ECE7850: Hybrid Systems: Theory and Applications

Lecture Note 12: Continuous Time Switched Optimal Control: Background

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Outline

- Variation Viewpoint of Finite-Dimensional Optimization
- Elementary Calculus of Variations
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- Elementary Calculus of Variations
Basic Building Block: 1D Optimization

\[
\begin{aligned}
\begin{cases}
\min_x & g(x) \\
\text{subj. to} & x \in [a, b]
\end{cases}
\end{aligned}
\]

- Necessary local min condition for interior point \(x^* \in (a, b)\)
  
  Taylor expansion around \(x^*\)
  
  \[
g(x) = g(x^*) + g'(x^*)(x-x^*) + o(|x-x^|)
\]
  
  \(x \in (a, b), \; x \in (-2x^*, x+2x^*) \Rightarrow g(x) \geq g(x^*) \Rightarrow g'(x^*) = 0\)

- Necessary local min condition for \(x^* = a\):
  
  \[
g(x) = g(x^*) + g'(x^*)(x-x^*) + o(|x-x^|)
\]
  
  \(x > x^*\)

  Need: \(g(x) \geq g(x^*), \; \forall x \in [a, a + x^*]\)

  \(\Rightarrow g'(x^*) \geq 0\)
Constrained Optimization Problem in $\mathbb{R}^n$

\[
\begin{aligned}
\min_{x} & \quad J(x) \\
\text{subj. to} & \quad x \in X
\end{aligned}
\]

- $X$ is a closed subset in $\mathbb{R}^n$ with boundary $\partial X$

**Definition 1 (Local minimizer).**

$x^* \in X$ is called a local min if there exists $r > 0$, such that

\[ J(x^*) \leq J(x), \forall x \in \mathcal{N}(x^*; r) \cap X \]  

(1)

- **Necessary conditions**: conditions weaker than (1) that can be used to eliminate non-optimal solutions
Necessary Optimality Condition: Feasible Direction

• We call $d \in \mathbb{R}^n$ a **feasible direction** if $\exists \bar{\alpha} > 0$ such that $x + \alpha d \in X$ for all $\alpha \in [0, \bar{\alpha}]$.

• Feasible cone at $x$: $F_X(x) = \{0\} \cup \{ \text{feasible directions} \}$

• An obvious necessary condition: cost does not increase along any feasible direction

\[
\forall d \in F_X(x^*), \text{ we have } J(x^* + \alpha d) \geq J(x^*), \text{ for sufficiently small } \alpha \quad (2)
\]
Necessary Optimality Condition: Directional Derivative

- **Directional derivative:** \( DJ(x^*; d) = \lim_{\alpha \to 0} \frac{J(x^* + \alpha d) - J(x^*)}{\alpha} \). \( \Rightarrow g'(0) \)

- 1-sided directional derivative: \( DJ^+(x^*; d) = \lim_{\alpha \downarrow 0} \frac{J(x^* + \alpha d) - J(x^*)}{\alpha} \). \( \Rightarrow g'(0^+) \)

- Optimality conditions in terms of directional derivatives:
  - Whenever the 1-sided directional derivative exists, (2) requires
    \[
    DJ^+(x^*; d) \geq 0, \quad \forall d \in F_X(x^*)
    \]
  - If \( DJ \) also exists, then we need \( DJ(x^*; d) = 0 \).

If \( J \) is differentiable \( \Rightarrow DJ^+(x^*, d) = (\partial J)^T(d) \)

\[
\begin{align*}
DJ^+(x^*; d) &\geq 0 \\
DJ^+(x^*; -d) &\geq 0
\end{align*}
\]

\( \Rightarrow DJ(x^*; d) = 0 \)
Example 1.

\[
\begin{aligned}
\min_x J(x) \\
\text{subj. to } h(x) \leq 0
\end{aligned}
\]

where \( J : \mathbb{R}^n \to \mathbb{R} \) and \( h : \mathbb{R}^n \to \mathbb{R} \) are differentiable.

\( X = \{ x \in \mathbb{R}^n : h(x) \leq 0 \} \)

1. Necessary cond. for interior \( x^* \in X^o \): \( h(x^*) < 0 \) \( \Rightarrow \) \( F_x(x^*) = \mathbb{R}^n \)

\[ DJ(x^*; d) \geq 0 , \forall d \in F_x(x^*) \Rightarrow DJ(x^*)^T d \geq 0 , \forall d \in \mathbb{R}^n \Rightarrow DJ(x^*) = 0 \]

2. For \( x^* \in \partial X \), \( h(x^*) = 0 \) we need \( DJ^+(x^*; d) \geq 0 , \forall d \in F_x(x^*) \)

\( \sup_{h(x^*)} \) \( F_x(x^*) = \{ d \in \mathbb{R}^n : \quad (h(x^*))^T d \leq 0 \} \)

we want \( DJ(x^*)^T d \geq 0 , \forall d \in F_x(x^*) \)

\( DJ \) \( \Rightarrow \) \( DJ^+ d \leq 0 \) but \( DJ^+ d \leq 0 \)

\( \Rightarrow \) such \( DJ \) is not possible

\( \Rightarrow DJ \) must be in opposite direction to \( oh \)

\( \Rightarrow DJ + \mu oh = 0 \)

for some \( \mu > 0 \)
From Feasible Cone to Tangent Cone

- Feasible direction can be restrictive (especially for nonlinear equality constraints)

\[ X = \{ x_1^2 + x_2^2 = 1 \} \]

\[ F_X(x^*) = \{0\} \]

- Need to replace \( F_X(x) \) with tangent cone \( T_X(x) \) in general: including all limiting tangent directions when approaching \( x \) inside \( X \)

- Note \( cl(F_X(x)) \subseteq T_X(x) \)
From Optimization to Optimal Control

A large class of optimal control problems can be viewed as optimization problem in infinite-dimensional space

- $X$ becomes a space of control input signals (function of time)

- $J$ becomes function of control signal (functional)

- But the results are still based on the same key concepts: necessary conditions, feasible direction, and directional derivatives

- We just need slight generalizations.
Outline

- Variation Viewpoint of Finite-Dimensional Optimization
- Elementary Calculus of Variations
Elementary Calculus of Variations

• Let $\mathcal{V}$ be a general normed vector space (may be infinite dimensional).

• Consider constrained optimization: $\min_{x \in \mathcal{X}} J(x)$, where $\mathcal{X} \subseteq \mathcal{V}$ and $J : \mathcal{X} \to \mathbb{R}$ is a functional.

• Generalization of directional derivative concept to function space: **Gateaux derivative**

$$\delta J(x; \eta) = \lim_{\alpha \to 0} \frac{J(x + \alpha \eta) - J(x)}{\alpha} - J'(0)$$

• It is also called the **first variation** of $J$ along $\eta$

• Gateaux derivative (or first variation) $\delta J(x; \eta)$ (according to above definition) may not be linear in $\eta$. Some people directly require Gateaux derivative to be linear in $\eta$. 
• One-sided **Gateaux derivative**:

\[
\delta J^+(x; \eta) = \lim_{\alpha \downarrow 0} \frac{J(x + \alpha \eta) - J(x)}{\alpha} \quad g'(0^+)
\]

• **Optimality conditions**:  
  - \(\delta J^+(x; \eta) \geq 0\) for all feasible/admissible directions \(\eta\)
  
  \(\delta J(x; \eta) = 0\), for all feasible/admissible directions \(\eta\)

• **Note**: \(\delta J(x; \eta) = g'(0)\), where \(g(\alpha) = J(x + \alpha \eta)\).
Basic Calculus of Variation (1/3)

Find a $C^1$ function $x : [a, b] \to \mathbb{R}$ with given $x(a) = x_0$ and $x(b) = x_f$ to minimize $J(x) = \int_a^b l(x, x', t)\, dt$.

- This curve optimization problem can be viewed as a control problem: find control $u \in C^0$ to minimize $J(u) = \int_a^b l(x, u, t)\, dt$ subject to dynamic constraint $\dot{x} = u$.
- Derive $\delta J(x; \eta)$ for admissible $\eta$ that satisfies $\eta(a) = \eta(b) = 0$.

$$g(\alpha) = J(x + \alpha \eta) = \int_a^b l(x + \alpha \eta, x' + \alpha \eta', t)\, dt = \int_a^b \left( l(x, x', t) + l_x(x, x', t) \alpha \eta + l_{x'}(x, x', t) \alpha \eta' \right)\, dt + o(\alpha)$$

$\eta$: admissible $\Rightarrow \alpha \eta$ is feasible for any feasible $x$, and small $\alpha \Rightarrow g(\alpha) = J(x) - J(x) = 0$

$$\delta J(x; \eta) = \lim_{\alpha \to 0} \frac{J(x + \alpha \eta) - J(x)}{\alpha} = \frac{1}{2} \left( \int_a^b \left( l_x(x, x', t) \eta + l_{x'}(x, x', t) \eta' \right)\, dt + o(\alpha) \right)$$

For optimal $x$, we need $\delta J(x; \eta) = 0$, $\forall$ admissible $\eta$
Lemma 1 (Fundamental lemma of calculus of variation).

If a continuous function \( \xi : [a, b] \to \mathbb{R} \) satisfies \( \int_a^b \xi(t)\eta(t)dt = 0 \) for all \( C^1 \) function \( \eta : [a, b] \to \mathbb{R} \) with \( \eta(a) = \eta(b) = 0 \), then \( \xi \equiv 0 \).

- Euler-Lagrange Equation: By Lemma 1

\[
\mathcal{L}_\alpha(x, x', t) = \frac{d}{dt} \mathcal{L}_\alpha(x, x', t), \quad \forall t \in [a, b]
\]
Example 2.

\[ J(x) = \int_0^{\pi/2} [\dot{x}^2(t) - x^2(t)] dt \] with boundary conditions \( x(0) = 0, x(\pi/2) = 1. \)

Provide solution later