ECE5463: Introduction to Robotics

Lecture Note 12: Dynamics of Open Chains: Lagrangian Formulation

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Outline

- Introduction
- Euler-Lagrange Equations
- Lagrangian Formulation of Open-Chain Dynamics
From Single Rigid Body to Open Chains

• Recall Newton-Euler Equation for a single rigid body:

\[ \mathcal{F}_b = \mathcal{G}_b \dot{\mathcal{V}}_b - [\text{ad}_{\mathcal{V}_b}]^T (\mathcal{G}_b \mathcal{V}_b) \]

• Open chains consist of multiple rigid links connected through joints

• Dynamics of adjacent links are coupled.

• We are concerned with modeling multi-body dynamics subject to constraints.
Preview of Open-Chain Dynamics

• Equations of Motion are a set of 2nd-order differential equations:

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

- $\theta \in \mathbb{R}^n$: vector of joint variables; $\tau \in \mathbb{R}^n$: vector of joint forces/torques
- $M(\theta) \in \mathbb{R}^{n \times n}$: mass matrix
- $h(\theta, \dot{\theta}) \in \mathbb{R}^n$: forces that lump together centripetal, Coriolis, gravity, and friction terms that depend on $\theta$ and $\dot{\theta}$

• **Forward dynamics:** Determine acceleration $\ddot{\theta}$ given the state $(\theta, \dot{\theta})$ and the joint forces/torques:

$$\ddot{\theta} = M^{-1}(\theta)(\tau - h(\theta, \dot{\theta}))$$

• **Inverse dynamics:** Finding torques/forces given state $(\theta, \dot{\theta})$ and desired acceleration $\ddot{\theta}$

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$
Lagrangian vs. Newton-Euler Methods

- There are typically two ways to derive the equation of motion for an open-chain robot: Lagrangian method and Newton-Euler method

<table>
<thead>
<tr>
<th>Lagrangian Formulation</th>
<th>Newton-Euler Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Energy-based method</td>
<td>- Balance of forces/torques</td>
</tr>
<tr>
<td>- Dynamic equations in closed form</td>
<td>- Dynamic equations in numeric/recursive form</td>
</tr>
<tr>
<td>- Often used for study of dynamic properties and analysis of control methods</td>
<td>- Often used for numerical solution of forward/inverse dynamics</td>
</tr>
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Generalized Coordinates and Forces

- Consider $k$ particles. Let $f_i$ be the force acting on the $i$th particle, $\bar{m}_i$ be its mass, $p_i$ be its position. Newton’s law: $f_i = \bar{m}_i \ddot{p}_i, \quad i = 1, \ldots, k$

- Now consider the case in which some particles are rigidly connected, imposing constraints on their positions

$$\alpha_j(p_1, \ldots, p_k) = 0, \quad j = 1, \ldots, n_c$$

- $k$ particles in $\mathbb{R}^3$ under $n_c$ constraints $\Rightarrow$ $3k - n_c$ degree of freedom

- Dynamics of this constrained $k$-particle system can be represented by $n \triangleq 3k - n_c$ independent variables $q_i$’s, called the generalized coordinates

$$\begin{cases}
\alpha_j(p_1, \ldots, p_k) = 0 \\
j = 1, \ldots, n_c
\end{cases} \quad \Leftrightarrow \quad \begin{cases}
p_i = \gamma_i(q_1, \ldots, q_n) \\
i = 1, \ldots, k
\end{cases}$$
Generalized Coordinates and Forces

• To describe equation of motion in terms of generalized coordinates, we also need to express external forces applied to the system in terms components along generalized coordinates. These “forces” are called **generalized forces**.

• Generalized force \( f_i \) and coordinate rate \( \dot{q}_i \) are dual to each other in the sense that \( f^T \dot{q} \) corresponds to power.

• The equation of motion of the \( k \)-particle system can thus be described in terms of \( 3k - n_c \) independent variables instead of the \( 3k \) position variables subject to \( n_c \) constraints.

• This idea of handling constraints can be extended to interconnected rigid bodies (open chains).
Euler-Lagrange Equation

- Now let $q \in \mathbb{R}^n$ be the generalized coordinates and $f \in \mathbb{R}^n$ be the generalized forces of some constrained dynamical system.

- **Lagrangian function:** $\mathcal{L}(q, \dot{q}) = \mathcal{K}(q, \dot{q}) - \mathcal{P}(q)$
  - $\mathcal{K}(q, \dot{q})$: kinetic energy of system
  - $\mathcal{P}(q)$: potential energy

- **Euler-Lagrange Equations:**

  $$f = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}$$  \hspace{1cm} (1)
Example: Spherical Pendulum

Consider an idealized spherical pendulum as shown in Figure 4.1. The system consists of a point with mass \( m \) attached to a spherical joint by a massless rod of length \( l \). We parameterize the configuration of the point mass by two scalars, \( \theta \) and \( \phi \), which measure the angular displacement from the \( z \)- and \( x \)-axes, respectively. We wish to solve for the motion of the mass under the influence of gravity.

The Lagrange's equations are an elegant formulation of the dynamics of a mechanical system. They reduce the number of equations needed to describe the motion of the system from \( n \), the number of particles in the system, to \( m \), the number of generalized coordinates. Note that if there are no constraints, then we can choose \( q \) to be the components of \( r \), giving \( T = \frac{1}{2} \sum_{i} \| \dot{r}_i \|^2 \), and equation (4.5) then reduces to equation (4.1). In fact, rearranging equation (4.5) as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} + \Upsilon
\]

is just a restatement of Newton's law in generalized coordinates:

\[
\frac{d}{dt} (\text{momentum}) = \text{applied force}
\]

The motion of the individual particles can be recovered through application of equation (4.4).
Example: Spherical Pendulum (Continued)
Outline

- Introduction
- Euler-Lagrange Equations
- Lagrangian Formulation of Open-Chain Dynamics
Lagrangian Formulation of Open Chains

- For open chains with \( n \) joints, it is convenient and always possible to choose the joint angles \( \theta = (\theta_1, \ldots, \theta_n) \) and the joint torques \( \tau = (\tau_1, \ldots, \tau_n) \) as the generalized coordinates and generalized forces, respectively.
  - If joint \( i \) is revolute: \( \theta_i \) joint angle and \( \tau_i \) is joint torque
  - If joint \( i \) is prismatic: \( \theta_i \) joint position and \( \tau_i \) is joint force

- Lagrangian function: \( \mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta, \dot{\theta}) \)

- Dynamic Equations:
  \[
  \tau_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i}
  \]

- To obtain the Lagrangian dynamics, we need to derive the kinetic and potential energies of the robot in terms of joint angles \( \theta \) and torques \( \tau \).
Some Notations

For each link $i = 1, \ldots, n$, Frame $\{i\}$ is attached to the center of mass of link $i$. All the following quantities are expressed in frame $\{i\}$

- $\mathcal{V}_i$: Twist of link $\{i\}$

- $\bar{m}_i$: mass; $\mathcal{I}_i$: rotational inertia matrix;

- $G_i = \begin{bmatrix} \mathcal{I}_i & 0 \\ 0 & \bar{m}_i I \end{bmatrix}$: Spatial inertia matrix

- Kinetic energy of link $i$: $\mathcal{K}_i = \frac{1}{2} \mathcal{V}_i^T G_i \mathcal{V}_i$

- $J_b^{(i)} \in \mathbb{R}^{6 \times i}$: body Jacobian of link $i$

$$J_b^{(i)} = \begin{bmatrix} J_b^{(i)} & \ldots & J_b^{(i)} \end{bmatrix}$$

where $J_b^{(i)} = \left[ \text{Ad}_{e^{-[\mathcal{B}_i] \theta_i} \cdots e^{-[\mathcal{B}_{j+1}] \theta_{j+1}}} \right] \mathcal{B}_j$, $j < i$ and $J_b^{(i)} = \mathcal{B}_i$
Kinetic and Potential Energies of Open Chains

• $J_{ib} = [J_b^{(i)} \ 0] \in \mathbb{R}^{6 \times n}$

• Total Kinetic Energy:

$$\mathcal{K}(\theta, \dot{\theta}) = \frac{1}{2} \sum_{i=1}^{n} \mathcal{V}_i^T \mathcal{G}_i \mathcal{V}_i = \frac{1}{2} \dot{\theta}^T \left( \sum_{i=1}^{n} (J_{ib}^T(\theta) \mathcal{G}_i J_{ib}(\theta)) \right) \dot{\theta} \triangleq \frac{1}{2} \dot{\theta}^T \mathbf{M}(\theta) \dot{\theta}$$

• Potential Energy:

$$\mathcal{P}(\theta) = \sum_{i=1}^{n} \bar{m}_i g h_i(\theta)$$

- $h_i(\theta)$: height of CoM of link $i$
Lagrangian Dynamic Equations of Open Chains

- Lagrangian: \( \mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta) \)

- \( \tau_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} \Rightarrow \)

\[
\tau_i = \sum_{j=1}^{n} M_{ij}(\theta) \ddot{\theta}_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \Gamma_{ijk}(\theta) \dot{\theta}_j \dot{\theta}_k + \frac{\partial \mathcal{P}}{\partial \theta_i}, \quad i = 1, \ldots, n
\]

- \( \Gamma_{ijk}(\theta) \) is called the **Christoffel symbols of the first kind**

\[
\Gamma_{ijk}(\theta) = \frac{1}{2} \left( \frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{jk}}{\partial \theta_i} \right)
\]
Lagrange's Dynamic Equations of Open Chains

- Dynamic equation in vector form:

\[ \tau = M(\theta) \ddot{\theta} + C(\theta, \dot{\theta}) \dot{\theta} + g(\theta) \]

\[ C_{ij}(\theta, \dot{\theta}) \triangleq \sum_{k=1}^{n} \Gamma_{ijk} \dot{\theta}_k \] is called the Coriolis matrix

\[ M_{ij} = I_{y2} s_2^2 + I_{y3} s_3^2 + I_{z1} + I_{z2} c_2^2 + I_{z3} c_3^2 + m_2 r_2^2 c_2^2 + m_3 (l_1 c_2 + r_2 c_3)^2 \]

\[ M_{12} = 0 \]

\[ M_{13} = 0 \]

\[ M_{21} = 0 \]

\[ M_{22} = I_{z2} + I_{x3} + m_3 r_1^2 + m_3 r_2^2 + m_3 r_3^2 + 2 m_3 l_1 r_2 c_3 \]

\[ M_{23} = I_{z3} + m_3 r_2^2 + m_3 l_1 r_2 c_3 \]

\[ M_{31} = 0 \]

\[ M_{32} = I_{x3} + m_3 r_2^2 + m_3 l_1 r_2 c_3 \]

\[ M_{33} = I_{x3} + m_3 r_2^2. \]

\[ \Gamma_{112} = (I_{y2} - I_{z2} - m_2 r_2^2) c_2 s_2 + (I_{y3} - I_{z3}) s_3 \]

\[ - m_3 (l_1 c_2 + r_2 c_3) (l_1 s_2 + r_2 s_3) \]

\[ \Gamma_{113} = (I_{y3} - I_{z3}) c_2 s_3 s_2 - m_3 r_2 s_2 s_3 (l_1 c_2 + r_2 c_3) \]

\[ \Gamma_{121} = (I_{y2} - I_{z2} - m_2 r_2^2) c_2 s_2 + (I_{y3} - I_{z3}) c_3 s_3 \]

\[ - m_3 (l_1 c_2 + r_2 c_3) (l_1 s_2 + r_2 s_3) \]

\[ \Gamma_{131} = (I_{y3} - I_{z3}) c_2 s_3 s_2 - m_3 r_2 s_2 s_3 (l_1 c_2 + r_2 c_3) \]

\[ \Gamma_{211} = (I_{z2} - I_{y2} + m_2 r_2^2) c_2 s_2 + (I_{z3} - I_{y3}) c_3 s_3 \]

\[ + m_3 (l_1 c_2 + r_2 c_3) (l_1 s_2 + r_2 s_3) \]

\[ \Gamma_{221} = -l_1 m_3 r_2 s_3 \]

\[ \Gamma_{232} = -l_1 m_3 r_2 s_3 \]

\[ \Gamma_{233} = -l_1 m_3 r_2 s_3 \]

\[ \Gamma_{311} = (I_{z3} - I_{y3}) c_3 s_3 s_2 + m_3 r_2 s_2 s_3 (l_1 c_2 + r_2 c_3) \]

\[ \Gamma_{322} = l_1 m_3 r_2 s_3 \]

\[ \begin{bmatrix} 0 \\
-m_2 g r_1 + m_3 g l_1 \cos \theta_2 - m_3 r_2 \cos (\theta_2 + \theta_3) \]

\[ -m_3 g r_2 \cos (\theta_2 + \theta_3) \end{bmatrix} \]
Lagrangian Dynamic Equations of Open Chains

- Dynamic model of PUMA 560 Arm:

**Table A4.** The expressions giving the elements of the kinetic energy matrix.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{16}$</td>
<td>$-37.2 \pm 0.00$</td>
</tr>
<tr>
<td>$\tau_{26}$</td>
<td>$-8.44 \pm 0.20$</td>
</tr>
<tr>
<td>$\tau_{45}$</td>
<td>$-2.49 \times 10^{-4} \pm 0.25 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\tau_{65}$</td>
<td>$-2.82 \times 10^{-4} \pm 0.56 \times 10^{-2}$</td>
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**Table A5.** The expressions for the terms of the centrifugal matrix.

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**Table A6.** The expressions for the terms of the Coriolis matrix.

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**Table A7.** The expressions for the terms of the gravity matrix.

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**Lagrangian Formulation**
More Discussions

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