation

  - $\hat{\omega}\theta$ is called the exponential coordinate for $SO(3)$. 
Skew Symmetric Matrices

- Recall that cross product is a special linear transformation.

- For any \( \omega \in \mathbb{R}^n \), there is a matrix \([\omega] \in \mathbb{R}^{n \times n}\) such that \( \omega \times p = [\omega]p \)

\[
\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}
\]

- Note that \([\omega] = -[\omega]^T \leftrightarrow \text{skew symmetric}\)

- \([\omega]\) is called a skew-symmetric matrix representation of the vector \(\omega\)

- The set of skew-symmetric matrices in: \(so(n) \triangleq \{S \in \mathbb{R}^{n \times n} : S^T = -S\}\)

- We are interested in case \(n = 2, 3\)
Find Rotation $\text{Rot}(\omega, \theta)$ via Differential Equation

- Consider a point $p$ with coordinate $p_0$ at time $t = 0$.

- Rotate the point with constant unit velocity around fixed axis $\omega$. The motion is described by

$$\dot{p}(t) = \omega \times p(t) = [\omega]p(t), \text{ with } p(0) = p_0$$

- This is a linear ODE with solution: $p(t) = e^{[\omega]t}p_0$

- Note $p(\theta) = \text{Rot}(\omega, \theta)p_0$, therefore

$$\text{Rot}(\omega, \theta) = e^{[\omega]\theta}$$
Find Rotation $\text{Rot}(\omega, \theta)$ via Differential Equation

- **Exponential Map**: By definition
  \[
e^{[\omega]\theta} = I + \theta [\omega] + \frac{\theta^2}{2!} [\omega]^2 + \frac{\theta^3}{3!} [\omega]^3 + \cdots \]

- **Rodrigues’ Formula**: Given any $[\omega] \in so(3)$, we have
  \[
e^{[\omega]\theta} = I + [\omega] \sin(\theta) + [\omega]^2 (1 - \cos(\theta))\]
Find Rotation $\text{Rot}(\omega, \theta)$ via Differential Equation

- **Proposition:** For any unit vector $[\hat{\omega}] \in so(3)$ and any $\theta \in \mathbb{R}$,

$$e^{[\hat{\omega}]\theta} \in SO(3)$$
Examples of Forward Exponential Map

• Rotation matrix $R_x(\theta)$ (corresponding to $\hat{x}\theta$)

• Rotation matrix corresponding to $(1, 0, 1)^T$
Proposition: For any $R \in SO(3)$, there exists $\hat{\omega} \in \mathbb{R}^3$ with $||\hat{\omega}|| = 1$ and $\theta \in \mathbb{R}$ such that $R = e^{[\hat{\omega}]\theta}$

- If $R = I$, then $\theta = 0$ and $\hat{\omega}$ is undefined.

- If $\text{tr}(R) = -1$, then $\theta = \pi$ and set $\hat{\omega}$ equal to one of the following

$$
\frac{1}{\sqrt{2}(1 + r_{33})} \begin{bmatrix} r_{13} \\
 r_{23} \\
 1 + r_{33}
\end{bmatrix}, \quad \frac{1}{\sqrt{2}(1 + r_{22})} \begin{bmatrix} r_{12} \\
 1 + r_{22} \\
 r_{32}
\end{bmatrix}, \quad \frac{1}{\sqrt{2}(1 + r_{11})} \begin{bmatrix} 1 + r_{11} \\
 r_{21} \\
 r_{31}
\end{bmatrix}
$$

- Otherwise, $\theta = \cos^{-1}\left(\frac{1}{2}(\text{tr}(R) - 1)\right) \in [0, \pi)$ and $[\hat{\omega}] = \frac{1}{2\sin(\theta)}(R - R^T)$
Exponential Coordinate of $SO(3)$

\[
\begin{align*}
\text{exp:} & \quad [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R \in SO(3) \\
\text{log:} & \quad R \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3)
\end{align*}
\]

- The vector $\hat{\omega}\theta$ is called the \textit{exponential coordinate} for $R$

- The exponential coordinates are also called the \textit{canonical coordinates} of the rotation group $SO(3)$
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Quaternions

• Quaternions generalize complex numbers and can be used to effectively represent rotations in $\mathbb{R}^3$.

• A quaternion is a vector quantity of the following form:

$$q = q_0 + q_1i + q_2j + q_3k$$

where $q_0$ is the scalar ("real") component and $\vec{q} = (q_1, q_2, q_3)$ is the vector ("imaginary") component.

• Addition and multiplication operations:
  - $p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k$
  - $i^2 = j^2 = k^2 = ijk = -1$
  - $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$

• In scalar-vector form, product of $p = (p_0, \vec{p})$ and $q = (q_0, \vec{q})$ is given by

$$pq = \left( p_0q_0 - \vec{p}^T\vec{q}, \quad p_0\vec{q} + q_0\vec{p} + \vec{p} \times \vec{q} \right)$$
Conjugate, Norm, and Inverse

• Given a quaternion $q = (q_0, \vec{q})$:
  - **Conjugate**: $q^* = (q_0, -\vec{q})$
  - **Norm**: $\|q\|^2 = q q^* = q^* q = q_0^2 + q_1^2 + q_2^2 + q_3^2$
  - **Inverse**: $q^{-1} \triangleq \frac{q^*}{\|q\|^2}$
Quat ernion Representation of Rotation

- For $\vec{v} \in \mathbb{R}^3$, we can associate it with a 0 scalar component to construct a purely imaginary quaternion:
  \[ \hat{v} = (0, \vec{v}) \]

- Each quaternion $q = (q_0, \vec{q})$ defines an operation on a vector $\vec{v} \in \mathbb{R}^3$: 
  \[ L_q(\vec{v}) = \text{Im}(q \hat{v} q^*) = (q_0^2 - \|\vec{q}\|^2)\vec{v} + 2(\vec{q}^T \vec{v})\vec{q} + 2q_0(\vec{q} \times \vec{v}) \]
Quaternion Representation of Rotation

- **Unit quaternion**: If \( \|q\| = 1 \), we can always find \( \theta \in [0, 2\pi) \) and unit vector \( \hat{\omega} \in \mathbb{R}^3 \) such that:

\[
q = \left( \cos \left( \frac{\theta}{2} \right), \hat{\omega} \sin \left( \frac{\theta}{2} \right) \right)
\]

In this case, \( L_q(\vec{v}) \) is the vector obtained by rotating \( \vec{v} \in \mathbb{R}^3 \) about the axis \( \hat{\omega} \) for \( \theta \) degree.

- Given a unit quaternion \( q = (q_0, \vec{q}) \), we can extract the rotation axis/angle by:

\[
\theta = 2 \cos^{-1}(q_0), \quad \hat{\omega} = \begin{cases} 
\frac{\vec{q}}{\sin(\theta/2)} & \text{if } \theta \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

- \( L_{q^*}(\vec{v}) \) rotates \( \vec{v} \) about \( \hat{\omega} \) for \(-\theta \) degree.

- Quaternion provides global parameterization of \( SO(3) \), which does not suffer from singularities.
Examples of Quaternions

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