Lecture Note 4: Basic Lyapunov Stability

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Outline

This lecture introduces basic concepts and results on Lyapunov stability of traditional nonlinear systems (non-hybrid).

- Lyapunov Stability Definitions
- Lyapunov Function Theorems
- Lyapunov Stability of Linear Systems
- Converse Lyapunov Function
- Conclusion
Lyapunov Stability Definitions

Consider a time-invariant autonomous (with no control) nonlinear system:

\[ \dot{x} = f(x) \text{ with I.C. } x(0) = x_0 \]  

(1)

- **Assumptions**: (i) \( f \) Lipschitz continuous; (ii) origin is an isolated equilibrium \( f(0) = 0 \)
- **Stability Definitions**: The equilibrium \( x = 0 \) is called
  - **stable** in the sense of Lyapunov, if

\[ \forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \forall t \geq 0 \]
Lyapunov Stability Definitions II

- **asymptotically stable** if it is stable and $\delta$ can be chosen so that

\[
\|x(0)\| \leq \delta \Rightarrow x(t) \to 0 \text{ as } t \to \infty
\]

- **exponentially stable** if there exist positive constants $\delta, \lambda, c$ such that

\[
\|x(t)\| \leq c\|x(0)\|e^{-\lambda t}, \quad \forall \|x(0)\| \leq \delta
\]

- **globally asymptotically/exponentially stable** if the above conditions holds for all $\delta > 0$

• **Region of Attraction**: $R_A \triangleq \{x \in \mathbb{R}^n : \text{whenever } x(0) = x, \text{ then } x(t) \to 0\}$
Lyapunov Stability Definitions III

- Does attractiveness implies stable in Lyapunov sense?
  - Answer is NO. e.g.:
    \[
    \begin{align*}
    \dot{x}_1 &= x_1^2 - x_2^2 \\
    \dot{x}_2 &= 2x_1x_2
    \end{align*}
    \]
  - By inspection of its vector field, we see that
    \( x(t) \to 0 \) for all \( x(0) \in \mathbb{R}^2 \)
  - However, there is no \( \delta \)-ball satisfying the Lyapunov stability condition
Stability Analysis

How to verify stability of a system:

• Find explicit solution of the ODE \( x(t) \) and check stability definitions
  - typically not possible for nonlinear systems

• Numerical simulations of ODE do not provide stability guarantees and offer limited insights

• Need to determine stability without explicitly solving the ODE

• Preferably, analysis only depends on the vector field

• The most powerful tool is: Lyapunov function
  - State trajectory \( x(t) \) governed by complex dynamics in \( \mathbb{R}^n \)
  - Lyapunov function \( V : \mathbb{R}^n \to \mathbb{R} \) maps \( x(t) \) to a scalar function of time \( V(x(t)) \)
  - If the function is designed such that: \([x(t) \to \text{equilibrium}] \iff [V(x(t)) \to 0]\).
    Then we can study \( V(x(t)) \) as function of time \( t \) to infer stability of the state trajectory in \( \mathbb{R}^n \).
Before giving a formal definition of Lyapunov function, we first introduce some classes of functions. Assume that $0 \in D \subseteq \mathbb{R}^n$

- $g : D \to \mathbb{R}$ is called positive semidefinite (PSD) on $D$ if $g(0) = 0$ and $g(x) \geq 0, \forall x \in D$
  - For quadratic function: $g(x) = x^T P x$: $[g$ is PSD] $\iff [P$ is a PSD matrix$]$

- $g : D \to \mathbb{R}$ is called positive definite (PD) on $D$ if $g(0) = 0$ and $g(x) > 0, \forall x \in D \setminus \{0\}$
  - Similarly, if $g(x) = x^T P x$ is quadratic, then $[g$ is PD] $\iff [P$ is a PD matrix$]$

- $g$ is negative semidefinite (NSD) if $-g$ is PSD

- $g : \mathbb{R}^n \to \mathbb{R}$ is radically unbounded if $g(x) \to \infty$ as $\|x\| \to \infty$
Lie Derivative

Definition 1 (Lie Derivative).

**Lie derivative** of a $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ along vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is:

$$\mathcal{L}_f V(x) \triangleq \left( \frac{\partial V}{\partial x}(x) \right)^T f(x)$$

- Sometimes the Lie derivative $\mathcal{L}_f V(x)$ is also denoted by $\frac{\partial V}{\partial x} f(x)$
- Let $x(t)$ be a solution to ODE $\dot{x}(t) = f(x(t))$. If we view $V(x(t))$ as a function of $t$, then

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial t} = \left( \frac{\partial V}{\partial x}(x(t)) \right)^T f(x(t)) = \mathcal{L}_f V(x(t))$$

- Therefore, the Lie derivative characterizes the time-course evolution of the value of $V$ along the solution trajectory of $\dot{x} = f(x)$
Lyapunov Stability Theorem

**Theorem 1 (Lyapunov Theorem).**

Let $D \subset \mathbb{R}^n$ be a set containing an open neighborhood of the origin. If there exists a PD function $V : D \to \mathbb{R}$ such that

$$\mathcal{L}_fV \text{ is NSD}$$  \hspace{1cm} (2)

then the origin is stable. If in addition,

$$\mathcal{L}_fV \text{ is ND}$$  \hspace{1cm} (3)

then the origin is asymptotically stable.

**Remarks:**

- A PD $C^1$ function satisfying (2) or (3) will be called a **Lyapunov function**

- Under condition (3), if $V$ is also radially unbounded
  \[ \Rightarrow \text{globally asymptotically stable} \]
Proof of Lyapunov Stability Theorem I

Sketch of proof of Lyapunov stability theorem:

- First show stability under condition (2):
  - Define sublevel set: \( \Omega_b = \{x \in \mathbb{R}^n : V(x) \leq b\} \). Condition (2) implies \( V(x(t)) \) nonincreasing along system trajectory \( \Rightarrow \) If \( x(0) \in \Omega_b \), then \( x(t) \in \Omega_b \), \( \forall t \).

  - Given arbitrary \( \epsilon > 0 \), if we can find \( \delta, b \) such that \( B(0, \delta) \subseteq \Omega_b \subseteq B(0, \epsilon) \). Then the Lyapunov stability conditions are satisfied. Below is to show how we can find such \( b \) and \( \delta \).

    - \( V \) is continuous \( \Rightarrow m = \min_{\|x\| = \epsilon} V(x) \) exists (due to Weierstrass theorem). In addition, \( V \) is PD \( \Rightarrow m > 0 \). Therefore, if we choose \( b \in (0, m) \), then \( \Omega_b \subseteq B(0, \epsilon) \).

    - \( V(x) \) is continuous at origin \( \Rightarrow \) for any \( b > 0 \), there exists \( \delta > 0 \) such that \( |V(x) - V(0)| = V(x) < b, \forall x \in B(0, \delta) \). This implies that \( B(0, \delta) \subseteq \Omega_b \).
Proof of Lyapunov Stability Theorem II

- Second, show asymptotic stability under condition (3):
  - We know $V(x(t))$ decreases monotonically as $t \to \infty$ and $V(x(t)) \geq 0$, $\forall t$. Therefore, $c = \lim_{t \to \infty} V(x(t))$ exists. So it suffices to show $c = 0$. Let us use a contradiction argument.

  - Suppose $c \neq 0$. Then $c > 0$. Therefore, $x(t) \notin \Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}, \forall t$. We can choose $\beta > 0$ such that $B(0, \beta) \subseteq \Omega_c$ (due to continuity of $V$ at 0).

  - Now let $a = -\max_{\beta \leq \|x\| \leq \epsilon} \dot{V}(x)$. Since $V$ is ND, then $a > 0$

  - $V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))ds \leq V(x(0)) - a \cdot t < 0$ for sufficiently large $t$. $\Rightarrow$ contradiction!
Definition 2 (Exponential Lyapunov Function).

$V : D \rightarrow \mathbb{R}$ is called an Exponential Lyapunov Function (ELF) on $D \subset \mathbb{R}^n$ if

$\exists k_1, k_2, k_3, \alpha > 0$ such that

\[
\begin{cases}
    k_1 \|x\|^\alpha \leq V(x) \leq k_2 \|x\|^\alpha \\
    \mathcal{L}_f V(x) \leq -k_3 V(x)
\end{cases}
\]

Theorem 2 (ELF Theorem).

If system (1) has an ELF, then it is exponentially stable.
Stability Analysis Examples I

Example 1.

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 + x_1x_2 \\
\dot{x}_2 &= x_1 - x_2 - x_1^2 - x_2^3
\end{align*}
\]

Try \( V(x) = \|x\|^2 \)
Stability Analysis Examples II

Example 2.

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1 x_2 \\
\dot{x}_2 &= -x_2
\end{align*}
\]

- Can we find a simple quadratic Lyapunov function? First try: \( V(x) = x_1^2 + x_2^2 \)

- In fact, the system does not have any (global) polynomial Lyapunov function. But it is GAS with a Lyapunov function \( V(x) = \ln(1 + x_1^2) + x_2^2 \).
Lyapunov Stability of Linear Systems I

Consider autonomous linear system: \( \dot{x} = f(x) = Ax \).

- Recall: \( A \) asymptotically stable \iff \( \Re(\lambda_i) < 0 \) for all eigenvalues \( \lambda_i \) of \( A \).

When does a linear system have a Lyapunov function?

- Consider a quadratic Lyapunov function candidate: \( V(x) = x^T P x \), with \( P \in \mathbb{R}^{n \times n} \)
  - \( V \) is PD \( \Rightarrow \) \( P > 0 \)
  - \( \mathcal{L}_f V \) is ND \( \Rightarrow \)
Theorem 3 (Stability Conditions for Linear System).

For an autonomous Linear system \( \dot{x} = Ax \). The following statements are equivalent.

- System is (globally) asymptotically stable
- System is (globally) exponentially stable
- \( \text{Re}(\lambda_i) < 0 \) for all eigenvalues \( \lambda_i \) of \( A \)
- System has a quadratic Lyapunov function
- For any symmetric \( Q \succ 0 \), there exists a symmetric \( P \succ 0 \) that solves the following Lyapunov equation:
  \[ PA + A^T P = -Q \]
  and \( V(x) = x^T Px \) is a Lyapunov function of the system.
When There is a Lyapunov Function

• Converse Lyapunov Theorem for Asymptotic Stability

\[ \begin{align*}
\text{origin asymptotically stable; } & \Rightarrow \exists V \text{ s.t. } \\
 f \text{ is locally Lipschitz on } D & \Rightarrow \exists V \text{ s.t.} \\
\text{with region of attraction } R_A & \Rightarrow \exists V \text{ s.t.} \\
& V \text{ is continuous and PD on } R_A \\
& \mathcal{L}_f V \text{ is ND on } R_A \\
& V(x) \to \infty \text{ as } x \to \partial R_A
\end{align*} \]

• Converse Lyapunov Theorem for Exponential Stability

\[ \begin{align*}
\text{origin exponentially stable on } D; & \Rightarrow \exists \text{ an ELF } V \text{ on } D \\
f \text{ is } C^1
\end{align*} \]

• Proofs are involved especially for the converse theorem for asymptotic stability

• \textbf{IMPORTANT}: proofs of converse theorems often assume the knowledge of system solution and hence are not constructive.
What about Discrete Time Systems?

- So far, all our definitions, results, examples are given using continuous time dynamical system models.
- All of them have discrete-time counterparts. The ideas and conclusions are the "same" (in spirit)
- For example, given autonomous discrete-time system: \( x(k + 1) = f(x(k)) \) with \( f(0) = 0 \) (origin is an equilibrium).
  - Rate of change of a function \( V(x) \) along system trajectory can be defined as:
    \[
    \Delta_f V(x) \triangleq V(f(x)) - V(x)
    \]
  - Asymptotically stable requires:
    \[ V \text{ is PD and } \Delta_f V \text{ is ND} \]
  - Exponentially stable requires:
    \[ k_1 \|x\|^\alpha \leq V(x) \leq k_2 \|x\|^\alpha \quad \text{and} \quad \Delta_f V(x) \leq -k_3 V(x) \]
Concluding Remarks

- We have learned different notions of internal stability, e.g. stability in Lyapunov sense, asymptotic stability, globally asymptotic stability (G.A.S), exponential stability, globally exponential stability (G.E.S)

- Sufficient condition to ensure stability is often the existence of a properly defined Lyapunov function

- Key requirements for a Lyapunov function:
  - positive definite and is zero at the system equilibrium
  - decrease along system trajectory

- For linear system: G.A.S ⇔ G.E.S ⇔ Existence of a quadratic Lyapunov function

- The definitions and results in this lecture have sometimes been stated in simplified forms to facilitate presentation. More general versions can be found in standard textbooks on nonlinear systems (e.g. [Kha96])

- **Next Lecture**: Semidefinite Programming and computational stability analysis
References