On Aspects of the Physical Realizability of Perfectly Matched Absorbers for Electromagnetic Waves

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Abstract. A discussion on the physical realizability of electromagnetic wave absorbers with perfectly matched impedances over arbitrary (i.e., doubly) curved, smooth surfaces is presented. The focus is on the analysis of the spectral characteristics (and their impact on the absorptive properties) of hypothetical material blueprints derived from the zero-reflection condition over such geometries.

1. Introduction

1.1. Background

Continual advances on microfabrication technology in recent years have made possible, particularly in two dimensions, the controlled design of novel material composites on scales down to tens of angstroms. For many cases in the microwave/millimeter-wave range, the observed electromagnetic behavior at such length scales and above is still essentially dominated by properties at a classical level, i.e., those governed by Maxwell's equations not augmented by any inherently quantum effects (or amenable to be described, for all practical purposes, by equivalent classical models). Motivated by such advances, entirely new classes of novel materials with superior electromagnetic properties in the microwave/millimeter-wave range have been recently proposed and/or demonstrated, including materials (ferroelectrics) with voltage tunable dielectric permittivities and (comparatively) small loss tangents at room temperature [Sengupta and Khushens, 1998], ultra-high dielectric materials (e.g., carbon

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nanotubes) [Garcia-Vidal et al., 1997; Slepyan et al., 1998], photonic bandgap materials (PBGs, also called electromagnetic bandgap materials, or EBGs) [Yablonovitch, 1993; Sievenpiper et al., 1996; Pendry, 1998; Centeno and Felbacq, 2000; Kyriazidou et al., 2001], antiferroelectrics [Lagerwall et al., 2001], (low-loss) magneto-dielectrics [Walser et al., 1998; Hansen and Burke, 2000], left-handed materials (also called double negative media or backward-wave media) [Veselago, 1967; Smith et al., 2000; Pendry, 2000; Smith et al., 2001; Ziolkowski, 2001; Lindell et al., 2001; Engheta, 2001; Tretyakov, 2001], and low-k dielectrics [Shch et al., 1998; Venugopal et al., 2000].

A particular area in which microfabrication technology can have a definitive impact is on the design of wideband electromagnetic wave absorbers. A number of applications such as stealth technologies for radar cross-section (RCS) reduction and control, critically depend on the performance of antireflection composites [Rozanov, 2000]. Traditional designs for electromagnetic absorbers described in the open literature include Salisbury/Dallenbach screens and Jaumann absorbers [Chambers and Ford, 2000; Fante and McCormack, 1998], as well as ferrite-based absorbers [Amin and James, 1981]. The main limitations of classical designs are narrowband performance and their dependence on geometric properties of the coated surface. For instance, designs for planar surfaces show a degradation of the antireflection properties when placed over realistic curved surfaces, requiring extensive empirical corrections [Hashimoto and Mizokami, 1990]. In the last decade, there has been much interest in the possible use of chiral materials [Jaggard et al., 1990; Jaggard and Engheta, 1989; Bohren et al., 1992], and synthetic (bi)anisotropic materials with elementary (small) embedded scatterers of complex shape (helices, omega shaped particles, etc.) [Brewitt-Taylor, 1994; Norrell, 1998; Simovski et al., 2000; Tretyakov, 1998]. This allows, in principle, greater flexibility in the design of electromagnetic wave absorbers through the addition of extra degrees of freedom on the constitutive parameters, e.g., magneto-electric effects.

1.2. PML absorbers: Planar case

Around 1994, the concept of a perfect matched layer (PML) absorber was introduced in computational electromagnetics [Berenger, 1994]. The PML corresponds to a reflectionless absorption layer (for all frequencies and incidence angles) developed as an
absorbing boundary condition (ABC) for computational purposes [Berenger, 1994; Katz et al., 1994; Chew and Weedon, 1994; Liu and He, 1998]. Initially, such concept relied upon the introduction of matched artificial electric and magnetic conductivities and a splitting of the electromagnetic fields into subcomponents. Because of this, the resulting fields inside the PML layer were rendered nonphysical (non-Maxwellian). This is not of major relevance for the intended computational purposes per se, but it ruled out any physical interpretation for the reflectionless mechanism. Shortly thereafter, however, the PML concept has found an interesting dual formulation (Maxwellian PML) with a clear physical interpretation whereby the PML corresponds to particular frequency dependent material tensors (blueprints) \( \hat{\mathbf{f}} \) and \( \hat{\mathbf{f}}' \) [Sacks et al., 1995; Gedney, 1996; Ziolkowski, 1997a; Ziolkowski, 1997b]. These tensors are material blueprints which exhibit large and matched imaginary parts for the absorption mechanism.

Because of the complex characteristics of the PML tensors, their actual physical realizability is (as expected) not a trivial matter. Nevertheless, physically realizable absorber concepts have been proposed to mimic the perfectly matched absorber behavior over a broad range of frequencies. For instance, a broad bandwidth absorbing material, based on the planar Maxwellian PML and hence potentially realizable with a proper engineering of materials, was introduced in [Ziolkowski, 1997a] and further studied in [Ziolkowski, 1997b]. This proposal is based upon a generalization of the Lorentz model for the polarization and magnetization fields that includes an extra time derivative of the driving fields. Moreover, it was shown then that this potentially realizable medium is both causal and passively absorbing. Another interesting proposal of a PML-like material is based on the use of uniaxial omega composites with higher order spatial dispersion effects [Tretyakov, 1998].

1.3. Curved PMLs

The PML was originally derived in Cartesian coordinates (planar PML). However, the effectiveness of a planar PML can be greatly reduced when applied to curved surfaces of electrically small radius of curvature. The PML concept was later extended to curvilinear coordinates [Chew et al., 1997; Maloney et al., 1997; Teixeira and Chew, 1997a; Collino and Monk, 1998; He and Liu, 1999; Hwang and Jin, 1999]. Although the first such extensions have dealt
with non-Maxwellian formulations only, it was later shown that Maxwellian PMLs could also be obtained in curvilinear geometries [Teixeira and Chev, 1998]. Such Maxwellian PMLs correspond to anisotropic material tensors with inhomogeneous properties depending on the local geometry (principal curvatures) of the termination (coated) surface. From the existence of such extensions the question naturally emerges whether or not they could provide blueprints for physically realizable absorbers over curved surfaces, similarly to the planar PML case.

For classification purposes, we may divide the study of the physical realizability of such materials into three basic levels. The first level, denoted here blueprint level, amounts to investigating whether basic electromagnetic properties (i.e., causality, passivity) of the hypothetical material blueprints (frequency domain constitutive tensors) derived from the zero reflection conditions are satisfied [Ziolkowski, 1997b; Norgren and He, 1997]. The second level amounts to the search of time-domain polarization models capable of approximating the desired response of the first level blueprints (ideal PML) at a given range of frequencies. At this second level, some compromise needs to be made on the frequency range under which such materials are expected to approximate the ideal PML behavior [Ziolkowski, 1997a; Ziolkowski and Auzanneau, 1997]. The third level consists on the investigation of specific material models (e.g., particulate composites with resonant, complex elementary scatterers) capable of furnishing the polarization responses derived at the second level [Auzanneau and Ziolkowski, 1998a; Auzanneau and Ziolkowski, 1998b; Tretyakov, 1998; Tretyakov and Khairina, 2000]. Because of microfabrication constraints, additional compromises on the intended material performance are also present at this third level.¹

As mentioned, for the planar PML, studies on the physical realizability of planar PML absorbers have been carried out at the different levels [Ziolkowski, 1997a; Tretyakov, 1998]. In this paper, we shall focus on the physical realizability of curved PML absorbers at the blueprint level.

2. Analysis

2.1. Curved PML blueprints

PML blueprints can be derived systematically for general geometries through a two step process. In the first step, a complexification of the metric of space
is employed to map ordinary solutions of Maxwell’s
equations continuously into non-Maxwellian fields
which exhibit a modified behavior (e.g., exponential
decay) inside the PML [Teixeira and Chew, 1999a].
This complexification can be carried out by an ana-
lytic continuation of the spatial coordinates [Teiz-
eira and Chew, 1998]. In the second step, an-
other field mapping is employed. This second map-
ing transforms the non-Maxwellian (analytic con-
tinued) fields into a third set of fields, which are
Maxwellian [Teixeira and Chew, 1999a]. This latter
map explores the metric invariance of Maxwell’s
equations (in the sense of [Deschamps, 1981; Teiz-
eira and Chew, 1999b; Bossavit, 2001]), to recast
the complexification of the metric as a change on
the constitutive parameters. The end result is a set
of Maxwell’s equations in anisotropic media, where
both the permeability and permittivity tensors are
frequency dispersive and depend on the local metric
coefficients of the PML surface (or, equivalently, the
radii of curvature of PML).

To describe these tensors, we attach a local ortho-
goal curvilinear coordinate system \((\xi_1, \xi_2, \xi_3)\) to a
point \(P\) on the PML surface, where \(\xi_1, \xi_2\) are
tangent coordinates to the surface and \(\xi_3\) is the normal
coordinate, such that \(\xi_3 = 0\) represents the PML surface
itself and \(\xi_3 > 0\) represents points inside the PML.
Using the convention \(e^{-i\omega t}\), the PML blueprints for
a doubly-curved surface are written as

\[
\overline{\mathbf{r}} = e^{i\lambda} \overline{\mathbf{r}}, \quad (1.1)
\]

\[
\overline{\mathbf{\mu}} = \mu \overline{\mathbf{r}}, \quad (1.2)
\]

with the PML tensor [Teixeira and Chew, 1998]

\[
\overline{\mathbf{\Lambda}} = \left( \frac{\hat{h}_1 h_2}{h_1 h_2} \right) \hat{t}_1 \hat{t}_1 + \left( \frac{\hat{h}_1 h_2}{h_1 h_2} \right) \hat{t}_2 \hat{t}_2 + \left( \frac{\hat{h}_1 h_2}{sh_1 h_2} \right) \hat{n}, \quad (1.3)
\]

In the above, \(s\) is the so-called complex stretching
variable [Chew and Weedon, 1994], which has a fre-
quency dependence on the form \(s(\omega, \xi_3) = a(\xi_3) +
\frac{i\sigma(\xi_3)}{\omega}\), with \(a(\xi_3) \geq 1\) and \(\sigma(\xi_3) \geq 0\) in the PML
\((\xi_3 > 0)\) (other functional dependencies of \(s\) in terms
of \(\omega\) are possible as long as they lead to an absorp-
tive behavior). The factors \(h_i, i = 1, 2\) are the
stretched and non-stretched local metric coefficients,
respectively, given by \(h_i = r_i/r_0\) and \(\hat{h}_i = \hat{r}_i/\hat{r}_0\),
\(i = 1, 2\), and \(h_3 = 1\), where \(r_0, \hat{r}_0, (\xi_1, \xi_2), i = 1, 2\),
are the local principal radii of curvature at the point
\((\xi_1, \xi_2, 0)\). The radii of curvature are defined from
the outside of the surface. Therefore, they are pos-
tive over concave surfaces and negative over convex
surfaces. Moreover, \( r_i = r_{0i} + \xi_3 \) and \( \bar{r}_i = r_{0i} + \bar{\xi}_3 \nabla \cdot \), with

\[
\bar{\xi}_3 = \int_0^{\xi_3} s(\zeta) d\zeta = \int_0^{\xi_3} \left( a(\zeta) + \frac{i\sigma(\zeta)}{\omega} \right) d\zeta \equiv b(\xi_3) + i \frac{\Delta(\xi_3)}{\omega}.
\]

being the analytic continuation of the coordinate \( \xi_3 \). The unit vectors \( \hat{t}_i, i = 1, 2 \), are tangential to \( S \) at \( P \) along the principal lines of curvature, and \( \hat{n} = \hat{t}_1 \times \hat{t}_2 \) is the unit vector normal to \( S \) at this point. In terms of the local coordinates \( \xi_1, \xi_2, \xi_3 \) of a local orthogonal system, we write \( \hat{t}_i = (\partial r/\partial \xi_i) / | (\partial r/\partial \xi_i) |, i = 1, 2 \), where \( r \) is the position vector, and analogously for \( \hat{n} \). The imaginary part of the complex stretching variable \( s \) is responsible for the loss mechanism inside the PML region.

The significance attached to Eq. (1) is that reflectionless absorption of incident waves on an curved surface can be obtained through by a hypothetical medium having properly chosen constitutive tensors. Strictly speaking, however, it is clear that the PML tensors given by Eq. (1) cannot represent the frequency behavior of real materials for all frequencies\(^2\), and, as a result, no physical significance can be attached to the absorber blueprint in Eq. (1) other than the (desired) constitutive behavior (objective function) to be approximated over some finite, pre-assigned bandwidth of interest. This has been the approach taken, for instance, in [Ziolkowski, 1997a; Ziolkowski, 1997b] for physically realizable planar PMLs.

### 2.2. Inhomogeneous properties and frequency behavior

In the case of planar PMLs, \( r_{0i} \to \infty, h_i \to 1, \) and \( \bar{h}_i \to 1, \) so that Eq. (1) reduces to the simpler form

\[
\bar{\Lambda} = s \left( \bar{T} - \hat{n} \hat{n} \right) + 1 \frac{1}{s} \hat{n} \hat{n}
\]

Meaningful microscopical time-domain models can be obtained for Eq. (3) by using, for example, Lorentz models for the polarization \( \bar{P} \) and magnetization \( \bar{M} \) fields near the resonance (which implies, however, a narrowband behavior if high absorption is desired), or, for more broadband behavior, through time-derivative Lorentz medium models for both \( \bar{P} \) and \( \bar{M} \) [Ziolkowski, 1997a].

The more complicated nature of the curved PML blueprint \( \bar{\Lambda} \) given by Eq. (1) when compared to the planar PML given by Eq. (3) is a consequence of the presence of the metric factors \( \bar{h}_i/h_i \). The properties of the curved PML differ in two major ways from
those of the planar PML:

(1) The curved PML tensor consist of rational functions of \( \omega \) involving higher order polynomials than the planar case. As a result, higher order time-derivatives will need to be present in equivalent polarization and magnetization time-domain models.

(2) The curved PML is, in general, an inhomogeneous medium in the transverse directions \((\xi_1, \xi_2)\) because the metric factors in Eq. (1) depend on the local radii of curvature \( r_0(\xi_1, \xi_2) \), \( i = 1, 2 \).

Both these facts lead to additional complexity for the necessary material engineering. A more fundamental aspect for the physical realizability, however, is the impact that the metric factors \( h_i/h_i \) have on the spectral properties (when treating \( \omega \) as a complex variable) of the PML blueprints. This is discussed next.

2.3. Spectral properties; limitations

An important mathematical property to be observed by any constitutive parameter of a real passive material is the Kramers-Kronig (KK) relations [Landau et al., 1984]. The KK relations are often assumed to be a necessary condition for a material response to satisfy the primitive causality conditions (i.e., an effect cannot precede its cause). In terms of the constitutive relations, this is equivalent to the requirement \( \mathcal{F}^{-1}(t) = \mathcal{F}^{-1}(t) = 0 \) for \( t < 0 \). However, the KK relations are indeed only a sufficient condition for causality, as will be discussed later.

The KK relations applied to the PML tensor in Eq. (1) read as

\[
\Re[\overline{\Lambda}(\omega)] - \overline{\Lambda}(\infty) = \mathcal{H}[\Im m(\overline{\Lambda}(\omega'))] = \frac{1}{\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\Im m(\overline{\Lambda}(\omega'))}{\omega' - \omega} d\omega'
\]

\[
\Im m(\overline{\Lambda}(\omega)) = -\frac{1}{\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\Re[\overline{\Lambda}(\omega']) - \overline{\Lambda}(\infty)}{\omega' - \omega} d\omega'
\]

for \( \omega \) and \( \omega' \) real and where \( \overline{\Lambda}(\infty) \equiv \lim_{\omega \to \infty} \overline{\Lambda}(\omega) \) is a real constant. In the above, \( \text{PV} \) denotes the Cauchy principal value, \( \mathcal{H} \) is a Hilbert transform, and \( \overline{\mathcal{F}} = -i \text{Res}[\overline{\Lambda}(\omega)]_{\omega=0} = -i \lim_{\omega \to 0} \omega \overline{\Lambda}(\omega) \). We call Eqs. (4) generalized KK relations because they include an extra term in Eq. (4.b) (residue contribution) due the pole at \( \omega = 0 \), originating from (static) conductive losses in the PML [Teixeira and Chew, 1999c].

In the case of the PML blueprints for curved surfaces, there is a major asymmetry between the spectral properties of \( \overline{\Lambda}(\omega) \) according to the local radii
of curvature. This asymmetry directly impacts the validity of the generalized KK relations for $\overline{\Lambda}(\omega)$. In order to study it more carefully, we first introduce some definitions.

**Definition 1.** A Concave or Planar (CoP) surface point is such that $\kappa_i \geq 0$ on it, for $i = 1, 2$ where $\kappa_i = 1/r_{ij}$ are the local curvatures.

**Definition 2.** A non-planar, non-concave (NPNC) surface point is such that $\kappa_0 < 0$ or $\kappa_0 < 0$ on it.

We have the following result for CoP surfaces.

**Proposition 1.** The PML tensor $\overline{\Lambda}(\omega)$ at a CoP surface point satisfies the generalized KK relations.

Proof. Eqs. (4) are a consequence of the application of the Cauchy's theorem to the function $(\overline{\Lambda}(\omega') - \overline{\Lambda}(\omega))/\omega'$ on the upper half of the complex $\omega$ plane (UHP), under the hypothesis that $\overline{\Lambda}(\omega')$ is analytic (holomorphic) there [Landau and Lifshitz, 1980]. Therefore, the proof just amounts to showing that $\overline{\Lambda}(\omega)$ is analytic on UHP.

In a concave or planar surface point $P = (\xi_1, \xi_2, \xi_3)$, we rewrite Eq. (1.c) explicitly as

$$\overline{\Lambda} = \Lambda_{11} \hat{\mathbf{t}}_1 \hat{\mathbf{t}}_1 + \Lambda_{22} \hat{\mathbf{t}}_2 \hat{\mathbf{t}}_2 + \Lambda_{33} \hat{\mathbf{n}} \hat{\mathbf{n}},$$

with

$$\Lambda_{11} = \frac{s h_1 h_2}{h_1 h_2} = \frac{(a(\xi_3) + i \sigma(\xi_3)/\omega)(r_{01}(\xi_1, \xi_2) + \xi_3)(r_{02}(\xi_1, \xi_2) + b(\xi_3) + i \Delta(\xi_3)/\omega)}{(r_{02}(\xi_1, \xi_2) + \xi_3)(r_{01}(\xi_1, \xi_2) + b(\xi_3) + i \Delta(\xi_3)/\omega)}\quad (6.a)$$

$$\Lambda_{22} = \frac{s h_2 h_1}{h_2 h_1} = \frac{(a(\xi_3) + i \sigma(\xi_3)/\omega)(r_{02}(\xi_1, \xi_2) + \xi_3)(r_{01}(\xi_1, \xi_2) + b(\xi_3) + i \Delta(\xi_3)/\omega)}{(r_{02}(\xi_1, \xi_2) + \xi_3)(r_{01}(\xi_1, \xi_2) + b(\xi_3) + i \Delta(\xi_3)/\omega)}\quad (6.b)$$

$$\Lambda_{33} = \frac{h_1 h_2}{s h_1 h_2} = \frac{(a(\xi_3) + i \sigma(\xi_3)/\omega)(r_{02}(\xi_1, \xi_2) + \xi_3)(r_{01}(\xi_1, \xi_2) + \xi_3)}{(a(\xi_3) + i \sigma(\xi_3)/\omega)(r_{02}(\xi_1, \xi_2) + \xi_3)(r_{01}(\xi_1, \xi_2) + \xi_3)}\quad (6.c)$$

From the above, the singularities of $\overline{\Lambda}(\omega)$ are simple poles located at

$$\omega_0 = 0 \quad (7.a)$$

$$\omega_1 = -i \frac{\Delta(\xi_3)}{r_{01}(\xi_1, \xi_2) + b(\xi_3)} \quad (7.b)$$

$$\omega_2 = -i \frac{\Delta(\xi_3)}{r_{02}(\xi_1, \xi_2) + b(\xi_3)} \quad (7.c)$$

$$\omega_4 = -i \frac{\sigma(\xi_3)}{a(\xi_3)} \quad (7.d)$$

For $\xi_3 \geq 0$, we have $a(\xi_3) \geq 1$ and $\sigma(\xi_3) \geq 0$ and therefore, from Eq. (2), $b(\xi_3) \geq \xi_3 \geq 0$, and $\Delta(\xi_3) \geq 0$. For a concave surface $r_{01}(\xi_1, \xi_2) > 0$ and $r_{02}(\xi_1, \xi_2) > 0$ by definition and hence, from Eqs. (7.a) and (7.b), both $\omega_1$, $\omega_2$, and $\omega_3$ are located on the lower half of the complex $\omega$ plane.
(LHP). For a planar surface, \( \Lambda_{11} = \Lambda_{22} = s \) and \( \Lambda_{33} = 1/s \) and hence the only poles are \( \omega_0 = 0 \) and \( \omega_1 = -i\sigma(\xi_3)/\alpha(\xi_3) \). Therefore, for a planar or concave surface no singularities for \( \Phi(\omega) \) are present on the UHP, and \( \Phi(\omega) \) is analytic there. 

A different conclusion is obtained for NPNC surfaces, as described by the next proposition.

**Proposition 2.** The PML tensor \( \Phi(\omega) \) at a NPNC surface point for which \( \kappa_0 \neq \kappa_2 \) violates the generalized KK relations.

**Proof:** In this case, the PML tensor again writes as Eqs. (5)-(6), and we still have, for \( \xi_3 \geq 0 \), that \( b(\xi_3) \geq \xi_3 \geq 0 \) and \( \Delta(\xi_3) \geq 0 \). However, by definition \( r_{01}(\xi_1, \xi_2) < 0 \) or \( r_{02}(\xi_1, \xi_2) < 0 \) for a NPNC surface point and therefore at least one of the poles \( \omega_1, \omega_2 \) will be located on the UHP. As a result, \( \Phi(\omega) \) is not analytic on the UHP. If in the UHP, we denote these poles \( \lambda_1 \) and \( \lambda_2 \) (the latter may not exist). Given a \( \Phi(\omega) \) tensor with simple poles on the UHP at \( \omega' = \lambda_k, \ k = 1, N \), the application of Cauchy’s theorem to the associated tensor \( \Phi(\omega, \omega') = (\Phi(\omega) - \Phi(\infty))/((\omega' - \omega)) \) on the UHP of \( \omega' \) gives

\[
P V \int_{-\infty}^{\infty} \Phi(\omega, \omega')d\omega' - \pi i \text{Res}[\Phi(\omega, \omega')]_{\omega' = 0} - \pi i \text{Res}[\Phi(\omega, \omega')]_{\omega' = \omega} + \int_{C_{\infty}} \Phi(\omega, \omega')d\omega' = 2\pi i \sum_{k=1}^{N} \text{Res}[\Phi(\omega, \omega')]_{\omega' = \lambda_k} \tag{8}
\]

where \( \omega \) is real, and the right hand side of Eq. (8) is a sum over all the residues of \( \Phi(\omega, \omega') \) on the UHP of \( \omega' \), \( N = 1 \) or \( N = 2 \) for \( \Phi(\omega) \) as in Eqs. (5)-(6). The last integral on the left hand side of Eq. (8) is carried out at a semi-circle at infinity, \( C_{\infty} \). Since \( \lim_{\omega' \to \infty} \Phi(\omega, \omega') = 0 \), the integration over \( C_{\infty} \) vanishes. The two residue contributions on the left hand side of Eq. (8) are due to small indentations above the singularities of \( \Phi(\omega, \omega') \) on the real axis at \( \omega' = 0 \) and \( \omega' = \omega \).

By writing

\[
\Phi(\omega) = \Phi_0(\omega) + i \frac{\Phi}{\omega}
\]

where \( \Phi_0(\omega) \) is analytic at \( \omega = 0 \), we have

\[
\Phi(\omega, \omega') = \Phi_0(\omega) - \Phi(\infty) + i \omega' \frac{\Phi}{\omega' - \omega} - \omega \frac{\Phi}{\omega' - \omega}
\]

and therefore

\[
\text{Res}[\Phi(\omega, \omega')]_{\omega' = 0} = -i \frac{\Phi}{\omega}
\]
\[ \text{Res} \left[ \bar{F}(\omega, \omega') \right]_{\omega' = \omega} = \overline{\Lambda}(0) - \Lambda(\infty) + i \frac{\overline{\Lambda}(\omega)}{\omega} = \overline{\Lambda}(\omega) - \Lambda(\infty) \]

\[ \text{Res} \left[ \bar{F}(\omega, \omega') \right]_{\omega' = \lambda_k} = - \frac{\text{Res} \left[ \overline{\Lambda}(\omega') \right]_{\omega' = \lambda_k}}{\omega - \lambda_k} \]

As a result, Eq. (8) becomes

\[ PV \int_{-\infty}^{+\infty} \frac{\overline{\Lambda}(\omega') - \overline{\Lambda}(\infty)}{\omega' - \omega} d\omega' - \pi i \frac{\overline{\Lambda}(\omega) - \overline{\Lambda}(\infty)}{\omega} = -2\pi i \sum_{k=1}^{N} \frac{\text{Res} \left[ \overline{\Lambda}(\omega') \right]_{\omega' = \lambda_k}}{\omega - \lambda_k} \]

(9)

Separating the real and imaginary part of the above we have

\[ \text{Re} \left[ \overline{\Lambda}(\omega) \right] - \overline{\Lambda}(\infty) = \frac{1}{\pi} PV \int_{-\infty}^{+\infty} \frac{3m [\overline{\Lambda}(\omega')]}{\omega' - \omega} d\omega' + 2 \sum_{k=1}^{N} \text{Re} \left[ \overline{\Lambda}(\omega') \right]_{\omega' = \lambda_k} \text{Re} \left[ \frac{1}{\omega - \lambda_k} \right] \]

\[ \text{Im} \left[ \overline{\Lambda}(\omega) \right] - \overline{\Lambda}(\infty) = - \frac{1}{\pi} PV \int_{-\infty}^{+\infty} \frac{\text{Re} \left[ \overline{\Lambda}(\omega') \right] - \overline{\Lambda}(\infty)}{\omega' - \omega} d\omega' \]

\[ + \frac{\overline{\Lambda}}{\omega} + 2 \sum_{k=1}^{N} \text{Res} \left[ \overline{\Lambda}(\omega') \right]_{\omega' = \lambda_k} \text{Im} \left[ \frac{1}{\omega - \lambda_k} \right] \]

(10.a)

(10.b)

where have used, from Eqs. (6)–(7), that \( \text{Res} \left[ \overline{\Lambda}(\omega') \right]_{\omega' = \lambda_k} \) are constant and purely real tensors. Eqs. (10) are the counterpart to the generalized KK relations at a NPNC surface point. Both the residues of \( \Lambda_{11}(\omega') \) and \( \Lambda_{22}(\omega') \) at \( \omega' = \omega_1 \) and \( \omega' = \omega_2 \), respectively, may contribute to \( \text{Res} \left[ \overline{\Lambda}(\omega') \right]_{\omega' = \lambda_k} \).

Furthermore, from Eqs. (5)–(6) we have

\[ \text{Res} [\Lambda_{11}(\omega')]_{\omega' = \omega_1} = (\gamma_0 - \gamma_0) \left( \frac{\gamma_0 + \xi_3}{\gamma_0 + \xi_3} \right) \left[ a - \sigma \left( \frac{\gamma_0 + b}{\Delta} \right) \right] \]

\[ \text{Res} [\Lambda_{22}(\omega')]_{\omega' = \omega_2} = (\gamma_0 - \gamma_0) \left( \frac{\gamma_0 + \xi_3}{\gamma_0 + \xi_3} \right) \left[ a - \sigma \left( \frac{\gamma_0 + b}{\Delta} \right) \right] \]

(11.a)

(11.b)

where, from Eq. (2), \( a, b, \sigma \) and \( \Delta \) are in general functions of position (\( \xi_3 \) coordinate). Therefore, unless we have \( \gamma_0 = \gamma_0 \), the summation terms in the right-hand side of Eqs. (10) are non-zero and Eqs. (10) do not reduce to Eqs. (4). ◊

2.4. Time-domain response

As mentioned before, violation of the KK relations for a function \( \overline{\Lambda}(\omega) \) does not necessarily render a corresponding time domain transform \( \overline{\Lambda}(t) \) noncausal. The actual behavior of \( \overline{\Lambda}(t) \) depends on the particular choice for the Fourier inversion contour used to invert \( \overline{\Lambda}(\omega) \). A noncausal \( \overline{\Lambda}(t) \) is obtained if an inverse Fourier contour along the real axis is used, as
illustrated in Fig. 1. On the other hand, a causal $\overline{\chi}(t)$ is obtained using a contour $C_\gamma$ taken above all singularities, as illustrated in Fig. 2. In the latter case, the inverse Fourier transform

$$\overline{\chi}(t) = \frac{1}{2\pi} \int_{C_\gamma} \overline{\chi}(\omega)e^{-i\omega t}d\omega$$

(12)
can be closed, when $t < 0$, from the above by the semi-circle at infinity $C_\infty \rightarrow C_\infty$ (Fig. 2), which yields $\overline{\chi}(t) = 0$ after applying Jordan's lemma. For $t > 0$, on the other hand, the contour $C_\gamma$ can be closed from below, and application of Cauchy's theorem and Jordan's lemma yields

$$\overline{\chi}(t) = \frac{1}{2\pi} \sum_{k=1}^{N_1} A_k e^{-i\lambda_k t} + \frac{1}{2\pi} \sum_{k=1}^{N_2} B_k e^{-i\nu_k t}$$

(13)

where we have assumed for simplicity [and according to Eqs.(5)-(6)], a $\overline{\chi}(\omega)$ tensor in Eq. (12) having simple poles $\omega = \lambda_k, \ k = 1, ..., N_1$ in the UHP ($3m[\lambda_k] > 0$) and single poles $\omega = \nu_k, \ k = 1, ..., N_2$ elsewhere ($3m[\nu_k] \leq 0$) as the only singularities 3.

The summation at the left in Eq. (13) corresponds to the residue contributions from the UHP poles (UHP associated eigenmodes), while the summation at the right corresponds to the residue contributions from the poles elsewhere. Because $3m[\lambda_k] > 0$, any term in the summation at the left is an unbounded function with exponential increase in time.

In the previous section, we have determined that the PML blueprint $\overline{\chi}(\omega)$ at a NPNC surface point with $\kappa_0 \neq \kappa_2$ has at least one pole in the UHP. Because of this, the expression for susceptibility-like kernel $\overline{\chi}(t)$ reads as Eq. (13) with at least one term in the summation at the left. As a result, $\overline{\chi}(t)$ is an unbounded function and such PML corresponds to an hypothetical medium which have internal sources of energy, i.e., artificially injects energy into the field (spurious active behavior), and not to an absorber medium. We also note here that for the degenerate case of NPNC surface points with $\kappa_0 = \kappa_2$, both residues in Eqs. (11) are zero and hence the generalized KK relations are satisfied. Despite of that, a spurious active behavior is still present. In this case, this behavior is associated with the presence of zeros for $\overline{\chi}(\omega)$ in the UHP [Aki and Richards, 1980; Landau and Lifshitz, 1980]. From Eq. (6.c) with $r_0 = r_0 = r_0$, we write

$$A_{33}(\omega) = \frac{1/\mu}{\omega(\omega + i\xi_3)} \left( \frac{r_0 + b}{r_0 + \xi_3} \right)^2 \left( \omega + i\frac{\Delta}{r_0 + b} \right)^2$$

(14)
and we see that a double zero is present on the imaginary $\Im\{\omega\}$ axis at the point $\omega = -ia/(r_0 + b)$. In a NPNC surface $r_0 < 0$ and this double zero is on the UHP. Hence, a similar analysis to the previous section can be made for the reciprocal kernel $\mathbf{A}^{-1}(\omega)$ with UHP singularities. On the contrary, for a CoP surface point, neither poles nor zeros are present on the UHP.

The spurious behavior of PML blueprints on NCNP surfaces discussed above has been observed in time domain numerical simulations employing such hypothetical media in curvilinear grids [Teixeira and Chew, 1999c; Teixeira et al., 2001]. In these simulations, the behavior has manifested itself in terms of a strong dynamic instability established as soon as the incident wave impinges upon a NCNP PML region.

It should be mentioned at this point that approximate PML blueprints can nevertheless still be obtained for NCNP geometries based on slight modifications from the exact PML blueprints. This could be done, for instance, by enforcing the poles and zeros in Eq. (6) to be on the LHP. Any such a posteriori modification would lead to blueprint absorbers suitable in principle for any geometries, but without reflectionless properties. In particular, enforcing $\Delta(\xi_0) = 0$ (planar PML) achieves this objective. Again, the performance of these blueprints will depend on the local radii of curvature, which large radii exhibiting better performances.

3. Conclusions

We have discussed some theoretical aspects for the physical realizability of material blueprints for the reflectionless absorption of electromagnetic waves on general surface geometries.

The results presented here have indicated that PML (i.e., reflectionless) absorber blueprints could not be established on all surface geometries. This is because the imposition of reflectionless conditions on some geometries (NPNC surface points) is theoretically irreconcilable with absorptive effects. The resulting blueprints on such geometries exhibit a spurious active behavior (internal sources of energy) which manifest as unbounded time domain susceptibilities.

This conclusion has been established against the backdrop of PML blueprint models that reduce to the usual planar PML in the limit of infinite radii of curvature [i.e., $\mathbf{\varepsilon} = c\mathbf{\varepsilon}(\omega)$ and $\mathbf{\mu} = \mu\mathbf{\mu}(\omega)$, with $\mathbf{\varepsilon}$ given by Eq. (3)], which is an important condition to ensure compatibility in complex objects composed of
both planar and curved surfaces. Consequently, this analysis does not rule out the existence of dissimilar reflectionless absorber blueprints for NCNP geometries (i.e., which do not reduce to the usual planar PML in the limit of infinite radii of curvature). Such possibility is currently under investigation.

Notes

1. An additional level of analysis, apart from physical realizability studies but equally important for practical purposes, is related to the physical feasibility of such absorbers. This essentially amounts to the study of the structural (mechanical) and thermal conditions of any physical level proposed model and specific manufacturing constraints (e.g., thickness, density). A detailed numerical investigation at this level requires a multiphysics simulation approach.

2. For instance, in the limit \( \omega \to \infty \) [in practice, this would correspond to the far ultra-violet for light element materials and X-ray frequencies for heavy element materials], a real material exhibits the following limiting approximate behavior for the permittivity (the fine structure is ignored for simplicity)

\[
\varepsilon_{ij}(\omega) \to \left( \varepsilon_0 \frac{\omega_0^2}{\omega^2} + \frac{\omega_0^2 \gamma}{\omega^2} \right) \delta_{ij},
\]

where \( \omega_0^2 = \frac{Ne^2}{mc} \), \( N \) is the number of atoms per unit volume, \( e \) and \( m \) are the electron charge and mass, respectively, and \( \gamma \) is a damping constant related to the collision rate. This approximate dependence is not exhibited by Eq. (1). In terms of the permeability, slowly decaying \( \mu_{ij} \) terms which decreases as \( 1/\omega \) may be present up to optical frequencies for some anisotropic media such as ferromagnets. Also, some polarization modes related to the core electrons in the atoms may give rise to variations on the \( \varepsilon(\omega) \) behavior other than above even at X-ray frequencies, but this again is not germane to the discussion here.

3. The discussion remains essentially the same if multiple poles or branch point are present. In the first case, the summation should include terms of the form \( \tilde{F}_k(t)e^{-\beta_k t} \) and/or \( \tilde{f}_k(t)e^{-\omega_k t} \), where \( \tilde{F}_k(t) \) is an \( [n-1] \)-th order tensor polynomial and \( n \) is the multiplicity of the pole at \( \omega = \lambda_k \) and/or \( \omega = \nu_k \), respectively. In the second case, the summation should include terms of the form \( \tilde{f}_k(t)e^{-\beta_k x} \), where \( \beta_k \) is a branch point at \( \omega = \beta_k \) and \( \tilde{f}_k(t) \) depends on the difference of \( \varepsilon(\omega) \) along the two sides of the corresponding branch cut.

4. For simplicity, we have been focusing exclusively on the impact of the spectral properties \( \Lambda \) on the constitutive equations. From the vector wave equation

\[
\nabla \times \frac{1}{\varepsilon} \cdot \nabla \times \mathbf{E} - \omega^2 \mu \cdot \mathbf{E} = 0
\]

we arrive, in the case of a PML medium, at

\[
\nabla \times \frac{1}{\Lambda} \cdot \nabla \times \mathbf{E} - \omega^2 \mu \cdot \Lambda \cdot \mathbf{E} = 0
\]

which depends explicitly both on \( \Lambda(\omega) \) and its inverse.
References


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Figure 1. Noncausal Fourier inversion contour $C_\eta$ for $\Lambda(\omega)$ with singularities in the upper-half of the complex $\omega$ plane.
Figure 2. Causal Fourier inversion contour $C_\gamma$ for $\Lambda(\omega)$ with singularities in the upper-half of the complex $\omega$ plane. No singularity is enclosed by $C_\gamma - C_\gamma'$. 