Outline

• Switched Optimal Control Problems

• Embedding Principle and Chattering Lemma

• Solving Relaxed Switched Optimal Control
Switched Optimal Control Problems (1/5)

• Switched nonlinear systems:

\[ \dot{x}(t) = f_{\sigma(t)}(t, x(t), u(t)), \text{ where } \sigma(t) \in Q = \{1, \ldots, q\} \]  \hspace{1cm} (1)

• Hybrid control: \( \xi(t) = (u(t), \sigma(t)) \) with constraints:

\[ u(t) \in \mathcal{U} \subset \mathbb{R}^m, \sigma(t) \in Q, \text{ where } \mathcal{U} \text{ bounded and convex} \]

• State trajectory driven by \( \xi \): \( x(t; \xi) \), or simply \( x(t) \).

• A finite time horizon, w.l.g., assume \( \mathcal{T} = [0, 1] \).

• State trajectory constraint: \( h_j(x(t)) \leq 0, \forall j \in \mathcal{J} = \{1, 2, \ldots, n_s\}, \forall t \in \mathcal{T} \).
Switched Optimal Control Problems (2/5)

- Cost function: $J(x(1; \xi))$
  - only penalize terminal state.
  - problems with running cost can be reduced to this form by introducing additional state.

- Notations to emphasize dependence on $\xi$:
  - $\phi_t(\xi) \triangleq x(t; \xi)$, $\psi_{j,t}(\xi) \triangleq h_j(x(t; \xi))$, $J(\xi) \triangleq J(x(1; \xi))$
  - Overall constraint functional: $\Psi(\xi) \triangleq \max_{j,t} \psi_{j,t}(\xi)$. 
Switched Optimal Control Problems (3/5)

Assumption 1.

- \( f_i(t, x, u), h_j(x), J(x) \) are Lipchitz continuous w.r.t all arguments
- \( \frac{\partial f_i}{\partial x}(t, x, u), \frac{\partial f_i}{\partial u}(t, x, u), \frac{\partial h_j}{\partial x}(x), \frac{\partial J}{\partial x}(x) \) exist and are Lipchitz continuous w.r.t. all arguments

• An equivalent way to write system dynamics:

\[
\dot{x} = f(t, x, u, d) \triangleq \sum_{i=1}^{q} d_i(t) f_i(t, x, u)
\]

where \( d(t) = [d_1(t), \ldots, d_q(t)]^T \) is a corner of the \( q \)-simplex:

\[
\sum_p^q = \left\{ (d_1, \ldots, d_q) \in \{0, 1\}^q | \sum_{i=1}^{q} d_i = 1 \right\}
\]
Switched Optimal Control Problems (4/5)

- Control Spaces:
  - We say $f : [0, 1] \rightarrow F$ belongs to $L_2([0, 1], F)$ if
    \[
    \|f\|_{L_2} = \left( \int_0^1 \|f(t)\|^2 dt \right)^{1/2} < \infty
    \]
  - Continuous input space: $U = L_2([0, 1], U)$
  - Discrete input space: $D_p = L_2([0, 1], \Sigma^q_p)$
  - Overall optimization space: $\mathcal{X} = L_2([0, 1], \mathbb{R}^m) \times L_2([0, 1], \mathbb{R}^q)$
  - Pure optimization space: $\mathcal{X}_p = U \times D_p$
Switched Optimal Control Problems (5/5)

• (Pure) Switched Optimal Control Problem:

\[
\mathcal{P}_p : \quad J^*_p = \begin{cases} 
\inf_{\xi} & J(\xi) \\
\text{subj. to} & \Psi(\xi) \leq 0, \quad \xi \in \mathcal{X}_p 
\end{cases}
\]  

(2)

• **Challenges**: space \( \mathcal{X}_p = \mathcal{U} \times \mathcal{D} \) is not a vector space due to \( \mathcal{D} \), on which gradient of \( J \) and \( \Psi \) are not well defined.
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Relaxed System (1/2)

- The **Key idea** for solving $\mathcal{P}_p$ is to “embed” the switched systems into a larger class of nonlinear systems for which $d$ takes values inside the entire $q$-simplex (not just the corner points).

- $q$-simplex: $\Sigma^q_r = \{(d_1, \ldots, d_q) \in [0, 1]^q | \sum_{i=1}^q d_i = 1\}$.

- **Relaxed System:**

  \[
  \dot{x}(t) = \sum_{i=\mathcal{Q}} d_i(t) f_i(t, x(t), u(t)), \text{ with } x(0) = x_0. \tag{3}
  \]

- $d(t) \in \Sigma^q_p \Rightarrow$ original switched systems

- $d(t) \in \Sigma^q_r \Rightarrow$ relaxed switched systems

- The set of all trajectories of the switched system is contained in that of the relaxed system.
Relaxed System (2/2)

- **Relaxed control spaces:**
  - Relaxed discrete input space: $\mathcal{D}_r = L^2([0, 1], \Sigma^q_r)$
  - Relaxed optimization space: $\mathcal{X}_r = \mathcal{U} \times \mathcal{D}_r$

- **Relaxed Switched Optimal Control Problem $\mathcal{P}_r$:**

\[
\mathcal{P}_r : \quad J_r^* = \begin{cases} 
\inf_{\xi} & J(\xi) \\
\text{subj. to} & \Psi(\xi) \leq 0, \quad \xi \in \mathcal{X}_r
\end{cases}
\]  

- Obviously: $J_r^* \leq J_p^*$

- Problem $\mathcal{P}_r$ can be solved using classical optimal control methods
Embedding-Based Approach (1/5)

• Solution through Embedding:
  - Solve $\mathcal{P}_r$, resulting in $\xi^*_r \in \mathcal{X}_r$
  - project back to pure space: $\Gamma(\xi^*_r) \rightarrow \xi^*_p \in \mathcal{X}_p$

• Question: can we find a good projection without losing much on performance?
  - Answer: Yes.
  - The cost of any relaxed control input $\xi_r$ can be approximated arbitrarily well by a pure control input $\xi_p$.
  - This is known as the Chattering Lemma.
Embedding-Based Approach (2/5)

Lemma 1 (Chattering Lemma).
\[ \forall \epsilon > 0, \forall \xi \in \mathcal{X}_r, \exists \xi_p \in \mathcal{X}_p \text{ s.t. } \| \phi_t(\xi_r) - \phi_t(\xi_p) \|_2 \leq \epsilon \]

Proof of chattering lemma
- We show the case with \( M = 2 \) with no continuous control. The result can be easily extended to the general case.

- Given an arbitrary \( \alpha(t) \in [0, 1] \). Let \( \phi_t \) be the solution to

\[
\dot{x} = f(t, x(t)) = \alpha(t) f_0(t, x(t)) + (1 - \alpha(t)) f_1(t, x(t))
\]

- We want to construct another \( \tilde{\alpha}(t) \in \{0, 1\} \) so that the corresponding solution \( \tilde{\phi}_t \) to

\[
\dot{x}(t) = \tilde{f}(t, x(t)) = \tilde{\alpha}(t) f_0(t, x(t)) + (1 - \tilde{\alpha}(t)) f_1(t, x(t))
\]

satisfies the desired inequality.
Embedding-Based Approach (3/5)

- Given partition \(0 = t_0 < t_1 < \cdots < t_n = 1\) with \(t_{k+1} - t_k = \Delta t\). Choose \(t'_k \in (t_k, t_{k+1})\) such that \(\int_{t_k}^{t'_k} (1 - \alpha(\tau)) d\tau = \int_{t'_k}^{t_{k+1}} \alpha(\tau) d\tau\). We propose to construct

\[
\tilde{\alpha}(t) = \begin{cases} 
0 & \text{if } t \in [t_k, t'_k) \\
1 & \text{if } t \in [t'_k, t_{k+1})
\end{cases}
\]

- Now let’s derive a bound for \(\|\phi_t - \tilde{\phi}_t\|\). Note that

\[
\phi_t - \tilde{\phi}_t = \int_0^t f(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) d\tau
\]

\[
= \int_0^t \left[ f(\tau, \phi_\tau) - \tilde{f}(\tau, \phi_\tau) \right] d\tau + \int_0^t \left[ \tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) \right] d\tau \tag{5}
\]

Define \(f^\Delta(t) = f_1(t, \phi_t) - f_0(t, \phi_t), \forall t \in [0, 1]\).
Embedding-Based Approach (4/5)

- first term of (5)

\[
\begin{align*}
&= \sum_k \left( \int_{t_k}^{t'_k} (1 - \alpha(\tau)) f^\Delta(\tau) d\tau - \int_{t'_k}^{t_{k+1}} \alpha(\tau) f^\Delta(\tau) d\tau \right) \\
&= \sum_k \left( f^\Delta(t_k) \left[ \int_{t_k}^{t'_k} (1 - \alpha(\tau)) d\tau \right] - f^\Delta(t_k) \left[ \int_{t'_k}^{t_{k+1}} \alpha(\tau) d\tau \right] + e_k \right) = \sum_k e_k
\end{align*}
\]

where

\[
\| e_k \| = \| \int_{t_k}^{t'_k} (1 - \alpha(\tau))(f^\Delta(\tau) - f^\Delta(t_k)) d\tau - \int_{t'_k}^{t_{k+1}} \alpha(\tau)(f^\Delta(\tau) - f^\Delta(t_k)) d\tau \| \leq \tilde{L} \Delta t^2
\]

Hence, choose \( \Delta t \) small enough: first term \( \leq \frac{\epsilon}{2} \)
Embedding-Based Approach (5/5)

- second term of (5):

\[
\| \int_0^t \left[ \tilde{f}(\tau, \phi_\tau) - \tilde{f}(\tau, \tilde{\phi}_\tau) \right] d\tau \| \leq \int_0^t L \| \phi_\tau - \tilde{\phi}_\tau \| d\tau
\]

\[
\Rightarrow \| \phi_t - \tilde{\phi}_t \| \leq \epsilon' + L \int_0^t \| \phi_\tau - \tilde{\phi}_\tau \| d\tau
\]

by Gronwall inequality \( \leq \epsilon \)

- The proof is constructive, but is not the best way to construct pure control input.

- A more effective projection strategy based on wavelet can be found in [vasudevan2013consistent1; vasudevan2013consistent2].
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How to solve the relaxed problem? (1/3)

• Now the question comes back to how to solve the relaxed optimal control $\xi^*_r$?

• This is a classical optimal control problem. Analytical solution usually does not exist.

• $\xi^*_r$ can be found using gradient type of algorithms in function space.
How to solve the relaxed problem? (2/3)

- The key is to compute the directional derivative: $DJ(\xi; \eta)$ and $D\psi_{j,t}(\xi; \eta)$

  - If $DJ(\xi; \eta) < 0$, we can decrease cost by moving in $\eta$ direction

  - If $\max_{j,t} D\psi_{j,t}(\xi, \eta) < 0$, we can reduce infeasibility by moving in $\eta$ direction

  - Once we have $DJ$ and $D\psi_{j,t}$, numerous algorithms are available to find local min (See [polak2012optimization])

- Since $d$ can be varied continuously for problem $\mathcal{P}_r$. We move $d$ into $u$ and deal with typical nonlinear system: $\dot{x} = f(t, x, u)$. In this case, $\xi = u$. 
How to solve the relaxed problem? (3/3)

- Directional derivative of state trajectory $D\phi_t(u; \eta)$ is given by

$$D\phi_t(u; \eta) = \int_0^t \Phi(t, \tau; u) \left( \frac{\partial f}{\partial u}(\tau, \phi_\tau(u), u(\tau)) \cdot \eta(\tau) \right) d\tau$$

where $\Phi(t, \tau; u)$ is the unique solution to

$$\frac{\partial \Phi}{\partial t}(t, \tau) = \frac{\partial f}{\partial x}(t, \phi_t(u), u(t)) \Phi(t, \tau)$$

- With $D\phi_t(u; \eta)$, we can easily find

$$DJ(u; \eta) = \frac{\partial J}{\partial x}(\phi_{t_f}(u)) D\phi_{t_f}(u; \eta), \quad D\psi_{j,t}(u; \eta) = \frac{\partial h_j}{\partial x}(\phi_t(u)) D\phi_t(u; \eta)$$
Example 1 (Switching Time Optimization Problem).

Consider switched linear system

\[ \dot{x} = A_{\sigma(t)} x(t), \quad \sigma(t) = \begin{cases} 
1 & t \in [0, u_1) \\
2 & t \in [u_1, u_2) \\
1 & t \in [u_2, 10] 
\end{cases} \]

\[ A_1 = \frac{1}{10} \begin{bmatrix} -1 & 0 \\
1 & 2 \end{bmatrix}, \quad A_2 = \frac{1}{10} \begin{bmatrix} 1 & 1 \\
1 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\
1 \end{bmatrix} \]

where mode sequence is known and the transition time \( u = [u_1, u_2] \) needs to be optimized with cost function

\[ J(u) = \frac{1}{20} \int_0^{10} \|x(t)\|^2 dt. \]
Solution to Example 1

- First note that this problem involves nontrivial running cost. To address this issue, we first augment the state space by introducing $z \in \mathbb{R}$ with dynamics $\dot{z}(t) = x_1^2(t) + x_2^2(t)$. It directly follows that $z(1) = \int_0^{10} \|x\|^2 dt$. Let

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},$$

and the augmented vector field $f$ is given as follows

$$\dot{\tilde{x}}(t) = f(t, \tilde{x}, u) = \begin{cases} 
A_1 x(t) \\ \|x(t)\|^2 
\end{cases} \triangleq f_1(t, \tilde{x}, u), \quad t \in [0, u_1) \\
A_2 x(t) \\ \|x(t)\|^2 
\end{cases} \triangleq f_2(t, \tilde{x}, u), \quad t \in [u_1, u_2) \\
A_1 x(t) \\ \|x(t)\|^2 
\end{cases} \triangleq f_1(t, \tilde{x}, u), \quad t \in [u_2, 10]$$

Augmented initial state is given by $\tilde{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and the cost is redefined as $J(\tilde{x}(1)) = \frac{1}{20} z(1)$. 
Example I: Switching Time Optimization Problem (3/4)

- From above discussion (6)-(8), we know that in order to obtain $D\phi_t(u; \eta)$, it remains to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial \tilde{x}}$.

- For $t \in (0, u_2)$, we have

$$f(t, \tilde{x}, u) = f_1(t, \tilde{x}, u) + U(t - u_1)[f_2(t, \tilde{x}, u) - f_1(t, \tilde{x}, u)],$$

where $U(a)$ is a step function. It follows that

$$\frac{\partial f}{\partial u_1} = \delta(t - u_1)[f_1(t, \tilde{x}, u_1) - f_2(t, \tilde{x}, u_1)].$$

Similarly,

$$\frac{\partial f}{\partial u_2} = \delta(t - u_2)[f_2(t, \tilde{x}, u_2) - f_1(t, \tilde{x}, u_2)].$$

- $\frac{\partial f}{\partial \tilde{x}} = \begin{bmatrix} A_i & 0 \\ 2x_1(t) & 2x_2(t) \end{bmatrix} \in \mathbb{R}^{3 \times 3}$. 
Example I: Switching Time Optimization Problem (4/4)

Plug into (6), we have

\[ D\phi_t(u; \eta) = 1_{t \geq u_1} \Phi(t, u_1)[f_1(\tilde{x}(u_1)) - f_2(\tilde{x}(u_1))]\eta_1 \]
\[ + 1_{t \geq u_2} \Phi(t, u_2)[f_2(\tilde{x}(u_2)) - f_1(\tilde{x}(u_2))]\eta_2 \]

where \(1_A\) is the indicator function of set \(A\). Therefore,

\[ D\phi_1(u; \eta) = \Phi(t, u_1)[f_1(\tilde{x}(u_1)) - f_2(\tilde{x}(u_1))]\eta_1 \]
\[ + \Phi(t, u_2)[f_2(\tilde{x}(u_2)) - f_1(\tilde{x}(u_2))]\eta_2 \]

Therefore \(DJ(u; \eta)\) is given by

\[ DJ(u; \eta) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{20} \end{bmatrix}^T D\phi_1(u; \eta). \]

Implement the above results with initial condition given by \(u_0 = [0.3, 0.5]\) and a simple gradient descent algorithm with fixed step size, we have the optimal transition time is given by \([0.2057, 0.7159]\) with a cost of 24.527.
Example II: Quadrotor (1/6)

Example 2 (Switched optimal control problem: Quadrotor Model).

\[ \dot{x} = f_i(x, u) \]

where

\[
f_1(x, u) = \begin{bmatrix}
x_4(t) \\
x_5(t) \\
x_6(t) \\
\frac{\sin x_3(t)}{M}(u(t) + Mg) \\
\frac{\cos x_3(t)}{M}(u(t) + Mg) - g \\
0
\end{bmatrix},
\]

\[
f_2(x, u) = \begin{bmatrix}
x_4(t) \\
x_5(t) \\
x_6(t) \\
g \sin x_3(t) \\
g \cos x_3(t) - g \\
\frac{-Lu(t)}{I}
\end{bmatrix},
\]

\[
f_3(x, u) = \begin{bmatrix}
x_4(t) \\
x_5(t) \\
x_6(t) \\
g \sin x_3(t) \\
g \cos x_3(t) - g \\
\frac{Lu(t)}{I}
\end{bmatrix}
\]

Cost function: \( \int_{0}^{t_f} 5u^2(t)dt + 5(x_1(t_f) - 6)^2 + 5(x_2(t_f) - 1)^2 + \sin \left( \frac{x_3(t_f)}{2} \right) \)

constraint: \( u(t) \in [0, 10^{-3}] \) and \( x_2(t) \geq 0 \)
Example II: Quadrotor (2/6)

Solution to Example 2:

- Similar to Example 1, we first augment the state space by defining \( z \) such that \( \dot{z}(t) = 5u^2(t) \) and \( z(0) = 0 \). Let the overall augmented state be 
\[
\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}
\]
and let 
\[
\tilde{f}_i = \begin{bmatrix} f_i \\ 5u^2 \end{bmatrix}
\]
be the augmented vector field corresponding to each \( f_i \).

- The augmented relaxed control variable is given by 
\[
\tilde{u} = \begin{bmatrix} u \\ \alpha \\ \beta \end{bmatrix}^T,
\]
and the augmented vector field is given by 
\[
f = \alpha \tilde{f}_1 + \beta \tilde{f}_2 + (1 - \alpha - \beta) \tilde{f}_3.
\]

- Cost function and constraint are then given by 
\[
J = z(t_f) + 5(x_1(t_f) - 6)^2 + 5(x_2(t_f) - 1)^2 + \sin\left(\frac{x_3(t_f)}{2}\right),
\]
and 
\[
\psi_t(\tilde{u}) = h(x(t)) = -x_2(t),
\]
respectively.
Example II: Quadrotor (3/6)

- Let $\tilde{\eta} = \begin{bmatrix} \eta & \alpha' & \beta' \end{bmatrix}^T$ be another control signal, from above discussion (6)-(8), we know that in order to obtain $DJ(\tilde{u}; \eta)$, it remains to compute $D\phi_t(\tilde{u}; \tilde{\eta})$, $\frac{\partial J}{\partial x}(\phi_{t_f}(\tilde{u}))$ and $\frac{\partial h}{\partial x}(\phi_t(\tilde{u}))$.

- The last two terms are easy to obtain, which are given by

$$\frac{\partial J}{\partial x}(\phi_{t_f}(\tilde{u})) = \begin{bmatrix} 10(x_1(t_f) - 6) & 10(x_2(t_f) - 1) & \frac{1}{2} \cos \frac{x_3(t_f)}{2} & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

and

$$\frac{\partial h}{\partial x}(\phi_t(\tilde{u})) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

- Now, we focus on $D\phi_t(\tilde{u}; \tilde{\eta})$. Similar to the previous example, in order to compute $D\phi_t(\tilde{u}; \tilde{\eta})$, we need to compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \tilde{u}}$. 
Example II: Quadrotor (4/6)

\[- \frac{\partial f}{\partial \bar{x}} = \alpha \frac{\partial \tilde{f}_1}{\partial x} + \beta \frac{\partial \tilde{f}_2}{\partial x} + (1 - \alpha - \beta) \frac{\partial \tilde{f}_3}{\partial x}, \text{ where} \]

\[
\frac{\partial \tilde{f}_1}{\partial \bar{x}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & u + Mg & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{u + Mg}{M} & \cos x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{Mg}{M} & \sin x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\frac{\partial \tilde{f}_2}{\partial \bar{x}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & g \cos x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & -g \sin x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Example II: Quadrotor (5/6)

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & g \cos x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -g \sin x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\frac{\partial f_3}{\partial \tilde{x}} = 
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & g \cos x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -g \sin x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Hence,

\[
\frac{\partial f}{\partial \tilde{x}} = 
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \alpha \frac{u + Mg}{M} \cos x_3 + (1 - \alpha) g \cos x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha \frac{u + Mg}{M} \sin x_3 - (1 - \alpha) g \sin x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[-\frac{\partial f}{\partial \tilde{u}} = \begin{pmatrix}
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial \alpha} & \frac{\partial f}{\partial \beta}
\end{pmatrix}, \text{ where } \frac{\partial f}{\partial u} = \alpha \frac{\partial f_1}{\partial u} + \beta \frac{\partial f_2}{\partial u} + (1 - \alpha - \beta) \frac{\partial f_3}{\partial u}.\]
Example II: Quadrotor (6/6)

\[
\frac{\partial \tilde{f}_1}{\partial u} = \begin{bmatrix}
0 \\
0 \\
0 \\
\frac{\sin x_3}{M} \\
\frac{\cos x_3}{M} \\
10u
\end{bmatrix}, \quad \frac{\partial \tilde{f}_2}{\partial u} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
10u
\end{bmatrix}, \quad \frac{\partial \tilde{f}_3}{\partial u} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
10u
\end{bmatrix}
\]

\[
\frac{\partial f}{\partial \alpha} = \tilde{f}_1 - \tilde{f}_3, \quad \frac{\partial f}{\partial \beta} = \tilde{f}_2 - \tilde{f}_3
\]

Hence, \[
\frac{\partial f}{\partial \tilde{u}} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\alpha \frac{\sin x_3}{M} & \frac{\sin x_3}{M} u & 0 \\
\alpha \frac{\cos x_3}{M} & \frac{\cos x_3}{M} u & 0 \\
(1 - \alpha - 2\beta) \frac{L}{I} & -\frac{L_u}{I} & -2 \frac{L_u}{I} \\
10u & 0 & 0
\end{bmatrix}
\]

Up to now, we have all the elements for computing the directional derivative \( DJ(\tilde{u}; \eta) \).
