Lattice electromagnetic theory from a topological viewpoint

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The language of differential forms and topological concepts are applied to study classical electromagnetic theory on a lattice. It is shown that differential forms and their discrete counterparts (cochains) provide a natural bridge between the continuum and the lattice versions of the theory, allowing for a natural factorization of the field equations into topological field equations (i.e., invariant under homeomorphisms) and metric field equations. The various potential sources of inconsistency in the discretization process are identified, distinguished, and discussed. A rationale for a consistent extension of the lattice theory to more general situations, such as to irregular lattices, is considered. © 1999 American Institute of Physics.

I. INTRODUCTION

For historical reasons, the prevalent approach to study initial and/or boundary value problems for Maxwell’s equations is based on the vector calculus language. Relatively few references make use of differential forms as an alternative mathematical language to describe classical electromagnetic (EM) field theory, despite its adequacy and strong geometrical content. This is in marked contrast with the current tendency of geometrization of other areas of physics.

The classical approach for deriving a lattice electromagnetic (EM) theory utilizes the vector calculus language. Such description assumes a space and time infinitely divisible. The discrete theory is then obtained by a finite-difference, finite-volume, or finite-element approximation. There has been a number of consistent (in the sense that properties exhibited by the continuum theory, such as divergence-preserving conditions, reciprocity, and conservation laws, are retained) and self-contained formulations of a lattice EM theory in the past. However, until recently, these formulations were restricted for the finite-difference case and regular lattices. Apart from exhibiting some interesting physical phenomena not present in its continuum counterpart (such as high-frequency cutoff and rotational symmetry breaking), the interest in the development of a discrete EM theory is driven basically by the recent surge of interest in the numerical simulation of EM fields in complex environments using differential equation solvers, made possible by advances in computer technology. The absence of consistency in more general lattices usually leads to harmful effects on the numerical simulations for hyperbolic equations (time-domain simulations), such as unconditional late-time instabilities. In the case of elliptic equations (frequency-domain simulations), they are usually associated with the presence of spurious modes.

Traditionally, the derivation of a lattice EM theory using the classical, vector calculus approach has some inherent drawbacks, which hamper its application for developing more general discrete models on irregular lattices. First, it involves an approximation whereby derivatives are replaced by finite differences (e.g., in the finite-difference method). Second, in the case of structured lattices, the discretization is dependent on the underlying coordinate system; in the case of
unstructured lattices, the differential operators are replaced by integral operators and the evaluation of fields involves an averaging procedure (e.g., in the finite-volume method) or a projection on a functional space (e.g., in the finite-element method). More importantly, the use of vector calculus implies that the placement of the theory on the lattice depends on the metric on which the continuum field theory was first cast. This leads, in most cases, to lengthy formulas. Furthermore, the underlying geometrical concepts and the metric independence of Maxwell’s equations are not explored. When written in the vector calculus language, the metric independence of Maxwell’s equations is hidden because the topological structure is intertwined with their metric structure.

More recently, an alternative approach to the finite-difference discretization of Maxwell’s equations for irregular quadrilateral lattices based on the support operator method (SOM) has been described in Ref. 30. Such an approach is still based on the vector calculus language, but, compared to traditional discretization schemes, has the distinct advantage of being consistent by construction and, therefore, free from spurious solutions and numerical instabilities. Moreover, such an approach also explores the metric independence of Maxwell’s equations, since the resulting discrete operators can be written as a composition of a topological part (formal differences) and a metric part.

Here, we explore and discuss the application of differential forms and their discrete counterparts (cochains) to study lattice EM theory. One of the advantages of using differential forms is that the metric independence of Maxwell’s equations is already factored out in the continuum, and, therefore, explicitly manifested.

This fact implies that the continuum Maxwell’s equations written in the differential forms language are invariant under diffeomorphisms, while their lattice counterparts are invariant under homeomorphisms (in Fig. 1, we illustrate the concepts of the topological and metric structure of a lattice). Metric concepts are present only in the so-called Hodge star operators, which also generalize the constitutive relations of the medium. In the lattice, these operators can be thought of either defining a priori the local metric structure of the lattice, or being defined a posteriori by a given metric structure of the lattice.

In the discrete counterpart of the differential forms language, the continuum derivative operations are replaced not by finite-difference approximations, but as exact exterior derivatives on the lattice cell complex. The discrete exterior derivative corresponds not to a discrete approximation, but to a discrete counterpart. The exterior derivative is an operator that can be related to a simple evaluation of quantities on the boundary of the elements of the lattice complex, and which makes no assumptions about differentiability. These observations illustrate yet another advantage of the
use of differential forms: their discrete counterparts are objects amenable to analysis using the powerful tools of algebraic topology.

Algebraic topological tools have been used to study discrete models for many years (see, e.g., Ref. 17 and references therein). The generality achieved by using algebraic topology was recently illustrated in Ref. 23, where it provided a conceptual basis to analyze the general similarities and differences among various discretization schemes, in the context of thermostatics.

The main objective of this paper is to tackle the general problem of the consistency of lattice EM theory within the framework of algebraic topology. By general, we mean lattices with arbitrary metric and topological structures. We distinguish three basic classes of consistency requirements. The first class (based on topological considerations only) is common to all field theories cast on a discrete form, and it is associated with the correct implementation of the boundary operator on the lattice. Discrete schemes that satisfy this first class can be classified as divergence-preserving schemes. The second class (also based on topological considerations only) is related to the topological structure of EM theory and the dual nature of ordinary and twisted cell complexes. The third class is the metric-dependent one, associated with the Hodge operators. We point out that each requirement is a separate, necessary condition for an overall consistent lattice EM theory.

The remainder of this work is organized as follows. In Sec. II, we write Maxwell’s equations using the language of differential forms and discuss their factorization into topological and metric equations. In Sec. III, we review the discretization of differential forms on a lattice using algebraic topological tools. In Sec. IV, we put Maxwell’s equations on the lattice using the concepts of the previous sections, stressing that it provides an exact counterpart to the continuum theory that is invariant under homeomorphisms. We also discuss the topological consistency requirements associated with the correct implementation of the boundary operator, and their connection with the usual theorems of vector calculus. In Sec. V, we discuss the concept of dual lattices and how it arises from the necessity of a proper discretization of the different geometrical objects representing the EM fields. In Sec. VI, we treat some additional algebraic properties of the resulting discrete Maxwell’s equations by discussing additional topological consistency requirements associated with the dual structure of the ordinary and twisted cell complexes (important to guarantee reciprocity of the discrete Maxwell’s equations). In Sec. VII, we discuss the problem of the discretization of the constitutive relations, where metric concepts are present and approximations are involved through the discretization of the Hodge operators. We do not present explicit constructions for the Hodge operators (these are highly problem specific); instead, we discuss general rationales for this, and describe basic requirements that any consistent version of the discrete Hodge should satisfy. Finally, in Sec. VIII, we summarize the conclusions. We use a \((3+1)\) representation with the \(e^{-i\omega t}\) time convention assumed. Throughout this work, the term discretization refers to spatial discretization, unless indicated otherwise.

II. Maxwell’s Equations and Differential Forms

In the language of differential forms, Maxwell’s equations are written as

\[
\begin{align*}
    dE & = i\omega B, \\
    dH & = -i\omega D + J_E, \\
    dB & = 0, \\
    dD & = \rho_E.
\end{align*}
\]

In the above, \(E\) and \(H\) are electric and magnetic field intensity 1-forms, \(D\) and \(B\) are electric and magnetic flux density 2-forms, \(J_E\) is the electric current density 2-form, and \(\rho_E\) is the electric charge density 3-form.

The operator \(d\) is the usual exterior derivative, which simultaneously plays the role of the curl and div operators of vector calculus. The exterior derivative is an operator applicable to any
differentiable manifold, even without a metric defined on it. This is in contrast to the vector calculus operators, which depend on metric factors and have different expressions when written in different coordinate systems. The Maxwell’s equations in the above form (27)–(30) are metric independent and retain the same form irrespective of the coordinate system used.\(^1\),\(^2\)

Constitutive parameters of a given medium relate the 1-forms \(E, H\) to the 2-forms \(D, B\) and are given in terms of Hodge operators, \(\star_e\) and \(\star_h\),\(^6\)–\(^8\) as

\[
D = \star_e E,
\]

\[
B = \star_h H.
\]

These relations close the Maxwell’s system. In this paper, the term Maxwell’s equations will refer to Eqs. (1)–(4), while the term Maxwell’s system will refer to (1)–(6). In the case of a three-dimensional manifold, the Hodge operator establishes a natural isomorphism between the space of 1-forms as \(E\) and \(H\) and the space of 2-forms as \(D\) and \(B\). This isomorphism is usually called a Hodge duality map. The Hodge operators depend on a metric and, in the equations (1)–(6), all the information about the metric of space is contained in the constitutive relations (5) and (6). Any modification on the metric tensor preserves the form of Maxwell’s equations.

The possibility of decomposing the field equations into a purely topological part and a metric one is not a special property of the EM theory. Such a decomposition is equally possible in the context of other classical field theories.\(^23\)

### III. DIFFERENTIAL FORMS ON A LATTICE

In this section, we will briefly review the correspondence between continuum and lattice equations provided by the known mapping of differential forms onto linear functions on the space of some lattice elements. For brevity, we have deliberately chosen a somewhat sloppy approach to any topological subtleties (these are discussed elsewhere, e.g., in Refs. 17–20), by focusing on the important concepts behind the terminology.

To make the right correspondence between the continuum and the lattice, the latter should be considered as a cell complex (or cell decomposition).\(^13\),\(^14\),\(^16\)–\(^20\),\(^22\),\(^23\) A cell complex is a partitioning of some space \(X\) into a finite number of \(k\) cells of different sizes, covering \(X\) without overlap, which form a set \(\chi\). In our case of interest, \(X\) is just a region of the three-dimensional Euclidean space. A \(k\)-cell \(s^k_i\) is an object homeomorphic to \(\mathbb{R}^k\) so that, a 0-cell is a point, a 1-cell is a link (edge), etc. In general, \(k\)-cells are \(k\)-dimensional \((k=0,1,2,3)\) elements of the lattice. The set of all \(k\)-cells is denoted by \(\chi^k\). The cell complex is the direct sum of such sets,

\[
\chi = \bigoplus_{k=0}^n \chi^k.
\]

Each \(k\)-dimensional element of \(\chi^k\), for \(k=0,1,2,3\), corresponds to a point (vertex), link (edge), plaquette (face), and polyhedron (volume), respectively.

We assume a fixed orientation for each \(k\)-cell \(s^k_i\) on \(\chi^k\), which results in an oriented cell complex. For brevity, the term complex will refer to oriented complex in the remaining of this work. With this assumption, a cell \(k\)-chain, or simply \(k\)-chain, is defined as a linear combination of \(k\)-cells in \(\chi^k\) through

\[
S^k = \sum_i \alpha_i s^k_i \in \chi^k.
\]

The weights \(\alpha_i\) belong, in general, to any additive Abelian group. For our purposes, we do not need such a generality and assume that they are just integer numbers. From this definition, a 0-chain is a linear combination of points, a 1-chain is a linear combination of links, etc. A chain is always one of these types; there are no mixed chains.
The $k$-cells $s^k$ form a basis for the space of $k$-chains. An arbitrary decomposition of $X$ is not a cell complex $\chi$. To characterize a cell complex, certain conditions must be observed. In particular, the boundary of any $k$-cell on $\chi$ should be the union of lower-dimensional cells in $\chi$, and no overlapping cells are allowed. The boundary operator $\partial$ is an operator on $\chi$, $\partial: \chi^k \rightarrow \chi^{k-1}$, which carries the usual geometric interpretation and is subject to the requirement $\partial^2 = \partial \circ \partial = 0$. It connects the algebra of $k$-chains with the algebra of $(k - 1)$-chains on the lattice: if $S^k$ is a $k$-chain, then $\partial S^k$ is a $(k - 1)$-chain. Moreover, the boundary operator acts linearly on the space of chains.

If $\Omega$ is a $k$-form and $\gamma$ is a $k$-dimensional integration surface, then integration defines a pairing,

$$\int_{\gamma} \Omega,$$

which gives a scalar as a result. Therefore, the space of $k$-forms can be thought of as being dual to the space of $k$-dimensional surfaces. This motivates the definition of cochains: if the space of $k$-dimensional surfaces on the continuum is identified with the space of $k$-chains on the lattice; then the space of $k$-forms in the continuum is identified as the space of cochains, which are linear functionals on the space of chains. Cochains constitute the discrete representation for the differential forms on the lattice, or the discrete counterparts of forms. Here, to emphasize the connection between the lattice and the continuum, we will also refer to cochains as lattice differential forms.

The continuum pairing given by the previous equation has the following exact counterpart on the lattice, in terms of a $k$-chain $S^k$ and a $k$-lattice form $\Theta^k$,

$$\int_{\gamma} \Omega \rightarrow \langle S^k, \Theta^k \rangle.$$

Such pairing defines a contraction between $S^k$ and $\Theta^k$. From the basis for $k$-chains, $s^k$, we define a dual basis of lattice forms, $\theta^k$, such that $\langle s^k, \theta^k \rangle = \delta_{ij}$. This basis generates the space of $k$-lattice forms so that the generic $k$-lattice form $\Theta^k$ is written as

$$\Theta^k = \sum_i \beta_i \theta^k_i.$$

In the above, $\beta_i \in G$, where $G$ is some Abelian group. To make the correspondence with the continuum, $G$ is assumed to be $\mathbb{R}$, and the composition law assumed to be the usual algebraic law of addition. The contraction of chains and lattice forms then gives

$$\langle S^k, \Theta^k \rangle = \sum_i \alpha_i \langle s^k_i, \theta^k \rangle = \sum_i \sum_j \alpha_i \beta_j \langle s^k_i, \theta^j \rangle = \sum_i \alpha_i \beta_i.$$

From Eqs. (1)–(4), we see that the only spatial operator present in the differential forms language version of Maxwell’s equations is the exterior derivative. The concept of duality makes the definition of the exterior derivative on a lattice very natural, usually called the coboundary operator. Here, to emphasize the connection between the lattice and the continuum, we will also refer to it as the lattice exterior derivative and write it with the same symbol, $d$, as in the continuum case. The lattice exterior derivative $d$ is defined in terms of its adjoint, the boundary operator $\partial$, as

$$\langle S^k, d\Theta^{k-1} \rangle = \langle \partial S^k, \Theta^{k-1} \rangle.$$

This definition has some interesting properties. First, it defines $d$ on a lattice in an exact manner. The coboundary does not correspond to an approximation to the continuum exterior derivative $d$, but instead, as a counterpart to it. Second, it is defined without any need for differentiability. Third, it automatically satisfies the generalized Stokes’ theorem.
In this framework, the usual differential operators of vector calculus are replaced by a single operator on the lattice: the boundary operator \( \partial \) acting on the elements of the lattice (cell complex). The \( \text{div} \) operator corresponds to the action of \( \partial \) on a 3-cell, the curl to the action of \( \partial \) on a 2-cell, and the grad to the action of \( \partial \) on a 1-cell. Furthermore, in this sense, these operators are distilled from an unnecessary metric structure, becoming purely topological operations.

In the next section, these remarks will be substantiated when discussing Maxwell’s equations on a lattice.

IV. MAXWELL’S EQUATIONS ON A LATTICE

In this section Maxwell’s equations are put on a lattice using the previously discussed concepts. The lattice counterpart to Maxwell’s equations \( 1 \)–\( 4 \) are formally the same as the continuum ones, written as

\[
\begin{align*}
\text{d}E &= i \omega B, \\
\text{d}H &= -i \omega D + J_E, \\
\text{d}B &= 0, \\
\text{d}D &= \rho_E,
\end{align*}
\]

but with \( E \) and \( H \) properly interpreted as lattice 1-forms, \( B \) and \( D \) interpreted as lattice 2-forms, and \( d \) interpreted as the coboundary operator. Since lattice forms are operators on the space of chains, we need to contract the above equations with any 2- and 3-chains \( S^2, \tilde{S}^2, S^3, \tilde{S}^3 \) to get actual numbers,

\[
\begin{align*}
\langle S^2, \text{d}E \rangle &= i \omega \langle S^2, B \rangle, \\
\langle \tilde{S}^2, \text{d}H \rangle &= -i \omega \langle \tilde{S}^2, D \rangle + \langle \tilde{S}^2, J_E \rangle, \\
\langle S^3, \text{d}B \rangle &= 0, \\
\langle \tilde{S}^3, \text{d}D \rangle &= \langle \tilde{S}^3, \rho_E \rangle.
\end{align*}
\]

Using the definition of the coboundary operator (the generalized Stokes’ theorem), we get

\[
\begin{align*}
\langle \partial S^2, E \rangle &= i \omega \langle S^2, B \rangle, \\
\langle \partial \tilde{S}^2, H \rangle &= -i \omega \langle \tilde{S}^2, D \rangle + \langle \tilde{S}^2, J_E \rangle, \\
\langle \partial S^3, B \rangle &= 0, \\
\langle \partial \tilde{S}^3, D \rangle &= \langle \tilde{S}^3, \rho_E \rangle.
\end{align*}
\]

We use an overtilde to distinguish between the chains belonging to the cell complex, \( \chi \) (i.e., the cell complex over which \( E \) and \( B \) live), from the chains belonging the cell complex, \( \bar{\chi} \) (i.e., the cell complex over which \( D \) and \( H \) live). These cell complexes are not necessarily the same since the pair of equations for \( (E,B) \) and for \( (D,H) \) are independent of each other. Indeed, we will show that there are strong reasons to use different cell complexes for these quantities (this is discussed in the next section). To find the lattice forms at each edge or face, we apply the above procedure for each edge or face of the cell complex, or, equivalently, to each element \( s_i^k \) of the basis of \( k \)-chains,

\[
\langle \partial s_i^2, E \rangle = i \omega \langle s_i^2, B \rangle.
\]
\[ \langle \partial s_i^2, H \rangle = -i \omega \langle \tilde{s}_i^2, D \rangle + \langle \tilde{s}_i^2, J_E \rangle, \]  
(27)

\[ \langle \partial s_i^3, B \rangle = 0, \]  
(28)

\[ \langle \partial s_i^3, D \rangle = \langle \tilde{s}_i^3, \rho_E \rangle. \]  
(29)

Since \( \partial \) is an operator from \( \chi^k \) to \( \chi^{k-1} \) (or \( \bar{\chi}^k \) to \( \bar{\chi}^{k-1} \)), the boundaries \( \partial s_i^1, \partial s_i^2, \partial s_i^3 \) (or \( \partial \tilde{s}_i^1, \partial \tilde{s}_i^2, \partial \tilde{s}_i^3 \)) can be expressed in terms of a basis of 0-, 1-, and 2-chains, respectively,

\[ \partial s_i^1 = \sum_j \alpha_{ij} s_j^0, \]  
(30)

\[ \partial s_i^2 = \sum_j \beta_{ij} s_j^1, \]  
(31)

\[ \partial s_i^3 = \sum_j \gamma_{ij} s_j^2, \]  
(32)

\[ \partial \tilde{s}_i^1 = \sum_j \bar{\alpha}_{ij} \tilde{s}_j^0, \]  
(33)

\[ \partial \tilde{s}_i^2 = \sum_j \bar{\beta}_{ij} \tilde{s}_j^1, \]  
(34)

\[ \partial \tilde{s}_i^3 = \sum_j \bar{\gamma}_{ij} \tilde{s}_j^2. \]  
(35)

The matrices \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \) (and \( \bar{\alpha}_{ij}, \bar{\beta}_{ij}, \bar{\gamma}_{ij} \)) define the incidence relations for the operator \( \partial \) in a given complex \( \chi \) (and \( \bar{\chi} \)). They are the discrete-topological counterpart to the grad, curl, and div operators, respectively. The elements of these matrices are integers having the values \( \pm 1 \) (depending on their relative chosen orientation) when \( s_j^{(k-1)} \in \partial s_i^k \), and zero otherwise. Furthermore, for any \( \chi \), the identity \( \partial^2 = 0 \) implies

\[ \sum_j \beta_{ij} \alpha_{jk} = 0, \]  
(36)

\[ \sum_j \gamma_{ij} \beta_{jk} = 0, \]  
(37)

and similar relations in \( \bar{\chi} \). Equations (36)–(37) are the topological equivalents on \( \chi \) of the familiar identities \( \text{curl} = 0 \) and \( \text{div curl} = 0 \), respectively. The fact that these identities are preserved in the numerical discretization scheme are necessary to ensure that the theorems of the continuum are preserved, although not sufficient. Schemes satisfying (36)–(37) may be classified as divergence-preserving schemes, for obvious reasons.

Substituting (30)–(35) in (26)–(29), we have

\[ \sum_j \beta_{ij} (s_j^1, E) = i \omega \langle s_i^2, B \rangle, \]  
(38)

\[ \sum_j \bar{\beta}_{ij} (\tilde{s}_j^1, H) = -i \omega \langle \tilde{s}_i^2, D \rangle + \langle \tilde{s}_i^2, J_E \rangle, \]  
(39)
These equations give the exact lattice counterparts of Maxwell’s equations in terms of the lattice variables \( s^2_i, B \), \( s^3_i, D \), \( s^3_i, H \), and \( s^1_i, E \). The lattice 2-forms \( B, D \) that live on 2-chains \( s^2_i, s^3_i \) are related to the lattice 1-forms \( E, D \), respectively, which live on 1-chains \( \partial s^2_i, \partial s^3_i \).

The above equations in terms of lattice variables are the same for any lattice with the same topological structure. Maxwell’s equations in this form are invariant under homeomorphisms. This topological equivalence leads to an equivalence relation among lattices.

We also note that the fundamental dynamic variables in the lattice theory are not the lattice forms \( E, H, D, B \) i.e., the field values themselves) anymore, but their contraction with the cell complex elements. The latter quantities are the usual ones of interest associated with a finite region of space (i.e., global quantities like electric voltages and magnetic fluxes). The discretization process just described can be viewed as a process of limiting the (originally infinite) degrees of freedom in accessing these global quantities.

The problem of obtaining a continuum representation for the continuum forms \( E, H, D, B \) over \( X \) from the knowledge of \( \langle s^1_i, E \rangle, \langle s^3_i, H \rangle, \langle s^2_i, D \rangle, \langle s^3_i, B \rangle \) over \( \chi, \bar{\chi} \), is nevertheless important for discretizing the Hodge operators (constitutive relations) in (5) and (6) and to achieve the full discretization of the Maxwell’s system (1)–(6). This is a part of the general problem of obtaining a consistent but approximate continuum representation for a differential \( k \)-form \( \Theta^k \) on \( \chi^k \). The discussion of this general problem is postponed until Sec. VII.

V. DUAL LATTICES AND TWISTED FORMS

In this section, we will discuss the concept of dual lattices for EM field simulations (such as in the Yee scheme or usual finite-volume discretizations) and show that its convenience arises not only for computational purposes, but also from geometrical reasons not obviated by the vector language. These reasons are connected with the concept of orientation.

We start by noting that there are two fundamental ways to define an orientation in three dimensions. This is illustrated in Fig. 2 in the case of a one-dimensional object (line) in a three-dimensional space. The first way is to specify a (inner) direction along the line. This does not make use of additional dimensions other than the one defined by the line itself (one dimensional) and is referred to as internal orientation. The second way is to specify a (transversal)
circulation along the line. In this case, additional dimensions are required and is referred to as external orientation. Internal and external orientations behave differently under coordinate reflection. These two kinds of orientations necessitate the definition of two different kinds of forms. Forms with internal orientation are called ordinary differential forms (for historical reasons only, since there is nothing about them to make them more "ordinary" than the twisted forms). Forms with external orientation are called twisted differential forms. In the case of vectors, this distinction is not present because the vector calculus language comes with a predefined screw sense (in addition to a metric structure). For instance, if the right-hand rule is used for the objects (1-forms) of Fig. 2, then their vector counterparts automatically will have the same orientation. However, using the concept of two different kinds of orientations, no a posteriori right-hand rules are necessary.

Figure 3 illustrates the concept of internal and external orientation for two-dimensional objects in the three-dimensional space (this concept may also be applied for zero and three-dimensional objects).

As expected, the boundary operator preserves orientation in the sense that the boundary of an ordinary/twisted form is another ordinary/twisted form. More interestingly, we note from Figs. 2 and 3 that the same concept (direction) that gives the internal orientation for one-dimensional objects gives the external orientation for two-dimensional objects. Similarly, the second concept (circulation), while giving internal orientation for two-dimensional objects, gives external orientation for one-dimensional ones.

If a given cell complex $\chi$ is chosen to discretize the space $X$ so that its links (edges) have internal orientation and its faces have external orientation (consistently through the boundary operator), then there is a dual cell complex $\tilde{\chi}$ where edge orientations are given by circulations and the face orientations given by directions. This is depicted in Fig. 4. The edges (faces) of the dual cell complex are associated with faces (edges) of the primary cell complex. If the lattice ordinary forms live on the primary cell complex $\chi$, then the lattice twisted forms should live on the dual cell complex $\tilde{\chi}$. This is one important distinction between differential forms in the continuum and the

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**FIG. 3.** The concept of two orientations now applied for two-dimensional objects on a three-dimensional space. The same observations made for Fig. 2 also apply here.

**FIG. 4.** Oriented cell complexes give rise to two distinct kinds of cell complexes. In the ordinary complex, each cell complex is endowed with an internal orientation and the associated lattice forms (cochains) are ordinary forms. In the twisted complex, each cell complex is endowed with an external orientation and the associated lattice forms (cochains) are twisted forms. On an EM lattice, these ordinary and twisted cell complexes are combined such that $k$-cells of one complex are associated to $(n-k)$-cells of the other, where $n$ is the dimensionality of the space. This gives rise to the concept of a dual lattice (complex). The fields $E$ and $B$ live on the ordinary complex, while $D$ and $H$ live on the twisted complex (also see Fig. 5).
lattice forms. For the continuum forms, ordinary and twisted forms live on the same space \( X \); but for the lattice forms, ordinary and twisted forms live on different spaces, \( \chi \) and \( \tilde{\chi} \). The role of the orientation concept in defining dual lattices was first discussed in Ref. 23.

To stress their geometrical properties, we will refer to the primary cell complex, \( \chi \), as the ordinary cell complex and to the dual cell complex, \( \tilde{\chi} \), as the twisted cell complex.

The concept of distinct orientations applies directly to three-dimensional EM fields. The electric field \( E \) is associated with internally oriented lines (ordinary 1-form), the magnetic field \( H \) is associated with externally oriented lines (twisted 1-form), the electric flux \( D \) with externally oriented surfaces (twisted 2-form), and the magnetic flux \( B \) with internally oriented surfaces (ordinary 2-form). In addition, electric charge density is associated with externally oriented volumes (twisted 3-form), and the electric current density is associated with externally oriented surfaces (twisted 2-form). Figure 5 illustrates this classification for EM fields. The \( E \) and \( B \) lattice forms live on \( \chi \), while \( D \) and \( H \) (and \( J_E, \rho_E \), which are also twisted forms) live on \( \tilde{\chi} \).

The ordinary EM forms \( (E \text{ and } B) \) are associated with the concept of forces (Lorentz formula), while the twisted EM forms \( (D \text{ and } H) \) are associated with the concept of sources \( (\rho_E \text{ and } J_E) \). Indeed, in the four-dimensional space–time notation, the forms \( E \) and \( B \) are components of a single two-form \( F \) (Faraday), while the forms \( D \) and \( H \) are components of its Hodge dual, \( \ast F \) (Maxwell two-form).

Here, we appreciate the amount of geometric structure that is lost when representing the EM fields in the vector language. Each EM field is a distinct geometric object (Fig. 5), but in vector calculus, all are under the same umbrella as three-dimensional (contravariant) vectors. The need for two cell complexes arises not only as a computational device but also to account for the inherent geometric differences among the electromagnetic fields.

VI. ALGEBRAIC PROPERTIES OF MAXWELL’S EQUATIONS ON A LATTICE

The expansions (30)–(35) should satisfy, by construction, some conditions resulting from the properties of the boundary operator \( \partial \) and from the dual complex construction. In this section, we shall describe and discuss these conditions. The fact that they are preserved in the lattice theory is important to preserve the continuum theorems and ensure an overall consistent theory.

The dual complex construction is such that, in the three-dimensional case, to each 2-cell of the ordinary cell complex, there corresponds a 1-cell on the twisted cell complex, and vice-versa. In this natural one-to-one pairing, the 1-cells (links), on the ordinary complex cross associate 2-cells (faces) on the dual complex, and vice-versa. A similar pairing also exist between 0-cells and 3-cells and vice-versa. For the two-dimensional case, the pairing is between ordinary 0-cells and twisted 2-cells (and vice-versa), and between ordinary and twisted 1-cells. This is true not only for hexahedral cells but for cell complexes with different cell topologies.

If the indices chosen for the basis elements of the cell complexes \( \chi \) and \( \tilde{\chi} \) reflects this natural pairing, i.e., if the cell \( s_i^k \) on the ordinary cell complex has the same \( i \) index of the associated cell \( \tilde{s}_i^{(n-k)} \) (on an \( n \)-dimensional space) on the twisted cell complex, then it is easy to show that the coefficients for the incidence relations in (30)–(35) are related through (see Fig. 6)

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FIG. 5. Each EM field is a different geometrical object with distinct properties. This is hidden in the vector calculus language and best revealed through the use of the differential form language. In the continuum theory this distinction (usually) does not have an important role, but on a lattice, where the degrees of freedom in accessing global quantities are limited and exhibit a specific interdependence (e.g., edges as a boundary of faces) each field must be associated with a proper geometric object (1- or 2-cell) on the proper lattice (ordinary or twisted).
These relations do not depend on the primary orientation chosen for $\chi$ and $\bar{\chi}$. Similar to Eqs. (36)–(37), Eqs. (42)–(44) are another example of consistency relations derived from topological considerations only. The fact that these relations are satisfied in a particular discretization scheme is of key importance to preserve symmetry and positive definiteness of the resulting matrix systems, and, consequently, stability in time–domain updates of numerical methods. Combined with symmetry properties to be observed on the discrete representation for the Hodge operators $\star_\epsilon, \star_\mu$ (discussed in Sec. VII), they mimic, on a lattice, the reciprocal nature of the continuum Maxwell’s equations. Equations (42)–(44) are also related with the consistency requirements for the boundary operator on the twisted complex $\bar{\chi}$. This can be seen by substituting Eqs. (42)–(44) into Eqs. (36), (37),

$$\sum_j \beta_{ij} \alpha_{kj} = \sum_j \bar{\beta}_{ji} \bar{\gamma}_{kj} = \sum_j \bar{\gamma}_{kj} \bar{\beta}_{ji} = 0;$$

(45)

$$\sum_j \gamma_{ij} \beta_{jk} = \sum_j \bar{\alpha}_{ji} \bar{\beta}_{kj} = \sum_j \bar{\beta}_{kj} \bar{\alpha}_{ji} = 0;$$

(46)

which is equivalent to having the identity $\bar{\partial}^2 = 0$ fulfilled on $\bar{\chi}$.

For lattices with simple topology, the relations (42)–(44) hold true because of the staggered nature of the ordinary and twisted cells. However, for more exotic lattices, such as those encountered in subgridding or in modeling curved boundaries through locally distorted lattice elements, they do not necessarily hold true in commonly employed discretizations schemes (with naive interpolatory rules), so that the reciprocity of the continuum Maxwell’s equations is lost. In these cases, the relations (42)–(44) should be enforced by construction to ensure that reciprocity is maintained. This is also discussed in Refs. 33–35, but from a completely different point of view.
VII. HODGE OPERATORS ON A LATTICE

The discretization of the Hodge operators $\star_e, \star_\mu$ (constitutive relations) is a central step for the formulation a general lattice EM theory. The discrete Hodge operators relate the lattice forms of the ordinary grid to the lattice forms of the twisted grid, and involve material properties of the particular medium. From the continuum equations, where the Hodge operators are metric-dependent objects, such relationship should also involve concepts such as lengths, angles, etc. Contrary to the topological equations treated before, the discrete Hodge operators are approximations to the continuum operators. The continuum Hodge operators are linear mapping of the space of $k$-forms into the space of $(n-k)$-forms, where $n$ is the dimensionality of the space. In the EM case, the constitutive relations are written in terms of those operators, $D = \star_e E$, $B = \star_\mu H$, connecting the 2-forms $D, B$ on one cell complex with the 1-forms $E, H$ on the other cell complex. Since they are linear mappings, the discrete version of the Hodge operators in the lattice can be represented as a generic linear mapping connecting the dynamical discrete variables on $\chi$ and $\bar{\chi}$ as follows:

\[
\star_e: \chi \rightarrow \bar{\chi},
\]

\[
\langle \tilde{s}_i^2, D \rangle = \sum_j [\star_e]_{ij} \langle s_j^1, E \rangle, \tag{47}
\]

\[
\star_\mu: \bar{\chi} \rightarrow \chi,
\]

\[
\langle s_i^2, B \rangle = \sum_j [\star_\mu]_{ij} \langle \tilde{s}_j^1, H \rangle. \tag{48}
\]

In the above, $[\star_e]$ and $[\star_\mu]$, are square, nonsingular, sparse matrices representing the discrete Hodge operators for a general dispersive and anisotropic linear media (in the general case of bianisotropic media, the following discussion remains essentially unchanged, except for the appearance of cross terms, $[\star_e]$ and $[\star_\mu]$, in the above equations, relating quantities on the same cell complex). These approximate equations close the Maxwell’s system (1)–(6) and, along with the exact discrete Maxwell’s equations (38)–(41), constitute the discrete approximation to the Maxwell’s system.

We now discuss some rationales for the construction of the Hodge operator on a lattice, but we do not claim such rationales to be unique. More importantly, we draw attention to the basic consistency requirements that any discrete Hodge should obey.

A rationale for a systematic construction of the discrete Hodge operators for a particular lattice geometry (metric) is described in Route $A$ below.

*Route A:*

(i) An approximate continuum representation for the electromagnetic forms $E$ and $H$ is built from the knowledge of the discrete quantities $\langle s_i^1, E \rangle$ and $\langle \tilde{s}_i^1, H \rangle$ over $\chi$ and $\bar{\chi}$, respectively (this will be discussed shortly).

(ii) The Hodge star operators $\star_e$ and $\star_\mu$ are applied to the resultant continuum representation for $E$ and $H$, respectively, to yield the corresponding approximate continuum representations for $D$ and $B$.

(iii) These resulting approximate representations for $D$ and $B$ are then paired with the elements of the cell complexes $\bar{\chi}$ and $\chi$, respectively, to yield $\langle \tilde{s}_i^2, D \rangle$ and $\langle s_i^2, B \rangle$. When $\langle \tilde{s}_i^2, D \rangle$ and $\langle s_i^2, B \rangle$ are written as functions of $\langle s_j^1, E \rangle$ and $\langle \tilde{s}_j^1, H \rangle$, respectively, we have determined the matrix elements $[\star_e]_{ij}$ and $[\star_\mu]_{ij}$. Note that in Eqs. (47), (48), the approximate continuum representations for the electromagnetic forms $E, B, D, H$ do not appear. They are used only as an auxiliary tool to obtain $[\star_e]$ and $[\star_\mu]$. This is only natural, since, in a true discrete theory, $E, B, D, H$ are not primary quantities and $[\star_e], [\star_\mu]$ should be treated as being given a priori, in the same manner as the material constitutive tensors, $\bar{\varepsilon}$ and $\bar{\mu}$, for the continuum theory.
As a result, an alternative route to (i)–(iii) can be followed by using, instead of $E,B,D,H$, their vector counterparts $E,B,D,H$, without changing the final formalism. This alternative route translates the metric dependency of the Hodge operators isolated in step (ii) to the metric-dependent notions of vector fields and surface integrals in modified versions of steps (i) and (iii). In this case, the modified step (ii) becomes just a tensorial product on vectors. This alternative route to determine the discrete Hodge operator may be summarized as follows.

**Route B:**

(i) An **approximate** continuum representation for the electromagnetic vector fields $E$ and $H$ is built from the knowledge of the discrete quantities $s_i^1, E$ and $\tilde{s}_i^1, H$ over $\chi$ and $\tilde{\chi}$, respectively.

(ii) The approximate continuum representations for $D$ and $B$ are found using the tensorial products $D = \epsilon \cdot E$ and $B = \mu \cdot H$.

(iii) These resulting approximate representations for the vector fields $D$ and $B$ are then integrated over 2-cells to give $\tilde{s}_i^2, D$ and $\tilde{s}_i^2, B$.

Variant schemes from the above are possible, where instead of first using the contractions of 1-forms, $\langle s_i^1, E \rangle$ and $\langle \tilde{s}_i^1, H \rangle$ over $\chi$ and $\tilde{\chi}$, to find the approximate continuum representations, the contractions used are located over the same cell complex, say $\chi$. In such a case, the continuum representations are derived from $\langle s_i^1, E \rangle$ and $\langle \tilde{s}_i^1, B \rangle$ and the discrete operators obtained are approximations to $*_e$ and $*_{\mu}^{-1}$. This does not result in equivalence, however, because, in general, $[*_{\mu}^{-1}] \neq [*_e]^{-1}$. In particular, the matrix $[*_{\mu}^{-1}]$ is sparse, but $[*_e]^{-1}$ is not. The use of a same cell complex to obtain the continuum field representation from the contractions is sometimes of interest because, in general, the primary and dual cell complexes have different topological structures (not only metric structure), as illustrated in Fig. 7. As will become clear shortly, a proper continuum representation construction at step (i) depends directly on the topology of the cell complex over which the contractions are defined.

Both Routes A and B above are inevitably metric dependent, and, in principle, there is no conceptual advantage in adopting one over another. However, since Route B involves vector calculus concepts only, which supposedly results in a more familiar operational approach, we will base our remaining discussion mainly on it.

To find the continuum approximation for the vector fields $E$ and $H$, a continuum basis of $k$-forms $\Omega_i^k$, counterpart to the basis of $k$-cochains $\theta_i^k$ (duals to $s_i^k$) should be constructed. Such a basis is built to obey the following properties.

(i) $\Omega_i^k = 0$ outside $s_i^k$ and its neighborhood (i.e., such forms should be compactly supported), which ensures the sparsity of $[*_e]$, $[*_\mu]$. 

![Fig. 7. A two-dimensional, irregular simplicial lattice (solid lines) and its dual lattice (dashed lines), which is not simplicial anymore (hexagonal elements).](image-url)
(ii) $\Omega_{s_i^k} = d\sigma_{s_i^k}$, consistent with Eq. (13) and indicating a proper relationship between the $\Omega_{s_i^k}$'s for the different $k$'s and a hierarchical construction that can be invoked.

(iii) The lattice form $\theta_i^k$ and its continuum counterpart $\nabla_i^k$ give the same results when contracted or integrated, respectively, over corresponding elements of $\chi$ and $X$, i.e.,

$$\int_{\gamma} \Omega_{s_i^k} = \langle \delta_j^k, \theta_i^k \rangle,$$

where $\gamma$ is the $k$-dimensional region in $X$ corresponding to the $k$-cell $s_i^k$ in $\chi$.

A basis for the space of forms obeying (i)–(iii) is introduced in Ref. 13 for a simplicial lattice (cell complex), and the resulting continuum forms are usually called Whitney forms. A simplicial lattice is one having the property that all its cell elements are simplices, i.e., cells whose boundaries are the union of a minimal number of lower-dimensional cells. Therefore, in a simplicial lattice, a 0-cell is a point (0-simplex), a 1-cell a link (1-simplex), a 2-cell a triangle (2-simplex), a 3-cell a tetrahedron (3-simplex), etc. The Whitney form associated with $k$-cell (simplex) $s_i^k$ is written as

$$\Omega_{s_i^k} = k! \sum_{j=0}^{k} (-1)^j \xi_{i,j} d\xi_{i,0} \wedge \cdots \wedge d\xi_{i,j-1} \wedge d\xi_{i,j+1} \wedge \cdots \wedge d\xi_{i,k},$$

where $\xi_{i,j}$, $0 \leq j \leq k$, are the barycentric coordinates of the simplex $s_i^k$, and the wedge denotes the usual exterior product. These are piecewise linear forms. Higher-order forms are also possible. The Whitney $k$-forms are just linear interpolants for simplicial cochains and are uniquely determined from their integration over the $k$-simplices, which completely defines their degrees of freedom.

Using a Euclidean metric for the continuum three-dimensional case, the Whitney forms for various degrees can be easily written using the vector calculus language as basis functions for $E$ and $H$ [step (i) of Route B above]. For the $k=0$ case, we simply have, for each node $i$,

$$\Omega_{s_i^0} \rightarrow \tau_{s_i^0} = \xi_{i,0},$$

where $\xi_{i,0}$ is the (single) barycentric coordinate associated with the 0-simplex $s_i^0$ (node), and $g_E$ denotes the isomorphism (0-form to scalar) governed by the Euclidean metric. In this case, the Whitney form is the barycentric coordinate itself (a scalar) and the continuum approximation for a lattice 0-form is the usual node (point-based) interpolation through scalar functions $\tau_{s_i^0}$. The value of this function is equal to unity at $s_i^0$, and equal to zero at all other 0-cells $s_j^0$, $j \neq i$.

For the $k=1$ case, we have, for each edge $i$,

$$\Omega_{s_i^1} \rightarrow \tau_{s_i^1} = \xi_{i,0} \nabla \xi_{i,1} - \xi_{i,1} \nabla \xi_{i,0},$$

where $\xi_{i,0}$ and $\xi_{i,1}$ are the barycentric coordinates associated with the two vertices of the 1-simplex $s_i^1$ (edge). The Whitney forms (1-forms) in this case translate to a vector field, $\tau_{s_i^1}$ (from the isomorphism between 1-forms and vectors governed by the Euclidean metric). The resultant interpolation scheme for a lattice 1-form using the above elements is the so-called edge interpolation. The line integral of this function is equal to unity along its associated edge $s_i^1$, and equal to zero at all other 1-cells $s_j^1$, $j \neq i$.

For the $k=2$ case, we have, for each face $i$,

$$\Omega_{s_i^2} \rightarrow \tau_{s_i^2} = 2(\xi_{i,0} \nabla \xi_{i,1} \nabla \xi_{i,2} + \xi_{i,1} \nabla \xi_{i,2} \nabla \xi_{i,0} + \xi_{i,2} \nabla \xi_{i,0} \nabla \xi_{i,1}),$$
where now \( \xi_{i,j} \), \( j = 0,1,2 \) are the barycentric coordinates associated with the three vertices of the 2-simplex \( s_i^2 \) (triangular face). The Whitney forms (2-forms) in this case translate to a (pseudo-) vector field, \( \tau_i^2 \) (from the isomorphism between 2-forms and pseudovectors governed by the Euclidean metric). The surface integral of this function is equal to unity over its associated face \( s_i^2 \), and equal to zero at all other 2-cells \( s_j^2 \), \( j \neq i \).

The \( k = 3 \) case, we have, for each volume \( i \),

\[
\Omega_{s_i^1} \rightarrow \tau_i^1 = 6(\xi_{i,0} \nabla \xi_{i,1} \times \nabla \xi_{i,2}) \cdot \nabla \xi_{i,3} + (\xi_{i,1} \nabla \xi_{i,2} \times \nabla \xi_{i,3}) \cdot \nabla \xi_{i,0} + (\xi_{i,2} \nabla \xi_{i,3} \times \nabla \xi_{i,0}) \cdot \nabla \xi_{i,1} + (\xi_{i,3} \nabla \xi_{i,0} \times \nabla \xi_{i,1}) \cdot \nabla \xi_{i,2}),
\]

which results in a (pseudo-)scalar function, \( \tau_i^1 \) (from the isomorphism between 3-forms and pseudoscalars governed by the Euclidean metric) associated with each 3-simplex (tetrahedron). Despite the complicated appearance of Eq. (54), these are simple, step-like functions, which are constant on the associated tetrahedron and zero elsewhere. The corresponding volume integral is equal to unity over its associated volume \( s_i^3 \), and equal to zero over all other 3-cells \( s_j^3 \), \( j \neq i \).

Using the Whitney forms, we may write the 1-chain approximation for the \( \mathbf{E} \) fields as a sum of the \( k = 1 \) vector basis functions running over all 1-simplices \( s_i^1 \) of the simplicial cell complex \( \chi \),

\[
\mathbf{E} = \sum_i \langle s_i^1, \mathbf{E} \rangle \tau_i^1.
\]

For the \( \mathbf{H} \) field, the sum runs over the 1-cells of the dual complex \( \bar{\chi} \) (not simplicial anymore),

\[
\mathbf{H} = \sum_i \langle \bar{s}_i^1, \mathbf{H} \rangle \tau_i^1.
\]

Alternatively, as discussed before, a discrete approximation may be first sought for the inverse operator \( \star_{\mu}^{-1} \), so that

\[
[\star_{\mu}^{-1}] \colon \chi \rightarrow \bar{\chi},
\]

\[
\langle \bar{s}_i^1, \mathbf{H} \rangle = \sum_j [\star_{\mu}^{-1}]_{i,j} \langle s_j^2, \mathbf{B} \rangle,
\]

and \( \mathbf{B} \) is expanded over 2-simplices \( s_j^2 \) of the simplicial cell complex \( \chi \),

\[
\mathbf{B} = \sum_i \langle s_i^2, \mathbf{B} \rangle \tau_i^2.
\]

Note that Eqs. (55), (56), and (58) are approximations for the total vector fields (and not for each of their components separately).

The Whitney functions are commonly used as vector basis functions for the finite-element method to avoid the appearance of spurious solutions.\(^3\) According to their order, \( k \), they are sometimes referred to as node interpolants \( (k = 0) \), edge elements \( (k = 1) \), or face elements \( (k = 2) \). Such elements have been generalized for other types of complexes also (e.g., with hexahedral cells). However, as opposed to the simplicial case, there are no established mathematical results behind such generalizations. More importantly, some of the basic properties of Whitney forms on a simplicial lattice are not preserved in more general lattices. Among them is the divergence-free condition. It can be shown that, although for regular hexahedral lattices, these vector basis functions are divergence-free inside each element, in the case of general hexahedral elements, this is not true.\(^36\) The divergence in this case is proportional to the amount of deviation from a regular lattice.
For the general cell complex case, i.e., not necessary simplicial [note that even for the simplicial case, its dual lattice is not simplicial anymore, as exemplified by Fig. 7 and Eq. (56)], and in the present absence of definitive mathematical results, the use of simpler ad hoc interpolatory schemes to obtain $[\star_{\mu}], [\star_{e}]$ are of interest for practical purposes. Any such scheme is highly dependent on the type of problem and geometry considered. However, any interpolatory scheme should meet some basic consistency requirements, described next.

For a reciprocal medium, $\tilde{\mu}=\tilde{\mu}', \tilde{e}=\tilde{e}'$, the continuum Hodge operators $\star_{\mu}, \star_{e}$ on a Riemannian manifold are symmetric, nondegenerate, and positive definite operators. These properties are a simple consequence from the fact that a Riemannian metric tensor itself is a symmetric, nondegenerate, positive definite tensor.

However, for nonorthogonal lattices, usual interpolatory schemes for the discrete Hodge operators do not generally preserve the symmetry of the continuum operators. This is because the discrete versions of the Hodge are not strictly local. The lattice variables $\langle \vec{s}_i^1, E \rangle, \langle \vec{x}_i^2, H \rangle, \langle \vec{s}_i^2, B \rangle, \langle \vec{x}_i^4, D \rangle$ are defined over different geometric elements that span finite regions of space. Elements of the ordinary (twisted) cell complex that contribute to a given element of the twisted (ordinary) cell complex (and therefore define the local interpolatory stencil) are associated with metric elements defined at different points of space. Since the metric itself is a function of position, the local symmetry of $\star_{\mu}, \star_{e}$ may be lost on $[\star_{\mu}], [\star_{e}]$, if not enforced by construction. In addition, even if symmetry is enforced by construction on $[\star_{\mu}], [\star_{e}]$, the positive definiteness condition on the discrete Hodge may be violated when highly skewed lattices are employed.

Symmetric, positive-definite discrete Hodge operators yield real, positive eigenvalues for the matrices $[\star_{\mu}], [\star_{e}]$. This means the resultant discrete Maxwell’s system will not contain spurious eigenmodes with exponential growth in time. Nonsymmetric, nonpositive definite matrices would give rise to negative or complex eigenvalues. Regardless of their magnitude, negative or complex eigenmodes with exponential growth in time. Nonsymmetric, nonpositive definite matrices would give rise to negative or complex eigenvalues. Regardless of their magnitude, negative or complex eigenvalues lead to spurious eigenmodes with unconditional exponential time growth that eventually contaminate the solution (late-time instabilities). This can be seen by substituting Eqs. (47), (48) into Eqs. (38)–(41) and solving, e.g., for $\langle \vec{s}_i^1, E \rangle$. As a result, we get

$$\sum_m \left( \sum_{j,k,l} ([\star_{e}]^{-1})_{ij} \tilde{\beta}_{jk} ([\star_{\mu}]^{-1})_{kl} \beta_{lm} - \omega^2 \delta_{im} \right) \langle \vec{s}_m^1, E \rangle = 0, \quad (59)$$

for all $i$. The eigenfunctions of the corresponding system of differential equations for $\langle \vec{s}_i^1, E \rangle$ in the time–domain are given by

$$\langle \vec{s}_i^1, E \rangle \rightarrow \phi(t) = e^{\pm i\tilde{\lambda}^{1/2}} \phi^0, \quad (60)$$

where $\phi(t)$ is the column vector of eigenfunctions, $\phi^0$ is the initial value of $\phi(t)$ at $t=0$, and the elements of the matrix $\tilde{\lambda}$ are given by

$$A_{im} = \sum_{j,k,l} ([\star_{e}]^{-1})_{ij} \tilde{\beta}_{jk} ([\star_{\mu}]^{-1})_{kl} \beta_{lm}, \quad (61)$$

The above functional operations on matrices are understood in the usual manner, by using similarity transformations and operating on the matrix eigenvalues. We let $\tilde{\lambda} = \tilde{W} \cdot \tilde{\lambda} \cdot \tilde{V}^\dagger$, or $\tilde{\lambda} = \tilde{W} \cdot \tilde{\lambda} \cdot \tilde{V}^\dagger \cdot \tilde{V}$, where $\tilde{W}$ contains the right eigenvectors of $\tilde{\lambda}$, while $\tilde{V}$ contains the left eigenvectors (for a symmetric $\tilde{\lambda}$, they are the same), and $\tilde{V}^\dagger \cdot \tilde{W} = \tilde{I}$. Consequently, $\tilde{\lambda} \cdot \tilde{W} \cdot \tilde{u} = \tilde{W} \cdot \tilde{\lambda} \cdot \tilde{u}$, and $\tilde{u}^\dagger \cdot \tilde{W} \cdot \tilde{u} = \tilde{W} \cdot \tilde{\lambda}^\dagger \cdot \tilde{u}$, or, in general, $f(\tilde{\lambda}) \cdot \tilde{W} \cdot \tilde{u} = \tilde{W} \cdot f(\tilde{\lambda}) \cdot \tilde{u}$, by using a Taylor expansion on $f(\cdot)$, where $\tilde{\lambda}$ is a diagonal matrix containing the eigenvalues of $\tilde{\lambda}$. Using $\tilde{V}^\dagger \cdot \tilde{W} = \tilde{I}$, we can let $\phi^0 = \tilde{V}^\dagger \cdot \tilde{W} \cdot \phi^0$, i.e., expand $\tilde{\lambda}^{1/2}$ in terms of the eigenvectors of $\tilde{W}$. Equation (60) then becomes

$$\phi(t) = \tilde{W} \cdot e^{\pm i\tilde{\lambda}^{1/2}} \tilde{V}^\dagger \cdot \phi^0. \quad (62)$$
Equation (62) gives the solutions of the semidiscrete problem, i.e., without considerations about the time discretization. For any convergent time-discretization scheme (e.g., independent of the time step chosen), Eq. (62) will lead to bounded solutions, $\phi(t)$, if all the eigenvalues of $\tilde{A}$ are real and positive (note that for a lossless, dispersionless media, $\tilde{A}$ is real and, therefore, any complex eigenvalues will occur in conjugate pairs). If $\tilde{B}_{ij} = \tilde{B}_{ji}$ [Eq. (43)], it can be easily shown that this is true if $[\star_e]^{-1}$ and $[\star_m]^{-1}$ (and, consequently, $[\star_e]$ and $[\star_m]$) are simultaneously nonsingular, symmetric, negative definite or positive definite. The positive definite is the case of interest to recover the continuum Hodge operators.

As observed in Sec. IV, the discretization process can be viewed as a process of limiting the degrees of freedom in accessing global dynamic quantities of interest. The original infinite degrees of freedom in the continuum theory are reduced to a finite number over the cell complex elements. In the semidiscrete dynamic equations, Eq. (59), this is reflected in the reduction of the spectral content of the solution to a finite number of poles (eigenfrequencies). Usually this may also be viewed as a low-pass filtering, which is determined by the lattice spacing size and nature of approximation; but in general terms corresponds to a rearrangement of the spectral content. The requirement for symmetric, positive definite discrete Hodge operators corresponds to assuring (in lossless media) that no spurious poles are introduced in the upper-half complex $\omega$ plane after discretization.

An additional, interesting point revealed by the differential forms language is that, since the metric is entirely defined in the Hodge operators, the simulation of Maxwell’s equations on an irregular lattice and homogeneous medium can be mimicked by a dual theory, where we view the simulation performed on a regular lattice, but on an inhomogeneous, particular class of orthotropic medium with electric and magnetic constitutive tensors proportional to each other, i.e., $\tilde{e} = e\tilde{\Gamma}$, $\tilde{\mu} = \mu\tilde{\Gamma}$. In this latter case, metric factors are incorporated into the medium properties so that lattice irregularities become orthotropic inhomogeneities. By making use of such an observation, the properties that should be explicitly enforced on the final, approximate matrix representation of the discrete Hodge operators on a general irregular lattice can be established on very simple physical grounds only. This is simply because the violation of symmetry or positive definiteness would render the orthotropic media of the dual theory nonreciprocal or active, giving rise to spurious numerical artifacts.

As mentioned, symmetry and positive definiteness for $[\star_e]$ and $[\star_m]$ (or their procedural equivalents in the vector calculus language) are not always met by some of the commonly employed interpolations for finite-volume or finite-difference simulations. Symmetry is guaranteed only if it is explicitly enforced at each lattice point (e.g., through a perfectly symmetric numerical evaluation of the metric coefficients\(^{17}\) and the positive definiteness is usually violated when using highly skewed or curved meshes.\(^{11,27}\)

It should be stressed that these (metric-dependent) conditions over $[\star_e]$ and $[\star_m]$ are not sufficient for the consistency of the lattice theory. They should be enforced in addition to the topological consistency conditions previously discussed in Secs. IV and VI.

**VIII. CONCLUSIONS**

In this work, we discussed the application of differential forms and topological concepts to the study of lattice EM theory. Differential forms provide a very concise and elegant language to treat the classical EM theory on a lattice. It allows for the factorization of the field equations into a topological part and a metric part. The resultant topological equations are invariant under homeomorphisms, viz., invariant for lattices with the same topological structure. All the usual vector calculus operators are unified by a single operator, the exterior derivative, which admits a trivial and exact discretization on an arbitrary lattice through the use of its discrete adjoint, the boundary operator. This allows for a more general interpretation for the derivative on the lattice, not as a finite-difference approximation, but as an evaluation of fields at boundaries.
Consistency conditions for the lattice theory, such as divergence-free conditions and reciprocity, are discussed in a very general setting using purely topological concepts. Metric concepts need to be invoked only in connection with the Hodge operators, which also generalize the constitutive relations of the medium. General consistency requirements on the discrete Hodge operator are also discussed.

Lattice differential forms provide a richer geometrical language to discuss some aspects of the discretization procedure. The potential sources of inconsistency can be adequately identified and classified. The treatment of the EM fields \( E \) and \( B \) as ordinary forms, and \( D \) and \( H \) as twisted forms reveals a geometric reason of the dual lattice construction, common to EM discretization schemes for numerical simulations, such as the celebrated Yee scheme.\(^{31}\)

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