This lecture introduces basic concepts and results on Lyapunov stability of traditional nonlinear systems (non-hybrid).

- Lyapunov Stability Definitions
- Lyapunov Function Theorems
- Lyapunov Stability of Linear Systems
- Converse Lyapunov Function
- Conclusion
Lyapunov Stability Definitions I

Consider a time-invariant autonomous (with no control) nonlinear system:

\[ \dot{x} = f(x) \text{ with I.C. } x(0) = x_0 \]  

(1)

- **Assumptions**: (i) \( f \) Lipschitz continuous; (ii) origin is an isolated equilibrium \( f(0) = 0 \)

\[ \| f(x) - f(y) \| \leq L \| x - y \| \]

- **Stability Definitions**: The equilibrium \( x = 0 \) is called **stable** in the sense of Lyapunov, if

\[ \forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \| x(0) \| \leq \delta \Rightarrow \| x(t) \| \leq \epsilon, \forall t \geq 0 \]
Lyapunov Stability Definitions II

- **asymptotically stable** if it is stable and $\delta$ can be chosen so that

\[ \|x(0)\| \leq \delta \Rightarrow \left[ x(t) \to 0 \text{ as } t \to \infty \right] \]

- **exponentially stable** if there exist positive constants $\delta, \lambda, c$ such that

\[ \|x(t)\| \leq c\|x(0)\|e^{-\lambda t}, \quad \forall \|x(0)\| \leq \delta \]

- **globally asymptotically/exponentially stable** if the above conditions hold for all $\delta > 0$

\[ (G.A.S.) \iff (G.E.S.) \]

• Region of Attraction: $R_A \triangleq \{ x \in \mathbb{R}^n : \text{whenever } x(0) = x, \text{ then } x(t) \to 0 \}$
Lyapunov Stability Definitions III

• Does attractiveness implies stable in Lyapunov sense?
  
  - Answer is NO. e.g.: \[
  \begin{cases}
    \dot{x}_1 = x_1^2 - x_2^2 \\
    \dot{x}_2 = 2x_1x_2
  \end{cases}
  \]

  - By inspection of its vector field, we see that \(x(t) \to 0\) for all \(x(0) \in \mathbb{R}^2\)

  - However, there is no \(\delta\)-ball satisfying the Lyapunov stability condition
Stability Analysis

How to verify stability of a system:

- Find explicit solution of the ODE $x(t)$ and check stability definitions
  - typically not possible for nonlinear systems

- Numerical simulations of ODE do not provide stability guarantees and offer limited insights

- Need to determine stability without explicitly solving the ODE

- Preferably, analysis only depends on the vector field

- The most powerful tool is: Lyapunov function
  - State trajectory $x(t)$ governed by complex dynamics in $\mathbb{R}^n$
  - Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ maps $x(t)$ to a scalar function of time $V(x(t))$
  - If the function is designed such that: $[x(t) \rightarrow \text{equilibrium}] \iff [V(x(t)) \rightarrow 0]$. Then we can study $V(x(t))$ as function of time $t$ to infer stability of the state trajectory in $\mathbb{R}^n$. 

Lyapunov Function Theorems

Before giving a formal definition of Lyapunov function, we first introduce some classes of functions. Assume that $0 \in D \subseteq \mathbb{R}^n$

- $g : D \to \mathbb{R}$ is called positive semidefinite (PSD) on $D$ if $g(0) = 0$ and $g(x) \geq 0, \forall x \in D$
  - For quadratic function: $g(x) = x^T P x$: $[g$ is PSD] $\iff [P$ is a PSD matrix$]$ 

- $g : D \to \mathbb{R}$ is called positive definite (PD) on $D$ if $g(0) = 0$ and $g(x) > 0, \forall x \in D \setminus \{0\}$
  - Similarly, if $g(x) = x^T P x$ is quadratic, then $[g$ is PD] $\iff [P$ is a PD matrix$]$ 

- $g$ is negative semidefinite (NSD) if $-g$ is PSD

- $g : \mathbb{R}^n \to \mathbb{R}$ is radically unbounded if $g(x) \to \infty$ as $\|x\| \to \infty$
  - e.g., $g(x) = \|x\|^3$
Lie Derivative

Definition 1 (Lie Derivative).

Lie derivative of a $C^1$ function $V: \mathbb{R}^n \to \mathbb{R}$ along vector field $f: \mathbb{R}^n \to \mathbb{R}^n$ is:

$$\mathcal{L}_f V(x) \triangleq \left( \frac{\partial V}{\partial x}(x) \right)^T f(x)$$

- Sometimes the Lie derivative $\mathcal{L}_f V(x)$ is also denoted by $\frac{\partial V}{\partial x} f(x)$
- Let $x(t)$ be a solution to ODE $\dot{x}(t) = f(x(t))$. If we view $V(x(t))$ as a function of $t$, then

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{dx_i}{dt} = \left( \frac{\partial V}{\partial x}(x(t)) \right)^T f(x(t)) = \mathcal{L}_f V(x(t))$$

- Therefore, the Lie derivative characterizes the time-course evolution of the value of $V$ along the solution trajectory of $\dot{x} = f(x)$
Lyapunov Stability Theorem

Theorem 1 (Lyapunov Theorem).

Let $D \subset \mathbb{R}^n$ be a set containing an open neighborhood of the origin. If there exists a PD function $V : D \rightarrow \mathbb{R}$ such that

\[ \mathcal{L}_fV \text{ is NSD} \Rightarrow \text{value of } V \text{ along sys trajectory nonincreasing (2)} \]

then the origin is stable. If in addition,

\[ \mathcal{L}_fV \text{ is ND} \Rightarrow \text{value of ... decreases (3)} \]

then the origin is asymptotically stable.

Remarks:

- A PD $C^1$ function satisfying (2) or (3) will be called a Lyapunov function

- Under condition (3), if $V$ is also radially unbounded
  \[ \Rightarrow \text{globally asymptotically stable} \]
Proof of Lyapunov Stability Theorem I

Sketch of proof of Lyapunov stability theorem:

• First show stability under condition (2):
  - Define sublevel set: \( \Omega_b = \{ x \in \mathbb{R}^n : V(x) \leq b \} \). Condition (2) implies \( V(x(t)) \) nonincreasing along system trajectory \( \Rightarrow \) If \( x(0) \in \Omega_b \), then \( x(t) \in \Omega_b \), \( \forall t \).

  - Given arbitrary \( \epsilon > 0 \), if we can find \( \delta, b \) such that \( B(0, \delta) \subseteq \Omega_b \subseteq B(0, \epsilon) \). Then the Lyapunov stability conditions are satisfied. Below is to show how we can find such \( b \) and \( \delta \).

  - \( V \) is continuous \( \Rightarrow \) \( m = \min_{\|x\| = \epsilon} V(x) \) exists (due to Weierstrass theorem). In addition, \( V \) is PD \( \Rightarrow \) \( m > 0 \). Therefore, if we choose \( b \in (0, m) \), then \( \Omega_b \subseteq B(0, \epsilon) \).

    \( \text{And } V(0) > 0 \)

  - \( V(x) \) is continuous at origin \( \Rightarrow \) for any \( b > 0 \), there exists \( \delta > 0 \) such that \( |V(x) - V(0)| = V(x) < b, \forall x \in B(0, \delta) \). This implies that \( B(0, \delta) \subseteq \Omega_b \).
Proof of Lyapunov Stability Theorem II

- Second, show asymptotic stability under condition (3):
  - We know $V(x(t))$ decreases monotonically as $t \to \infty$ and $V(x(t)) \geq 0$, $\forall t$. Therefore, $c = \lim_{t \to \infty} V(x(t))$ exists. So it suffices to show $c = 0$. Let us use a contradiction argument.

- Suppose $c \neq 0$. Then $c > 0$. Therefore, $x(t) \notin \Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$, $\forall t$. We can choose $\beta > 0$ such that $B(0, \beta) \subseteq \Omega_c$ (due to continuity of $V$ at 0).

- Now let $a = -\max_{\beta \leq \|x\| \leq \epsilon} \dot{V}(x)$. Since $V$ is ND, then $a > 0$

- $V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))ds \leq V(x(0)) - a \cdot t < 0$ for sufficiently large $t$. \Rightarrow contradiction!
### Exponential Lyapunov Function

**Definition 2 (Exponential Lyapunov Function).**

Let $V : D \to \mathbb{R}$ be a function defined on a domain $D \subseteq \mathbb{R}^n$. Then $V$ is called an Exponential Lyapunov Function (ELF) on $D$ if there exist positive constants $k_1, k_2, k_3, \alpha > 0$ such that

\[
\begin{align*}
    k_1 \|x\|^\alpha &\leq V(x) \leq k_2 \|x\|^\alpha \\
    \mathcal{L}_f V(x) &\leq -k_3 V(x)
\end{align*}
\]

**Theorem 2 (ELF Theorem).**

If system (1) has an ELF, then it is exponentially stable.

**Proof sketch:**

Recall:

\[
\dot{x} = -k_3 x \implies x(t) = e^{-k_3 t} x(0)
\]

By Comparison Lemma:

\[
\dot{V}(x) \leq -k_3 V(x)
\]

\[
\|x(t)\|^\alpha \leq \frac{1}{k_1} V(x(0)) \leq \frac{1}{k_1} e^{-k_3 t} V(0) \leq \left(\frac{k_2}{k_1}\right) e^{-\lambda t} \|x(0)\|^\alpha
\]

where $\lambda = k_3$. 

**Note:**

$L_f V(x) = -k_3 V(x)$ along system trajectory.
Stability Analysis Examples I

Example 1.

\[ \begin{align*}
\dot{x}_1 &= -x_1 + x_2 + x_1 x_2 \leq f_1(x) \\
\dot{x}_2 &= x_1 - x_2 - x_1^2 - x_2^2 \leq f_2(x)
\end{align*} \]

Try \( V(x) = \|x\|^2 = x_1^2 + x_2^2 \)

4) \( V(x) = \|x\|^2 \) is P.D. and is \( C^1 \)

2) \( 2f V(x) = (\nabla V(x))^T f(x) = \left[ \frac{2x_1}{\|x\|}, \frac{2x_2}{\|x\|} \right]^T \left[ f_1(x), f_2(x) \right] = \left[ 2x_1, 2x_2 \right]^T \begin{bmatrix} -x_1 + x_2 + x_1 x_2 \\ x_1 - x_2 - x_1^2 - x_2^2 \end{bmatrix} \)

\[ \begin{align*}
&= 2\left[ -(x_1 - x_2)^2 - x_2^4 \right] \text{ is N.D. after algebra}
\end{align*} \]

\( \Rightarrow \) sys is asymptotically stable
Example 2.

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1 x_2 \\
\dot{x}_2 &= -x_2
\end{align*}
\]

- Can we find a simple quadratic Lyapunov function? First try: \( V(x) = x_1^2 + x_2^2 \)

1. \( V(x) \) is P.D.
2. \( \dot{V}(x) = (\frac{\partial V}{\partial x})^T f(x) = (2x_1 \ 2x_2) \begin{bmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1^2 (1 - x_2) + x_2^2 \\ -x_2 \end{bmatrix} \)

Is this \( \dot{V}(x) \) N.D.? Let \( x_1 = \sqrt{8} \) \( \Rightarrow \dot{V}(x) = -2 \begin{bmatrix} 8 (1 - x_2) + x_2^2 \\ -x_2 \end{bmatrix} \)

\[ = -2 \begin{bmatrix} x_2 - x_2 \sqrt{8} - 8 \end{bmatrix} \]

Not N.D.

- In fact, the system does not have any (global) polynomial Lyapunov function. But it is GAS with a Lyapunov function \( V(x) = \ln(1 + x_1^2) + x_2^2 \).
Consider autonomous linear system: \( \dot{x} = f(x) = Ax. \)

- Recall: \( A \) asymptotic stable \( \iff \) \( \text{Re}(\lambda_i) < 0 \) for all eigenvalues \( \lambda_i \) of \( A \)

When does a linear system have a Lyapunov function?

- Consider a quadratic Lyapunov function candidate: \( V(x) = x^T P x \), with \( P \in \mathbb{R}^{n \times n} \)
  - \( V \) is PD \( \Rightarrow \) \( P > 0 \)
  - \( \mathcal{L}_f V \) is ND \( \Rightarrow \)

\( \mathcal{L}_f V(x) = (\frac{\partial V}{\partial x})(Ax) = (2Px)^T Ax = 2x^T PAx \) \( \quad \text{(a)} \)

or equivalently:

\[
\dot{V}(x) = x^T Px + x^T P \dot{x} = (Ax)^T P x + x^T PAx = x^T (A^T P + PA) x \quad \text{(b)}
\]

\( (a) \equiv (b) \quad \forall x \) . \( \overbrace{x^T A^T P x}^{(c)} = (\bullet)' = x^T PAx \)

\( \implies V \) is L.F. if \( P > 0 \) and \( A^T P + PA < 0 \) \( \iff \exists \theta > 0 \text{ s.t. } A^T P + PA \preceq 0 \)
Theorem 3 (Stability Conditions for Linear System).

For an autonomous Linear system \( \dot{x} = Ax \). The following statements are equivalent.

- System is (globally) asymptotically stable
- System is (globally) exponentially stable
- \( \text{Re}(\lambda_i) < 0 \) for all eigenvalues \( \lambda_i \) of \( A \)
- System has a quadratic Lyapunov function
- For any symmetric \( Q \succ 0 \), there exists a symmetric \( P \succ 0 \) that solves the following Lyapunov equation:
  \[
P A + A^T P = -Q
  \]
  and \( V(x) = x^T P x \) is a Lyapunov function of the system.
When There is a Lyapunov Function

- **Converse Lyapunov Theorem for Asymptotic Stability**

  \[
  \begin{align*}
  &\text{origin asymptotically stable; } \\
  &f \text{ is locally Lipschitz on } D \\
  &\text{with region of attraction } R_A
  \end{align*}
  \Rightarrow \exists V \text{ s.t. }
  \begin{align*}
  &V \text{ is continuous and PD on } R_A \\
  &\mathcal{L}_f V \text{ is ND on } R_A \\
  &V(x) \to \infty \text{ as } x \to \partial R_A
  \end{align*}
  \]

- **Converse Lyapunov Theorem for Exponential Stability**

  \[
  \begin{align*}
  &\text{origin exponentially stable on } D; \\
  &f \text{ is } C^1
  \end{align*}
  \Rightarrow \exists \text{ an ELF } V \text{ on } D
  \]

- **Proofs are involved especially for the converse theorem for asymptotic stability**

- **IMPORTANT**: proofs of converse theorems often assume the knowledge of system solution and hence are not constructive.
What about Discrete Time Systems?

- So far, all our definitions, results, examples are given using continuous time
  dynamical system models.

- All of them have discrete-time counterparts. The ideas and conclusions are
  the "same" (in spirit)

- For example, given autonomous discrete-time system: \( x(k+1) = f(x(k)) \)
  with \( f(0) = 0 \) (origin is an equilibrium).
  - Rate of change of a function \( V(x) \) along system trajectory can be defined as:
    \[
    \Delta_f V(x) \triangleq V(f(x)) - V(x)
    \]
  - Asymptotically stable requires:
    \( V \) is PD and \( \Delta_f V \) is ND
  - Exponentially stable requires:
    \[
    k_1 \|x\|^\alpha \leq V(x) \leq k_2 \|x\|^\alpha \quad \text{and} \quad \Delta_f V(x) \leq -k_3 V(x)
    \]
    - 

Concluding Remarks

- We have learned different notions of internal stability, e.g. stability in Lyapunov sense, asymptotic stability, globally asymptotic stability (G.A.S), exponential stability, globally exponential stability (G.E.S).

- Sufficient condition to ensure stability is often the existence of a properly defined Lyapunov function.

- Key requirements for a Lyapunov function:
  - positive definite and is zero at the system equilibrium
  - decrease along system trajectory


- The definitions and results in this lecture have sometimes been stated in simplified forms to facilitate presentation. More general versions can be found in standard textbooks on nonlinear systems (e.g. [Kha96]).

- **Next Lecture**: Semidefinite Programming and computational stability analysis.
References