Outline

• Switched Optimal Control Problems

• Embedding Principle and Chattering Lemma

• Solving Relaxed Switched Optimal Control
Switched Optimal Control Problems (1/5)

- Switched nonlinear systems:

\[ \dot{x}(t) = f_{\sigma(t)}(t, x(t), u(t)), \text{ where } \sigma(t) \in Q = \{1, \ldots, q\} \]  

(1)

- Hybrid control: \( \xi(t) = (u(t), \sigma(t)) \) with constraints:

\[ u(t) \in U \subset \mathbb{R}^m, \sigma(t) \in Q, \text{ where } U \text{ bounded and convex} \]

- State trajectory driven by \( \xi \): \( x(t; \xi) \), or simply \( x(t) \).

- A finite time horizon, w.l.g., assume \( \mathcal{T} = [0, 1] \).

- State trajectory constraint: \( h_j(x(t)) \leq 0, \forall j \in J = \{1, 2, \ldots n_s\}, \forall t \in \mathcal{T} \).

\[
\Rightarrow \max_{t \in \mathcal{T}, j \in J} h_j(x(t)) \leq 0
\]
Switched Optimal Control Problems (2/5)

- Cost function: \( J(x(1; \xi)) \)
  - only penalize terminal state.

- problems with running cost can be reduced to this form by introducing additional state.

\[
\text{In general, } J(\xi) = \int_0^1 l(x, j, t) \, dt + J_f(x(1; \xi))
\]

Introduce new state \( \dot{\xi} = (x, j, t) \Rightarrow \xi(t) = \int_0^t l(x, j, s) \, ds \)

\[\Rightarrow J(\xi) = J_f(\xi) \triangleq J_f([x(1; \xi)]) \]

- Overall constraint functional: \( \Psi(\xi) \triangleq \max_{j,t} \psi_{j,t}(\xi) \).

- Notations to emphasize dependence on \( \xi \):
  - \( \phi_t(\xi) \triangleq x(t; \xi), \psi_{j,t}(\xi) \triangleq h_j(x(t; \xi)), J(\xi) \triangleq J(x(1; \xi)) \)

  \[
  \begin{bmatrix}
  \dot{\xi} \\
  \dot{\xi}
  \end{bmatrix} = \begin{bmatrix}
  x \\
  l(x, j, t)
  \end{bmatrix}
  \]

  new state
  \[
  \begin{bmatrix}
  x \\
  z
  \end{bmatrix}
  \]

  new state
Switched Optimal Control Problems (3/5)

**Assumption 1.**

- $f_i(t, x, u), h_j(x), J(x)$ are Lipschitz continuous w.r.t all arguments
- $\frac{\partial f_i}{\partial x}(t, x, u), \frac{\partial f_i}{\partial u}(t, x, u), \frac{\partial h_j}{\partial x}(x), \frac{\partial J}{\partial x}(x)$ exist and are Lipschitz continuous w.r.t. all arguments

- An equivalent way to write system dynamics: $\dot{x} = \sum_{i=1}^{q} d_i(t) f_i(t, x, u)$

where $d(t) = [d_1(t), \ldots, d_q(t)]^T$ is a corner of the $q$-simplex: $\Sigma_p = \left\{ (d_1, \ldots, d_q) \in \{0, 1\}^q \mid \sum_{i=1}^{q} d_i = 1 \right\}$

\begin{align*}
\text{(q=3, s=4)} & \quad (d_1, d_2, d_3, d_4) \\
\text{(q=2, s=4)} & \quad \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}
\end{align*}
Switched Optimal Control Problems (4/5)

- Control Spaces:
  - We say $f : [0, 1] \rightarrow F$ belongs to $L_2([0, 1], F')$ if
    \[
    \|f\|_{L_2} = \left(\int_0^1 \|f(t)\|_2^2 dt\right)^{1/2} < \infty
    \]
  - Continuous input space: $\mathcal{U} = L_2([0, 1], U)$
  - Discrete input space: $\mathcal{D}_p = L_2([0, 1], \Sigma_p)$
  - Overall optimization space: $\mathcal{X} = L_2([0, 1], \mathbb{R}^m) \times L_2([0, 1], \mathbb{R}^q)$
  - Pure optimization space: $\mathcal{X}_p = \mathcal{U} \times \mathcal{D}_p$
Switched Optimal Control Problems (5/5)

• (Pure) Switched Optimal Control Problem:

\[
P_p : \quad J_p^* = \begin{cases} 
\inf_{\xi} J(\xi) \\
\text{subj. to } \Psi(\xi) \leq 0, \quad \xi \in \mathcal{X}_p
\end{cases}
\] (2)

• **Challenges:** space \( \mathcal{X}_p = \mathcal{U} \times \mathcal{D} \) is not a vector space due to \( \mathcal{D} \), on which gradient of \( J \) and \( \Psi \) are not well defined.
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- Solving Relaxed Switched Optimal Control
Relaxed System (1/2)

- The **Key idea** for solving $P_p$ is to “embed” the switched systems into a larger class of nonlinear systems for which $d$ takes values inside the entire $q$-simplex (not just the corner points)

- $q$-simplex: $\Sigma_q = \{(d_1, \ldots, d_q) \in [0, 1]^q | \sum_{i=1}^{q} d_i = 1\}$.

- **Relaxed System:**

\[
\dot{x}(t) = \sum_{i=Q} d_i(t) f_i(t, x(t), u(t)), \text{ with } x(0) = x_0. \tag{3}
\]

- $d(t) \in \Sigma_p \Rightarrow$ original switched systems

- $d(t) \in \Sigma_r \Rightarrow$ relaxed switched systems

- The set of all trajectories of the switched system is contained in that of the relaxed system.
Relaxed System (2/2)

- **Relaxed control spaces:**
  - Relaxed discrete input space: \( \mathcal{D}_r = L^2([0, 1], \Sigma_r^q) \)
  - Relaxed optimization space: \( \mathcal{X}_r = \mathcal{U} \times \mathcal{D}_r \)

- **Relaxed Switched Optimal Control Problem** \( \mathcal{P}_r \):

\[
\mathcal{P}_r : \quad J^*_r = \begin{cases}
\inf_{\xi} & J(\xi) \\
\text{subj. to} & \Psi(\xi) \leq 0, \quad \xi \in \mathcal{X}_r
\end{cases}
\]  

(4)

- Obviously: \( J^*_r \leq J^*_p \)

- Problem \( \mathcal{P}_r \) can be solved using classical optimal control methods
Embedding-Based Approach (1/5)

• Solution through Embedding:
  - Solve $P_r$, resulting in $\xi_r^* \in X_r$

\[
\xi_r = \begin{bmatrix} u \\ \phi_r \end{bmatrix}
\]

- project back to pure space: $\Gamma(\xi_r^*) \rightarrow \xi_p^* \in X_p$

• Question: can we find a good projection without losing much on performance?

  - Answer: Yes.

  - The cost of any relaxed control input $\xi_r$ can be approximated arbitrarily well by a pure control input $\xi_p$.

  - This is known as the Chattering Lemma.
Lemma 1 (Chattering Lemma).
\[ \forall \epsilon > 0, \forall \xi \in \mathcal{X}_r, \exists \xi_p \in \mathcal{X}_p \text{ s.t. } \| \phi_t(\xi_r) - \phi_t(\xi_p) \|_2 \leq \epsilon \]

Proof of chattering lemma
- We show the case with \( M = 2 \) with no continuous control. The result can be easily extended to the general case.

- Given an arbitrary \( \alpha(t) \in [0, 1] \). Let \( \phi_t \) be the solution to

\[
\dot{x} = f(t, x(t)) = \alpha(t)f_0(t, x(t)) + (1 - \alpha(t))f_1(t, x(t)).
\]

- We want to construct another \( \tilde{\alpha}(t) \in \{0, 1\} \) so that the corresponding solution \( \tilde{\phi}_t \) to

\[
\dot{x}(t) = \tilde{f}(t, x(t)) = \tilde{\alpha}(t)f_0(t, x(t)) + (1 - \tilde{\alpha}(t))f_1(t, x(t))
\]

satisfies the desired inequality.
Embedding-Based Approach (3/5)

- Given partition $0 = t_0 < t_1 < \cdots < t_n = 1$ with $t_{k+1} - t_k = \Delta t$. Choose $t_k' \in (t_k, t_{k+1})$ such that $\int_{t_k}^{t_k'} (1 - \alpha(\tau)) d\tau = \int_{t_k'}^{t_{k+1}} \alpha(\tau) d\tau$. We propose to construct

$$
\tilde{\alpha}(t) = \begin{cases} 
0 & \text{if } t \in [t_k, t_k') \\
1 & \text{if } t \in [t_k', t_{k+1}) 
\end{cases}
$$

- Now let’s derive a bound for $\|\phi_t - \tilde{\phi}_t\|$. Note that

$$
\phi_t - \tilde{\phi}_t = \int_0^t f(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) d\tau \\
= \int_0^t \left[ f(\tau, \phi_\tau) - \tilde{f}(\tau, \phi_\tau) \right] d\tau + \int_0^t \left[ \tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) \right] d\tau \tag{5}
$$

Define $f^\Delta(t) = f_1(t, \phi_t) - f_0(t, \phi_t), \forall t \in [0, 1]$. 

Embedding-Based Approach (4/5)

- first term of (5)

\[
\sum_k \left( \int_{t_k}^{t'_k} (1 - \alpha(\tau)) f^\Delta(\tau) d\tau - \int_{t'_k}^{t_{k+1}} \alpha(\tau) f^\Delta(\tau) d\tau \right)
\]

\[
= \sum_k \left( f^\Delta(t_k) \left[ \int_{t_k}^{t'_k} (1 - \alpha(\tau)) d\tau \right] - f^\Delta(t_k) \left[ \int_{t'_k}^{t_{k+1}} \alpha(\tau) d\tau \right] + e_k \right) = \sum_k e_k
\]

where

\[
\|e_k\| = \| \int_{t_k}^{t'_k} (1 - \alpha(\tau))(f^\Delta(\tau) - f^\Delta(t_k)) d\tau - \int_{t'_k}^{t_{k+1}} \alpha(\tau)(f^\Delta(\tau) - f^\Delta(t_k)) d\tau \| \leq \tilde{L} \Delta t^2
\]

Hence, choose \( \Delta t \) small enough: first term \( \leq \frac{\epsilon}{2} \)
Embedding-Based Approach (5/5)

- second term of (5):

\[
\| \int_0^t \left[ \tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) \right] d\tau \| \leq \int_0^t L \| \phi_\tau - \tilde{\phi}_\tau \| d\tau
\]

\[
\Rightarrow \| \phi_t - \tilde{\phi}_t \| \leq \epsilon' + L \int_0^t \| \phi_\tau - \tilde{\phi}_\tau \| d\tau
\]

by Gronwall inequality \( \leq \epsilon \)

- The proof is constructive, but is not the best way to construct pure control input.

- A more effective projection strategy based on wavelet can be found in [Vas+13a; Vas+13b].
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How to solve the relaxed problem? (1/3)

• Now the question comes back to how to solve the relaxed optimal control $\xi^*_r$?

• This is a classical optimal control problem. Analytical solution usually does not exist.

• $\xi^*_r$ can be found using gradient type of algorithms in function space.
How to solve the relaxed problem? (2/3)

- The key is to compute the directional derivative: $DJ(\xi; \eta)$ and $D\psi_{j,t}(\xi; \eta)

\[
\frac{\partial}{\partial \eta} \left( J(\xi + d\eta) - J(\xi) \right)
\]

- If $DJ(\xi; \eta) < 0$, we can decrease cost by moving in $\eta$ direction

- If $\max_{j,t} D\psi_{j,t}(\xi, \eta) < 0$, we can reduce infeasibility by moving in $\eta$ direction

- Once we have $DJ$ and $D\psi_{j,t}$, numerous algorithms are available to find local min (See [Pol12])

- Since $d$ can be varied continuously for problem $P_r$. We move $d$ into $u$ and deal with typical nonlinear system: $\dot{x} = f(t, x, u)$. In this case, $\xi = u$. 

\[
\begin{align*}
    u & \in \mathcal{U} \\
    d & \in \mathcal{D}_r
\end{align*}
\]
How to solve the relaxed problem? (3/3)

- Directional derivative of state trajectory \( D\phi_t(u; \eta) \) is given by

\[
D\phi_t(u; \eta) = \int_0^t \Phi(t, \tau; u) \left( \frac{\partial f}{\partial u} (\tau, \phi_\tau(u), u(\tau)) \cdot \eta(\tau) \right) d\tau
\]

where \( \Phi(t, \tau; u) \) is the unique solution to

\[
\frac{\partial \Phi}{\partial t} (t, \tau) = \frac{\partial f}{\partial x} (t, \phi_t(u), u(t)) \Phi(t, \tau)
\]

- With \( D\phi_t(u; \eta) \), we can easily find

\[
DJ(u; \eta) = \frac{\partial J}{\partial x} (\phi_{tf}(u)) D\phi_{tf}(u; \eta), \quad \psi_{j,t}(u; \eta) = \frac{\partial h_j}{\partial x} (\phi_t(u)) D\phi_t(u; \eta)
\]
Example 1 (Switching Time Optimization Problem).

Consider switched linear system

\[
\dot{x} = A_{\sigma(t)}x(t), \quad \sigma(t) = \begin{cases} 
1 & t \in [0, u_1) \\
2 & t \in [u_1, u_2) \\
1 & t \in [u_2, 10] 
\end{cases},
\]

\[
A_1 = \frac{1}{10} \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \frac{1}{10} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

where mode sequence is known and the transition time \( u = [u_1, u_2] \) needs to be optimized with cost function \( J(u) = \frac{1}{20} \int_0^{10} \|x(t)\|^2 dt \).
Example I: Switching Time Optimization Problem (2/4)

Solution to Example 1

- First note that this problem involves nontrivial running cost. To address this issue, we first augment the state space by introducing $z \in \mathbb{R}$ with dynamics 

$$
\dot{z}(t) = x_1^2(t) + x_2^2(t).
$$

It directly follows that 

$$
z(1) = \int_0^{10} \|x\|^2 dt.
$$

Let 

$$
\tilde{x}(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},
$$

and the augmented vector field $f$ is given as follows

$$
\dot{\tilde{x}}(t) = f(t, \tilde{x}, u) = \begin{cases} 
A_1 x(t) \\ A_2 x(t) \\ A_1 x(t) \\
\|x(t)\|^2 \|x(t)\|^2 \|x(t)\|^2 \\
\end{cases} \triangleq f_1(t, \tilde{x}, u), \quad t \in [0, u_1) \\
f_2(t, \tilde{x}, u), \quad t \in [u_1, u_2) \\
f_1(t, \tilde{x}, u), \quad t \in [u_2, 10] \\
\end{cases}
$$

(9)

Augmented initial state is given by 

$$
\tilde{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and the cost is redefined as } J(\tilde{x}(1)) = \frac{1}{20} z(1).
$$
Example 1: Switching Time Optimization Problem (3/4)

- From above discussion (6)-(8), we know that in order to obtain $D\phi_t(u; \eta)$, it remains to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial \tilde{x}}$.

- For $t \in (0, u_2)$, we have

  $$f(t, \tilde{x}, u) = f_1(t, \tilde{x}, u) + U(t - u_1)[f_2(t, \tilde{x}, u) - f_1(t, \tilde{x}, u)],$$

  where $U(a)$ is a step function. It follows that

  $$\frac{\partial f}{\partial u_1} = \delta(t - u_1)[f_1(t, \tilde{x}, u_1) - f_2(t, \tilde{x}, u_1)].$$

  Similarly,

  $$\frac{\partial f}{\partial u_2} = \delta(t - u_2)[f_2(t, \tilde{x}, u_2) - f_1(t, \tilde{x}, u_2)].$$

- $\frac{\partial f}{\partial \tilde{x}} = \begin{bmatrix} A_i & 0 \\ 2x_1(t) & 2x_2(t) \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$. 
Example I: Switching Time Optimization Problem (4/4)

Plug into (6), we have

\[
D\phi_t(u; \eta) = 1_{t \geq u_1} \Phi(t, u_1)\left[f_1(\tilde{x}(u_1)) - f_2(\tilde{x}(u_1))\right]\eta_1 \\
+ 1_{t \geq u_2} \Phi(t, u_2)\left[f_2(\tilde{x}(u_2)) - f_1(\tilde{x}(u_2))\right]\eta_2
\]

where \(1_A\) is the indicator function of set \(A\). Therefore,

\[
D\phi_1(u; \eta) = \Phi(t, u_1)\left[f_1(\tilde{x}(u_1)) - f_2(\tilde{x}(u_1))\right]\eta_1 \\
+ \Phi(t, u_2)\left[f_2(\tilde{x}(u_2)) - f_1(\tilde{x}(u_2))\right]\eta_2
\]

Therefore \(DJ(u; \eta)\) is given by

\[
DJ(u; \eta) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{20} \end{bmatrix}^T D\phi_1(u; \eta).
\]

Implement the above results with initial condition given by \(u_0 = [0.3, 0.5]\) and a simple gradient descent algorithm with fixed step size, we have the optimal transition time is given by \([0.2057, 0.7159]\) with a cost of 24.527.
Example II: Quadrotor (1/6)

Example 2 (Switched optimal control problem: Quadrotor Model).

\[ \dot{x} = f_i(x, u) \] where

\[ f_1(x, u) = \begin{bmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \\ \frac{\sin x_3(t)}{M} (u(t) + Mg) \\ \frac{\cos x_3(t)}{M} (u(t) + Mg) - g \end{bmatrix}, \]

\[ f_2(x, u) = \begin{bmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \\ g \sin x_3(t) \\ g \cos x_3(t) - g \end{bmatrix}, \]

\[ f_3(x, u) = \begin{bmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \\ g \sin x_3(t) \\ g \cos x_3(t) - g \end{bmatrix}. \]

Cost function: \[ \int_0^{t_f} 5u^2(t) dt + 5(x_1(t_f) - 6)^2 + 5(x_2(t_f) - 1)^2 + \sin \left( \frac{x_3(t_f)}{2} \right) \]

constraint: \( u(t) \in [0, 10^{-3}] \) and \( x_2(t) \geq 0 \)
Solution to Example 2:

- Similar to Example 1, we first augment the state space by defining $z$ such that $\dot{z}(t) = 5u^2(t)$ and $z(0) = 0$. Let the overall augmented state be $\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}$ and let $\tilde{f}_i = \begin{bmatrix} f_i \\ 5u^2 \end{bmatrix}$ be the augmented vector field corresponding to each $f_i$.

- The augmented relaxed control variable is given by $\tilde{u} = \begin{bmatrix} u \\ \alpha \\ \beta \end{bmatrix}^T$, and the augmented vector field is given by $f = \alpha \tilde{f}_1 + \beta \tilde{f}_2 + (1 - \alpha - \beta) \tilde{f}_3$.

- Cost function and constraint are then given by

$$J = z(t_f) + 5(x_1(t_f) - 6)^2 + 5(x_2(t_f) - 1)^2 + \sin\left(\frac{x_3(t_f)}{2}\right),$$

and

$$\psi_t(\tilde{u}) = h(x(t)) = -x_2(t),$$

respectively.
Example II: Quadrotor (3/6)

- Let \( \tilde{\eta} = \begin{bmatrix} \eta & \alpha' & \beta' \end{bmatrix}^T \) be another control signal, from above discussion (6)-(8), we know that in order to obtain \( DJ(\tilde{u}; \eta) \), it remains to compute \( D\phi_t(\tilde{u}; \tilde{\eta}) \), \( \frac{\partial J}{\partial x}(\phi_{t_f}(\tilde{u})) \) and \( \frac{\partial h}{\partial x}(\phi_t(\tilde{u})) \).

- The last two terms are easy to obtain, which are given by

\[
\frac{\partial J}{\partial x}(\phi_{t_f}(\tilde{u})) = \begin{bmatrix} 10(x_1(t_f) - 6) & 10(x_2(t_f) - 1) & \frac{1}{2} \cos \frac{x_3(t_f)}{2} & 0 & 0 & 0 & 1 \end{bmatrix}^T
\]

and

\[
\frac{\partial h}{\partial x}(\phi_t(\tilde{u})) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\]

- Now, we focus on \( D\phi_t(\tilde{u}; \tilde{\eta}) \). Similar to the previous example, in order to compute \( D\phi_t(\tilde{u}; \tilde{\eta}) \), we need to compute \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial \tilde{u}} \).
Example II: Quadrotor (4/6)

\[-\frac{\partial f}{\partial \tilde{x}} = \alpha \frac{\partial \tilde{f}_1}{\partial \tilde{x}} + \beta \frac{\partial \tilde{f}_2}{\partial \tilde{x}} + (1 - \alpha - \beta) \frac{\partial \tilde{f}_3}{\partial \tilde{x}}, \text{ where}\]

\[
\frac{\partial \tilde{f}_1}{\partial \tilde{x}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{u+Mg}{M} \cos x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{u+Mg}{M} \sin x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\frac{\partial \tilde{f}_2}{\partial \tilde{x}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & g \cos x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -g \sin x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Example II: Quadrotor (5/6)

\[
\frac{\partial \tilde{f}_3}{\partial \tilde{x}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & g \cos x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -g \sin x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

Hence,

\[
\frac{\partial f}{\partial \tilde{x}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{u+M}M g \cos x_3 + (1 - \alpha)g \cos x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{u+M}M g \sin x_3 - (1 - \alpha)g \sin x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

- \frac{\partial f}{\partial \tilde{u}} = \begin{bmatrix}
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial \alpha} & \frac{\partial f}{\partial \beta}
\end{bmatrix}, \text{ where } \frac{\partial f}{\partial u} = \alpha \frac{\partial \tilde{f}_1}{\partial u} + \beta \frac{\partial \tilde{f}_2}{\partial u} + (1 - \alpha - \beta) \frac{\partial \tilde{f}_3}{\partial u}.
Example II: Quadrotor (6/6)

\[
\begin{align*}
\frac{\partial \tilde{f}_1}{\partial u} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin x_3 \\ \frac{M}{\cos x_3} \\ M \\ 0 \\ 10u \end{bmatrix}, \\
\frac{\partial \tilde{f}_2}{\partial u} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{L}{I} \\ 10u \end{bmatrix}, \\
\frac{\partial \tilde{f}_3}{\partial u} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L}{I} \\ 10u \end{bmatrix}, \\
\end{align*}
\]

\[
\frac{\partial f}{\partial \alpha} = \tilde{f}_1 - \tilde{f}_3, \quad \frac{\partial f}{\partial \beta} = \tilde{f}_2 - \tilde{f}_3
\]

Hence,

\[
\frac{\partial f}{\partial \tilde{u}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha \frac{\sin x_3}{M} & \frac{\sin x_3}{M} u & 0 \\ \alpha \frac{\cos x_3}{M} & \frac{\cos x_3}{M} u & 0 \\ (1 - \alpha - 2\beta) \frac{L}{I} & -\frac{Lu}{I} & -2\frac{Lu}{I} \\ 10u & 0 & 0 \end{bmatrix}
\]

Up to now, we have all the elements for computing the directional derivative \( DJ(\tilde{u}; \eta) \).
References

