Moment-method analysis of circularly symmetric reflectors using bandlimited basis functions

F.L. Teixeira
J.R. Bergmann

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Abstract: To reduce the computational requirements, a quasilocalised, spatially bandlimited set of basis functions is adapted and explored in the context of the moment-method (MM) analysis of circularly symmetric reflectors. By considering the analysis of representative geometries, its performance is evaluated against results produced by entire-domain and local-domain basis functions. It is shown that the use of bandlimited functions is particularly useful in this context, improving the computational efficiency in terms of both storage and CPU time requirements.

1 Introduction

To provide a more economical representation of induced currents and to extend the applicability of integral equation analysis to larger reflector antennas, entire-domain basis functions (EDFs) were applied in the context of the method of moments to treat scattering by circularly symmetric reflectors [1]. Convergent solutions were obtained with relatively few unknowns. Although this represented a major improvement in terms of memory requirements when compared to the usual analysis via local basis functions (LFs), this occurred at the expense of a dramatic increase in the CPU time required to evaluate each impedance matrix element. The impedance matrix overall fill-time was somewhat alleviated by the reduction in the size of the matrix, but a major drawback arose due to the manner in which the fill-time scaled with the electrical size of the object. While the impedance matrix overall fill-time for LFs grows with the square of the electrical size ($D^2$) in a full-Galerkin method, the matrix fill-time for EDFs is proportional to $D^4$.

The EDF expansion employed in [1] consisted of a finite series of sinusoids (Fourier series). The reduced number of terms then needed for adequate induced current description is based on the hypothesis that, over smooth surfaces, the electric current can be approximated by a spatially bandlimited function, even more so in the context of far-field scattering. The use of bandlimited expansions have proved useful in the computation of radiation integrals and near-field to far-field transformations [2–4]. More detailed discussions on this aspect and some heuristic arguments supporting this hypothesis along with numerical experiments can be found in [5–7]. The bandwidth of the electromagnetic radiation is extensively discussed in [8]. As consequence of this hypothesis, the same economy of representation shown by EDFs in the MM can be achieved with functions of spatially bandlimited nature but with compact support. In contrast to a Fourier series, these functions can be truncated without introducing relevant high frequency components. This alternative was first introduced by Hermann [5, 6], in the 2-D scattering context, using an expansion in terms of quasilocalised, interpolational basis functions, hereinafter referred to as bandlimited functions (BLFs).

In this work, BLFs are adapted to the analysis of circularly symmetric reflectors. The main objective here is to show the suitability and computational attractiveness of BLFs to MM reflector antenna analysis. The motivation is to extend MM analysis to larger reflector antennas without incurring dramatic increases in the CPU time to evaluate impedance matrix elements. This objective is sought by means of a comparative study among results derived from application of LFs, EDFs and BLFs to the analysis of representative reflectors. Computational performances are assessed by comparing the storage requirements and impedance matrix fill-time. It is shown that BLFs are particularly useful and easily adapted to circularly symmetric reflectors, since, besides maintaining a simple analytical form, they yield an economy of representation similar to the EDFs presented in [1], with the additional bonus of providing much faster computation of the integrals involved.

2 Formulation

The problem of electromagnetic scattering by a circularly symmetric PEC (perfect electric conductor) has been thoroughly studied by a number of authors [1, 9–11]. Its surface can be parametrised by two coordinates: \( r \), the arclength along the generating arc; and \( \phi \), the azimuth angle. In the MM, the induced current is first expanded in a series of previously chosen basis functions, with unknown coefficients. A convenient expansion has the general form:

\[
\mathbf{J}_e(t, \phi) = \sum_{m=-M}^{M} \left\{ \sum_{i=1}^{N} I_{i m}^e b_{i m}^e(t, \phi) + \sum_{j=1}^{N} I_{j m}^e b_{j m}^e(t, \phi) \right\}
\]

(1)
\[ b_{m}^{i}(t, \phi) = b_{m}^{i}(t) \exp(jm\phi)t \]  
\[ b_{m}^{i}(t, \phi) = b_{m}^{i}(t) \exp(jm\phi)\phi \]

where \( NF \) is the number of basis functions \( b_{m}^{i} \) for the \( \phi \) component. The number of basis functions \( b_{m}^{i} \) is \( NF - 1 \) as the \( t \) component is null at the edge.

The discussion in this Section is concise as it follows the general steps outlined in [1], Section II. The MM solution proceeds by substituting the above expansion in the EFIE and performing an inner product upon EFIE using the Galerkin alternative, i.e. with a set of test functions, complex conjugates to the basis functions. A linear system of equations in terms of the coefficients of eqn. 1 is then obtained. By virtue of the circular symmetry, a natural decoupling occurs among different Fourier modes \( m \). The resultant linear system shows a block diagonal form and the original problem is decomposed to a set of smaller ones for each mode. Explicit expressions for the impedance matrix and excitation vectors can be found in [9].

3 Basis functions

(a) Local functions. LF's previously used for bodies of revolution included staggered pulses [10, 11] and a combination of triangles for \( t \)-component and pulses for \( \phi \)-component [1, 9], where the general rule of approximately ten basis functions per wavelength was observed. The local representation used for comparisons in this work is the choice in [1]. It is the simplest choice not involving approximations in the differential operators of the EFIE and satisfies, a priori, the boundary condition \( J\cdot i = 0 \) at the reflector rim.

(b) Entire domain functions. The EDFs considered here are sinusoidal functions as defined in [1]. This choice also models a priori the longitudinal current behaviour at the rim \( (J\cdot i = 0) \). At the axis, these functions have first derivative null.

(c) Bandlimited functions. For circularly symmetric reflectors, a possible choice is an adapted version of the functions introduced by Hermann in [5] and written as:

\[ b_{m}^{i}(t) = \frac{t}{\rho(t)} \frac{\sin(c(\alpha_{x}i)\sin(x)}{\sin(\alpha_{x}i)} \quad i = 1, \ldots, NF - 1 \]  
\[ b_{m}^{i} = \sin(c(\alpha_{x}i)\sin(\alpha_{x}i)) \quad i = 1, \ldots, NF \]

In the above equations, \( x = R_{M}(t/R_{N} - \tau_{e}) \), and \( \sin(x) = \sin(\alpha_{x}i)/\sin(x) \). The function \( \rho(t) \) is the transverse distance to the symmetry axis. \( R_{N} = (NF - 1) \) is associated with the sampling rate given by \( \sigma = (NF - 1)/T_{p} \), and \( \tau_{e} = (i - 1)/R_{N} \) defines the sample locations. \( T_{p} \) is the total arclength of the generating arc.

In many practical cases, the reflector surface is smooth at the axis symmetry and the first derivative of the current is null at \( t = 0 \). To enforce this condition on the first derivative of the basis functions, a modification can be introduced in eqns. 3a and 3b by adding sampling functions centred at the negative portions of the radial arclength and written as follows:

\[ b_{m}^{i}(t) = \frac{t}{\rho(t)} \frac{\sin(c(\alpha_{x}i)\sin(x)}{\sin(\alpha_{x}i)} \quad i = 1, \ldots, NF - 1 \]  
\[ b_{m}^{i} = \sin(c(\alpha_{x}i)\sin(\alpha_{x}i)) \quad i = 1, \ldots, NF \]

The above sets form an exact expansion for any bandlimited function with spectrum limited to \((1 - \alpha)/(2 \times 2 \leq \alpha < 1)\) and thus, it is oversampled by a \( 1/(1 - \alpha) \) factor relative to the Nyquist rate. The factor \( \alpha \) controls the trade-off between the required sampling rate (which affects the impedance matrix dimensions) and the degree of overlapping (which affects the impedance matrix fill time). The choice of \( \alpha = 0.3 \) [5] is used throughout this work. The Fourier transform (in spatial frequency \( \lambda^{1} \)) of these nontruncated functions has a compact support and a trapezoidal form. They have a quadratic decay and subsequent truncation at the first zero of \( \sin(c(\alpha x)) \) (or \( \sin(c(\alpha x')) \)) does not introduce relevant high frequency components. Truncation makes the CPU time required to calculate each element of the impedance matrix independent of the reflector dimension. The choice of \( R_{N} \) equal to an integer \((NF - 1)\) implies \( J\cdot i = 0 \) at the rim. These expansions are sufficiently general to treat a wide range of practical reflector configurations.

4 Comparative numerical results

Two confocal conical reflectors were used to study the relative computational performance of BLFs. As these reflectors are smooth at the axis, the current has first derivative null at \( t = 0 \), a condition that is enforced by EDFs and by BLFs given by expressions eqns. 4a and 4b. The illumination is a spherical-wave point source excitation with origin at \( r = 0 \), pointing toward \( -z \) and with radiation limited to \( 0 < \theta' < \pi/2, \theta = \pi - \theta' \).

\[ E = \eta H_{0} \left( \frac{\sin(\phi')}{F_{E}(\phi')} + \frac{\cos(\phi')}{F_{H}(\phi')} \right) \frac{exp(-jkr)}{R} \]

\( F_{E} \) and \( F_{H} \) are, respectively, the \( E \)-plane and the \( H \)-plane patterns of this source. This field exists only \( m = \pm 1 \) modes. \( \eta H_{0} \) is a normalisation factor. In all examples considered here, \( F_{E}(\theta) = F_{H}(\theta) = \cos^{2}(\theta), p = 1 \). The following results are checked against a reference solution obtained by solving the same problems using a high density MM, with nearly 20 LFs per wavelength.

The first reflector studied was a hyperboloidal reflector with diameter \( D = 10\lambda \), eccentricity \( \varepsilon = 2 \) and inter focal distance \( 2c = 18.334 \). The illumination eqn. 5 produces \(-4.3 \) dB attenuation at the reflector edge. Figs. 1 and 2 show the normalised current (longitudinal and azimuthal components, respectively) using EDFs with a sampling rate of \( \sigma_{1} = 3.6 \) \((NF = 19)\) and BLFs with two different sampling rates: \( \sigma_{1} = 3.6 \) and \( \sigma_{2} = 8.7 \) \((NF = 45)\). For the smaller sampling rate \( \sigma_{1} \), the generating wave is discretised by \( 41 \) points (40 segments) and for \( \sigma_{2} \), by \( 101 \) points. The full-line reference result uses LFs with \( \sigma_{1} = 19.5 \) \((NF = 100)\). It is observed that MM currents given by BLFs converge to the reference result, and are capable of estimating the induced currents with an accuracy comparable to that furnished by EDFs, even with a sampling rate as low as \( \sigma_{1} \approx 3.6 \).

The small discrepancies observed have minor effects on far-field quantities, as will be discussed later. The singular behaviour in the close vicinity of the rim (for the \( \phi \) component) is not reproduced by neither EDFs or BLFs due to its high frequency content. The current description at this specific region could be improved by using a higher sampling rate over the whole object, as observed for the reference result. However, due to the localized nature of the singular behaviour, it does not seem to be a worthwhile procedure.

for $\sigma_2$, by 101 points. The full-line reference result (high density MM) uses LFs with $\sigma_1 \approx 18.8$ (NF = 100). When compared to the hyperboloidal case, the lower relative illumination of the paraboloidal reflector edge induces a less abrupt decay on the longitudinal component near the rim. As a consequence, higher spatial frequency components are de-emphasised, and an even lower sampling rate ($\sigma_1 \approx 2.7$) can be used for adequate description of the induced currents (except from the singularity at the rim for the $\phi$ component), as can be observed on Fig. 3.

Table 1 shows a comparison of storage and matrix fill-time requirements between EDFs and BLFs. The times are normalised to the CPU time required by the reference result (high density MM). For impedance matrices having comparable dimensions, note the dramatic increase in the computational cost to fill the impedance matrix with EDFs as the sampling rate is increased. For $\sigma = 2.7$ (paraboloidal case), the fill-time using BLFs was 16.1% of that using EDFs. For a larger number of contour points and a higher sampling rate ($\sigma_1 = 8.4$), this relative number is even smaller (2.27%), since an increase in the sampling rate of the BLFs is compensated by a reduction in their spatial support. Similar results are found for larger reflectors [12].

Table 1: Storage and matrix fill-time requirements

<table>
<thead>
<tr>
<th>Basis function</th>
<th>Contour points</th>
<th>NF</th>
<th>$\sigma$</th>
<th>Matrix size (storage requirements)</th>
<th>Normalised fill-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paraboloidal reflector, $D = 10$, $f = 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LF</td>
<td>101</td>
<td>100</td>
<td>18.8</td>
<td>199 x 199</td>
<td>1.00</td>
</tr>
<tr>
<td>EDF</td>
<td>101</td>
<td>45</td>
<td>8.4</td>
<td>89 x 89</td>
<td>879.03</td>
</tr>
<tr>
<td>BLF</td>
<td>101</td>
<td>45</td>
<td>8.4</td>
<td>89 x 89</td>
<td>18.55</td>
</tr>
<tr>
<td>EDF</td>
<td>41</td>
<td>15</td>
<td>2.7</td>
<td>29 x 29</td>
<td>14.93</td>
</tr>
<tr>
<td>BLF</td>
<td>41</td>
<td>15</td>
<td>2.7</td>
<td>29 x 29</td>
<td>2.39</td>
</tr>
<tr>
<td>Hyperboloidal reflector, $D = 10$, $\phi = 2$, $c = 9.167$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LF</td>
<td>101</td>
<td>100</td>
<td>19.5</td>
<td>199 x 199</td>
<td>1.00</td>
</tr>
<tr>
<td>EDF</td>
<td>101</td>
<td>45</td>
<td>8.7</td>
<td>89 x 89</td>
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<td>BLF</td>
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<td>18.55</td>
</tr>
<tr>
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<td>3.6</td>
<td>37 x 37</td>
<td>24.31</td>
</tr>
<tr>
<td>BLF</td>
<td>41</td>
<td>19</td>
<td>3.6</td>
<td>37 x 37</td>
<td>2.45</td>
</tr>
</tbody>
</table>

$N_F = 1$ Number of basis functions for $t$-component
$N_F$ Number of basis functions for $\phi$-component
$\sigma$ Sampling rate ($\lambda^{-1}$)
For the above reflectors, the use of the basis functions given by eqns. 4a and 4b permits one to enforce a condition on the first derivative of current at the axis. This strategy depends on the geometry and it may not be applicable as a general procedure, specially if conically-tipped surfaces are considered. To illustrate the effects of relaxing this condition, Figs. 5 and 6 compare the current components obtained for the hyperboloid case by employing eqns. 3a and 3b with those shown in Figs. 1 and 2. As observed, in both cases the truncation of the expansion at $t = 0$ gives rise to poor behavior of the current representation near the axis but the ripples can be reduced by enforcing the proper condition on the first derivative. However, the far-field depends primarily on the current moment $\rho \delta$ instead of the current itself, and discrepancies near the $z$ axis ($\rho = 0$) will have minor consequences. Moreover, the stationarity of the far-field radiation integral implies that current discrepancies will have only a second-order effect on the scattered far-field. Fig. 7 superposes the $E$-plane ($\phi = 0$) scattered field (primary field not included) by the hyperboloidal reflector for the high density MM result and for BLFs and EDFs with $\sigma_1 = 3.6$. Higher spatial frequency components eventually not reproduced by the bandlimited expansion do not significantly affect the far-fields, as evidenced by the good agreement between the radiation patterns. Fig. 8 shows an analogous evaluation for the $E$-plane copolar pattern (primary field included) of the paraboloidal reflector. The agreement in the $H$-plane is even better for both reflectors but is not shown here.

5 Conclusions

This work has presented a comparison of basis functions for the MM analysis of circularly symmetric reflectors. The basic motivation was to extend the MM technique to larger reflector antennas (as achieved in [1] using EDFs) without incurring a dramatic increase in the CPU time to evaluate the impedance matrix elements (as presented by these functions).

From the results it is concluded that an adapted version of the bandlimited basis functions (BLFs) introduced by Hermann [5, 6] for 2D problems can efficiently handle the MM analysis of circularly symmetric reflectors. They take full advantage of the spectral characteristics of induced currents and are easily adapted to this problem. While maintaining a simple analytical form, they reduce both the storage requirements and the CPU time involved in the solution.

In terms of storage requirements, it is shown for representative geometries that the usual rule of ten basis functions per wavelength (in the case of LFs) can be substituted by a less expensive sampling rate of four (or less) basis functions per wavelength, when using BLFs.

In terms of the CPU time involved, the impedance matrix fill-time for BLFs is proportional to $D^2$ (the square of the electric size of the scatterer, a dependency equal to that for LFs) in contrast to $D^4$ for EDFs. As an example of the economy achieved, application to a 10$^\circ$ paraboloidal reflector resulted in a fill-time of 16.1% and 2.7% of that produced by EDFs for impedance matrix sizes of 29 x 29 and 89 x 89, respectively.
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