ECE5463: Introduction to Robotics

Lecture Note 11: Dynamics of a Single Rigid Body

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Outline

- Kinetic Energy of a Rigid Body
- Rotational Inertia Matrix
- Newton Euler Equation
- Twist-Wrench Formulation of Rigid-Body Dynamics
Robot Dynamic Model Can Be Complicated

- Dynamic model of PUMA 560 Arm:

<table>
<thead>
<tr>
<th>Table A4. The representation giving the elements of the kinetic energy matrix. (The Abbreviated Expressed have units of kg m^2)</th>
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<td>( a_{11} = a_{11} = a_{12} = 0.0325 + 0.0235 + 0.0235 = 0.0325 )</td>
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<th>Part II. Gravitational Constants</th>
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<tr>
<th>Table A4. Computed Values for the Constants Appearing in the Equation of Forces of Motion. (Nautical constants have units of kg m^2)</th>
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<td>( a_{11} = \frac{14.31}{102} )</td>
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<td>( a_{12} = \frac{13.15}{102} )</td>
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<td>( a_{15} = \frac{12.52}{102} )</td>
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<td>( a_{19} = \frac{11.78}{102} )</td>
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<th>Table A3. The representation for the terms in the rotational kinetic. (The Abbreviated Expressed have units of kg m^2)</th>
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<th>Table A2. The representation giving the elements of the kinetic energy matrix. (The Abbreviated Expressed have units of kg m^2)</th>
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<th>Lecture 11 (ECE5463 Sp18)</th>
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<tr>
<td>Wei Zhang (OSU)</td>
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Kinetic Energy

- Consider a point mass $\bar{m}$ with $\{s\}$-frame coordinate $p(t)$, its kinetic energy is given by
  \[
  K = \frac{1}{2} \bar{m} \|\dot{p}\|^2
  \]

- Note: $m$ denotes moment (vector) and $\bar{m}$ denotes mass (scalar).

- Question: given a moving rigid body with configuration $T(t) = (R(t), p(t))$, how to compute its kinetic energy?
  - Rigid body consists of infinitely many “particles” with different $\{s\}$-frame velocities
    \[
    K \approx \frac{1}{2} \sum_i \bar{m}_i \|\dot{p}_i\|^2
    \]
  - Velocities of particles $\dot{p}_i$ are caused by the rigid body velocity (twist)
  - The overall kinetic energy should depend on the rigid body velocity as well as the geometry and mass distribution of the body
Recall: Rigid Body Velocity

Given rigid body $T(t) = (R(t), p(t))$:

- Spatial twist:

- Body twist:
Recall: Rigid Body Velocity

• Consider a particle $i$ on the body with $\{b\}$-frame coordinate $r_i$ and $\{s\}$-frame coordinate $p_i$
  - Velocity of particle $i$:

  - Acceleration of particle $i$:

  - Velocity of the origin of $\{b\}$:
Rigid Body Kinetic Energy

- **Kinetic Energy**: Given a rigid body $T(t) = (R(t), p(t))$ with body twist $\mathcal{V}_b = (\omega_b, v_b)$. Suppose the $\{b\}$-frame origin coincides with the center of mass of the body. Then its kinetic energy is given by:

$$K = \frac{1}{2} \bar{m} \|v_b\|^2 + \frac{1}{2} \omega_b^T \mathcal{I}_b \omega_b,$$

with $\mathcal{I}_b = \int_B \rho(r) [r]^T [r] dV$

where $\mathcal{I}_b$ is the rotational inertia matrix in body frame

**Derivation**: Divide the body into small point masses, where point $i$ has mass $\bar{m}_i$, $\{b\}$-frame coordinate $r_i$, and $\{s\}$-frame coordinate $p_i$
Derivation of Kinetic Energy (Continued)

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Rotational Inertia Matrix in Body Frame

\[ \mathcal{I}_b \triangleq \int_{\mathcal{B}} \rho(r)[r]^T[r]dV \]

- Individual entries of \( \mathcal{I}_b \):

\[
\mathcal{I}_b = \begin{bmatrix}
\mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\
\mathcal{I}_{yx} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\
\mathcal{I}_{zx} & \mathcal{I}_{zy} & \mathcal{I}_{zz}
\end{bmatrix}
\]

where

\[
\mathcal{I}_{xx} = \int_{\mathcal{B}} (y^2 + z^2)\rho(x, y, z)dV, \quad \mathcal{I}_{yy} = \int_{\mathcal{B}} (x^2 + z^2)\rho(x, y, z)dV
\]

\[
\mathcal{I}_{zz} = \int_{\mathcal{B}} (x^2 + y^2)\rho(x, y, z)dV, \quad \mathcal{I}_{xy} = \mathcal{I}_{yx} = -\int_{\mathcal{B}} xy\rho(x, y, z)dV
\]

\[
\mathcal{I}_{xz} = \mathcal{I}_{zx} = -\int_{\mathcal{B}} xz\rho(x, y, z)dV, \quad \mathcal{I}_{yz} = \mathcal{I}_{zy} = -\int_{\mathcal{B}} yz\rho(x, y, z)dV
\]

- If the body has a uniform density, then \( \mathcal{I}_b \) is determined exclusively by the shape of the rigid body
Principal Axes of Inertia

Let $v_1, v_2, v_3$ and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvectors and eigenvalues of $I_b$, respectively. They are called the **principal axes of inertia**

- The principal axes of inertia are in the directions of $v_1, v_2, v_3$

- The principal moments of inertia about these axes are $\lambda_1, \lambda_2, \lambda_3$

- All the eigenvalues are nonnegative. The largest one maximizes the moment of inertia among all the axes passing through the center of mass of the body.

- If the principal axes of inertia are aligned with the axes of $\{b\}$, the off-diagonal terms of $I_b$ are all zero.
Examples of Inertia Matrix

Rotational Inertia Matrix

Examples of common uniform-density solid bodies, their principal axes of inertia, and the principal moments of inertia obtained by solving the integrals (8.24), are given in Figure 8.5.

The principal axes and the inertia about the principal axes for uniform-density bodies

rectangular parallelepiped:
- volume $= abc$,
- $I_{xx} = m(w^2 + h^2)/12$,
- $I_{yy} = m(l^2 + h^2)/12$,
- $I_{zz} = m(l^2 + w^2)/12$

circular cylinder:
- volume $= \pi r^2 h$,
- $I_{xx} = m(3r^2 + h^2)/12$,
- $I_{yy} = m(3r^2 + h^2)/12$,
- $I_{zz} = mr^2/2$

ellipsoid:
- volume $= 4\pi abc/3$,
- $I_{xx} = m(b^2 + c^2)/5$,
- $I_{yy} = m(a^2 + c^2)/5$,
- $I_{zz} = m(a^2 + b^2)/5$
Inertia Matrix in a Different Frame

- Consider another frame \( \{c\} \) with relative orientation \( R_{bc} \)

- The origin of both frames is located at the CoM of the body. The rotational inertia matrix in \( \{c\} \) frame is defined as \( \mathcal{I}_c = \int_B \rho(r_c)[r_c]^T[r_c]dV \)

- Kinetic energy is independent of reference frames \( \Rightarrow \mathcal{I}_c = R_{bc}^T\mathcal{I}_b R_{bc} \)

- **Steiner's Theorem:** The inertia matrix \( I_q \) about a frame aligned with \( \{b\} \), but at a point \( q = (q_x, q_y, q_z) \) in \( \{b\} \), is related to \( \mathcal{I}_b \) by

\[
\mathcal{I}_q = \mathcal{I}_b + \bar{m} \left( q^T q I - qq^T \right)
\]
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Newton Euler Equation

• Recall that for a point mass $\bar{m}$ with a fixed-frame coordinate $p(t)$, Newton’s second law of motion: $f = \bar{m} \ddot{p}(t)$

• A rigid body consists of infinitely many point masses. The collective motion of these particles depend on the linear and rotational velocities of the body, and on the total force and moment acting on the body.

• **Euler-Newton Equation of Motion:** Given rigid body $T(t) = (R(t), p(t))$ with rotational inertia matrix $\mathcal{I}_b$ and body twist $\mathcal{V}_b = (\omega_b, v_b)$:

$$\begin{cases}
    m_b = \mathcal{I}_b \dot{\omega}_b + \omega_b \times \mathcal{I}_b \omega_b \\
    f_b = \bar{m} \dot{v}_b + \omega_b \times \bar{m} v_b
\end{cases}$$

1. $\bar{m}$: mass of the body; assume origin of $\{b\} = \text{CoM}$
2. $f_b, m_b$: total force and moment (expressed in $\{b\}$) acting on the body
3. $\bar{m} \dot{v}_b$: is the **linear momentum** of the body
4. $\mathcal{I}_b \omega_b$: is the **angular momentum** of the body
Derivation of Newton Euler Equation
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**Lie Bracket**

- **Lie Bracket:** Given two twists $V_1 = (\omega_1, v_1)$ and $V_2 = (\omega_2, v_2)$, the Lie bracket of $V_1$ and $V_2$, written as $[\text{ad}_{V_1}] V_2$, is defined as follows:

$$
[\text{ad}_{V_1}] V_2 = \begin{bmatrix}
\omega_1 & 0 \\
v_1 & [\omega_1]
\end{bmatrix}
\begin{bmatrix}
\omega_2 \\
v_2
\end{bmatrix} \in \mathbb{R}^6
$$

where $[\text{ad}_V] \triangleq \begin{bmatrix}
[\omega] & 0 \\
[v] & [\omega]
\end{bmatrix}$ for any $V = (\omega, v) \in se(3)$

- Lie Bracket can be viewed as a generalization of the cross-product operation of two vectors to two twists

- Given a twist $V = (\omega, v)$ and a wrench $F = (m, f)$, we define the mapping:

$$
[\text{ad}_V]^T F = \begin{bmatrix}
[\omega] & 0 \\
[v] & [\omega]
\end{bmatrix}^T \begin{bmatrix}
m \\
f
\end{bmatrix}
$$
Twist-Wrench Formulation

- Rigid body with body twist $\mathcal{V}_b = (\omega_b, v_b)$ and body wrench $\mathcal{F}_b = (m_b, f_b)$

- Spatial inertia matrix $\mathcal{G}_b \in \mathbb{R}^{6 \times 6}$: $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \bar{m}I \end{bmatrix}$

- Spatial momentum $\mathcal{P}_b \in \mathbb{R}^6$: $\mathcal{P}_b = \mathcal{G}_b \mathcal{V}_b$

- Kinetic energy: $\mathcal{K} = \frac{1}{2} \mathcal{V}_b^T \mathcal{G}_b \mathcal{V}_b$
Twist-Wrench Formulation

- Newton-Euler Equation (1) can be written in twist-wrench form:

\[
\mathcal{F}_b = \mathcal{G}_b \dot{\nu}_b - [\text{ad}_{\mathcal{V}_b}]^T \mathcal{P}_b = \mathcal{G}_b \dot{\nu}_b - [\text{ad}_{\mathcal{V}_b}]^T \mathcal{G}_b \mathcal{V}_b
\]
Dynamics in Other Frames

- Our derivation of dynamics relies on using CoM $\{b\}$ frame. We can also write dynamics in another frame, say $\{a\}$, with relative configuration $T_{ba}$

- Kinetic energy is independent of reference frame: $\frac{1}{2} V_b^T G_b V_b = \frac{1}{2} V_a^T G_a V_a$

- This implies that the spatial inertia matrix $G_a$ is related to $G_b$ by

$$G_a = [\text{Ad}_{T_{ba}}]^T G_b [\text{Ad}_{T_{ba}}]$$

- One can show the Newton-Euler equation (1) can be written equivalently in frame $\{a\}$ as:

$$\mathcal{F}_a = G_a \dot{V}_a - [\text{ad}_{V_a}]^T G_a V_a$$
More Discussions

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