Semi-analytical modeling of arbitrarily distributed quantum emitters embedded in nano-patterned hyperbolic metamaterials

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Abstract: Nano-patterned hyperbolic metamaterials (NPHMs) have proven to be very efficient structures for enhancing the spontaneous emission rate (Γ) and quantum efficiency (η) of quantum emitters (QEs). However, much of the NPHM designs still rely on computationally costly three-dimensional (3D) numerical simulations. In this context, we propose a fast, semi-analytical method capable of calculating both Γ and η of QEs placed inside a medium bounded by nanopatterned structures. The low computational cost of our approach makes it attractive for optimizing the NPHMs geometrical parameters that maximize η for a desired Γ. Furthermore, we suggest a more realistic procedure to calculate the decay behavior of multiple QEs arbitrarily distributed in the NPHM. This calculation is only possible with the knowledge of Γ and η mapped for all possible positions of the QEs, which is easily achieved with the proposed model. As validation procedure, we compare the model results with those obtained by the finite difference time domain (FDTD) method. We apply the proposed model to an NPHM composed of nine Ag/SiO₂ layers, with the polymer host layer embedded with Rhodamine 6G, to maximize η for a specified tenfold increase of Γ. This procedure allowed η to be increased by 69% and 170% for one- and two-dimensional nano-patterning, respectively. The time required to build the Γ and η maps (used in the calculation of the decay behavior) is reduced by approximately 96% when compared to those numerically calculated via FDTD.

1. Introduction

In the past few years, much research efforts has been dedicated to the study of new approaches to efficiently extract photons from quantum emitters (QEs) in applications as diverse as lasers [1], LEDs [2,3], quantum sensors [4], quantum communication systems [5,6] and single-photon-sources [7]. All these systems rely on tailoring the QE’s radiation, usually assessed via well-known figures-of-merits such as spontaneous emission rate Γ, radiated power Q and external quantum efficiency η. The most promising approach for optimizing these parameters is to engineer the photonic density of states (PDoS), since it affects the QE’s emission properties [8–10]. This task can be accomplished by placing the QE either close or inside photonic structures, such as photonic crystal cavities [11,12], dielectric nanoholes [13], nano-antennas [14,15], and hyperbolic metamaterials (HMM) [9,10]. The use of HMMs is particularly attractive because of its simplicity and broadband spectral characteristics [9,10]. HMMs are well-known for exhibiting extreme anisotropy, i.e., the parallel (ε∥) and perpendicular (ε⊥) and
permittivity tensor elements (with respect to the anisotropy axis) present opposite signs leading to an open hyperboloidal isofrequency surface [10]. Thus, an ideal HMM allows the propagation of an indefinite number of photonic modes [9]. Most of these modes have wavenumber \( k \) much larger than the free space wavenumber \( k_0 \) and are therefore known as high-\( k \) modes. By engineering these modes through the choice of materials and geometrical parameters it is possible to manipulate the system’s PDoS and thus to control QE’s radiation parameters.

HMMs can be obtained from wire media [10,16–18] and from planar stacks of thin alternating metal/dielectric layers [8–10,19–22]. The latter is preferred because its dispersion profile can be easily tuned by changing materials, layer thicknesses, or wavelength [9]. Its large PDoS allows faster spontaneous emission rate [23], shorter lifetimes and higher QE dissipated power. Unfortunately, the spatial dispersion induced by high-\( k \) modes in real HMMs limits the number of supported modes and, consequently, the PDoS [24]. In addition, most of the dissipated power coupled into the HMM is due to the high-\( k \) modes which, in turn, do not couple to free-space modes [23,25]. Consequently, the energy of these modes is mostly thermally dissipated inside the HMM, reducing the QE’s \( \eta \). Some authors have resorted to nano-patterned HMMs (NPHM) to convert high-\( k \) modes into free-space modes \((k < k_0)\) to increase \( \eta \). For instance, in [2,26–30] the authors have used a patterned HMM surrounded by QEs, while in [31,32] the QEs are placed inside the HMM. Metallic or dielectric gratings on the top of the HMM have also been used for this purpose, where QEs are in the layer below the HMM [25], or in, where they are inside it [33–35]. Regardless of the geometry, the optimization of the NPHM geometrical parameters is usually carried out via numerical simulations, such as finite-difference time-domain (FDTD) and finite elements (FEM) methods, which are computationally costly. An alternative to improve the optimization process is to use semi-analytical techniques, such as in [36,37] where QEs are allowed in any layer of a non-patterned structure, or in [25] where QEs are placed close to the NPHM either in the top or bottom semi-infinite layer. Moreover, only a single (dipole) source can be used with these techniques, rendering them not general enough.

In this context, we propose here a general semi-analytical model to calculate, both in two and three dimensions (2D and 3D), all radiation parameters of any electromagnetic source type (including QEs modeled as a dipole) embedded in any of the layers of a nano-patterned stratified media (NSM). This method allows multiple electromagnetic sources arbitrarily distributed in the cover, substrate, and any inner layer to be used simultaneously. The flexibility of placing the source inside any layer (nano-patterned or not) allows us to very efficiently obtain the radiation emission parameters of any previously published HMM/source configuration [2,20-34]. The validation of the proposed method is carried out by calculating \( \Gamma \) and \( Q \) of a two-source system consisting of a QE embedded in a medium bounded by nano-patterned layers (QE\(_L\)) and a dipole source in the cover layer (QE\(_C\)). We also investigate the influence of QE\(_L\) on QE\(_C\)’s emission parameters and compare the results with FDTD simulations.

Once validated, we apply the proposed model to a more realistic situation where a NPHM is designed to enhance both the \( \Gamma \) and \( \eta \) of a QE. These parameters are mapped here as function of the QE and NPHM relative position, which is facilitated by the method’s low computational cost. This mapping is a helpful tool to understand the decay behavior of the whole system [13,26] since QEs are randomly distributed and oriented inside the NPHM [2,26–29,31–35]. By knowing the QE’s \( \Gamma \) and \( \eta \) at each position allows the decay curve \( g(t) \) of a system with multiple QEs to be analytically calculated. This procedure is fundamentally different from those currently used in the literature, where the decay curve is either fitted as a sum of exponentials [28,35,38] or calculated from \( \Gamma \) and \( \eta \) at each position [13,26]. These approaches resort to computationally costly numerical simulations to calculate \( \Gamma \) and \( \eta \). Our procedure, in contrast, is not only more accurate but also computationally more attractive and can be applied to any NSM. Furthermore, the analytically calculated \( g(t) \) allows the system’s effective external quantum efficiency \( \eta_{\text{eff}} \) and emission rate \( \Gamma_{\text{eff}} \) to be directly obtained assuming multiple
arbitrarily distributed electromagnetic sources in the cover, substrate, and any inner layer. In this sense, we also propose here a new procedure to optimize the NPHM geometrical parameters to maximize $\eta_{\text{eff}}$ while achieving the desired $\Gamma_{\text{eff}}$. Differently from previous approaches based on single QE at a fixed position [25], the proposed optimization procedure takes the behavior of all QEs embedded in the system into account. The optimization procedure is divided into three steps, as follows: Step 1) The 3D semi-analytical model (3D-SAM) is used to optimize the layer thicknesses of a non-patterned HMM to maximize $\eta_{\text{eff}}$ while achieving the desired $\Gamma_{\text{eff}}$. Differently from previous approaches based on single QE at a fixed position [25], the proposed optimization procedure takes the behavior of all QEs embedded in the system into account. The optimization procedure is divided into three steps, as follows: Step 2) The 2D semi-analytical model (2D-SAM) is employed to optimize the nano-pattern's period and fill-factor. This helps hasten the final optimization step based on the more realistic 3D model. Step 3) The 2D optimized parameters are fed into the 3D model for final tuning. In this step, the full $\Gamma_{\text{fp}}$ and $\eta_{\text{fp}}$ maps are built for a few different scenarios to evaluate the ones capable of achieving the desired $\Gamma_{\text{eff}}$. The proposed procedure is then applied to maximize the $\eta_{\text{eff}}$ of a system whose host (polymer) layer is embedded with Rhodamine 6G (R6G), a fluorescent molecule, while increasing $\Gamma_{\text{eff}}$ by a factor of 10 (actually, it can be any value). We show that a NPHM consisting of 9 alternating layers of silver (Ag) and silicon dioxide (SiO$_2$) can be optimized to enhance $\eta_{\text{eff}}$ from 4.3% (non-patterned) up to 7.3% and 11.6% if patterned along one (1D-NPHM) and two axes (2D-NPHM), respectively. We also show that the computational time required to build the $\Gamma_{\text{fp}}$ and $\eta_{\text{fp}}$ maps, used in the calculation of $g(t)$, is reduced by 96% (1D-NPHMs) and 98% (2D-NPHMs) when compared to FDTD simulations. All the MATLAB codes used to solve the scenarios proposed in this paper are available in Code 1 [39].

This article is organized as follows: Section 2 presents the mathematical model used to calculate $\eta_{\text{eff}}$ and $\Gamma_{\text{eff}}$, both in 2D and 3D. Section 3 begins with the validation procedure which consists in calculating $\eta_{\text{fp}}$ and $\Gamma_{\text{fp}}$ of a two-source double nano-patterned structure and comparing the results with those of FDTD simulations. With the validation procedure complete, we then follow steps 1 and 2 of the optimization procedure to find the best NPHM layer thicknesses, period and fill-factor to maximize $\eta_{\text{eff}}$. Finally, we carry out step 3 to map $\Gamma_{\text{fp}}$ and $\eta_{\text{fp}}$ for all possible QE positions for a 1D- and 2D-NPHM using the 3D-SAM. These maps are then utilized to calculate $\eta_{\text{eff}}$ and $\Gamma_{\text{eff}}$ for different source scenarios. Section 4 presents some concluding remarks.

2. Mathematical model
The proposed model assumes a \( W \)-layer stack sandwiched between two semi-infinite slabs, i.e., a substrate and a cover layer. The depicted nano-pattern profiles are just an illustration of possible geometries. \( A_x^w \) and \( A_y^w \) represent the \( w \)th layer unit cell sizes along \( x \) and \( y \)-directions, respectively. The sources \( S \), \( L \), and \( C \) are embedded within the substrate, \( w \)th layer, and cover, respectively. The electric fields radiated by the sources \( S \) (\( E_p^S \) and \( E_z^S \)) and \( C \) (\( E_p^C \) and \( E_z^C \)) are transmitted to layer \( w \) giving rise to the fields \( E_{sw} \) and \( E_{cw} \), respectively. These fields, combined with the electric field radiated by the source \( L \) (\( E_p^L \) and \( E_z^L \)), are reflected by layers \( w+1 \) and \( w-1 \), creating two electric fields (\( A \) and \( B \)) inside layer \( w \). Moreover, the sum of the electric fields propagating along the \( +z \) and \(-z \) directions inside layer \( w \) are transmitted to the cover \( E_{TC} \) and substrate \( E_{TS} \), respectively. A list of symbols utilized in this paper is provided in Visualization 1.

The electric fields radiated by the source \( A \) (where \( j= S \), \( L \), or \( C \)) along \( +z \) and \(-z \) directions, respectively, \( E_p^X \) and \( E_z^X \) are \( X \)-polarized (\( X= s \) or \( p \) for transversal electric (TE) and magnetic (TM) polarizations, respectively) electric field amplitudes propagating along \( +z \) and \(-z \) directions, respectively. Note that we use bold and italic non-bold notation to represent vectors and their respective magnitude, respectively (e.g. vector \( \mathbf{A} \) has magnitude \( A \)). \( \mathbf{k}_j = k_x \mathbf{x} + k_y \mathbf{y} \) is the parallel component of the wavevector \( \mathbf{k}_A = k_\parallel \mathbf{A} \). The electric field in the substrate \( \mathbf{E}_A(z) \) is given by:

\[
\mathbf{E}_A(z) = \sum_{\chi=s,p} \frac{E_{A_\chi}(k_\parallel, z)}{k_\parallel} \left[ \frac{\chi^+}{k_\parallel} \left( k_\parallel \right) P_{A_s}^+(k_\parallel) + \frac{\chi^-}{k_\parallel} \left( k_\parallel \right) P_{A_p}^+(k_\parallel) \right] e^{i(k_\parallel z - \epsilon_\parallel z)}(z-z_\parallel) > 0
\]

where \( E_{A_\chi} \) are the electric field vectors radiated by the source \( \chi= S, L, \) or \( C \) along \( +z \) and \(-z \) directions, respectively, \( E_{X_\chi} \) and \( E_{Z_\chi} \) are \( X \)-polarized (\( X= s \) or \( p \) for transversal electric (TE) and magnetic (TM) polarizations, respectively) electric field amplitudes propagating along \( +z \) and \(-z \) directions, respectively. Note that we use bold and italic non-bold notation to represent vectors and their respective magnitude, respectively (e.g. vector \( \mathbf{A} \) has magnitude \( A \)). \( \mathbf{k}_j = k_x \mathbf{x} + k_y \mathbf{y} \) is the parallel component of the wavevector \( \mathbf{k}_A = k_\parallel \mathbf{A} \). The electric field in the substrate \( \mathbf{E}_A(z) \) is given by:

\[
\mathbf{E}_A(z) = \sum_{\chi=s,p} \frac{E_{A_\chi}(k_\parallel, z)}{k_\parallel} \left[ \frac{\chi^+}{k_\parallel} \left( k_\parallel \right) P_{A_s}^+(k_\parallel) + \frac{\chi^-}{k_\parallel} \left( k_\parallel \right) P_{A_p}^+(k_\parallel) \right] e^{i(k_\parallel z - \epsilon_\parallel z)}(z-z_\parallel) > 0
\]
which the source $A$ is located. Assuming temporal dependence $e^{-j\omega t}$, where $\omega$ is the angular frequency, $P_\chi$ is written as [25]:

$$
P_\chi^\star(k_d) = \begin{bmatrix} k_x & -k_y \\
\sqrt{k_x^2 + k_y^2} & \sqrt{k_x^2 + k_y^2} \end{bmatrix}^T,
$$

and

$$
P_\chi(k_d) = \begin{bmatrix} k_x k_y & k_x k_y \\
k_x \sqrt{k_x^2 + k_y^2} & k_y \sqrt{k_x^2 + k_y^2} \end{bmatrix}^T.
$$

As seen in Fig. 1, $E_x^S$ and $E_C$ are transmitted to layer $w$ generating the electric field vector $E_{\Delta w} = [E_{\Delta w}^P, E_{\Delta w}^A] e^{i(k_{\Delta w} z - \omega t)}$, where $z_0 = -a$ or $b$ for $A = S$ or $C$, respectively, $a$ and $b$ are the distances from source $L$ to the layer $w-1$ and $w+1$, respectively ($a+b=h$, where $h$ is $w$th layer thickness). Furthermore, $E_{Lw}^T$, $E_{Lw}^A$, $E_{Sw}$ and $E_{Cw}$ are reflected at the upper and bottom interfaces creating two electric fields, one propagating along $+z$ ($A = [A_\psi^P + A_\nu^P favored]$) and the other propagating along $-z$ ($B = [B_\psi^P + B_\nu^P favored]$). Note also that the blue and red arrows in Fig. 1 represent the wavector and the electric field, respectively. If the nano-patterning is not present, then the fields for each $k_i$ point can be solved individually [23]. However, the nano-patterning does not affect the light (with moment $k_0^b$) into the $n$th and $m$th orders ($k_0^b + ik_k x + mK_y y$) in the $k_x$ and $k_y$ directions, respectively, creating a coupling mechanism between the harmonics. Consequently, all harmonics need to be solved at once. Hence, we propose to write the amplitude of the 2D Fourier transformed electric field $\Psi^X = \Psi^X_{\psi} = A', B', E_{\Delta w}$ as a modulated comb function ($\Psi^X_{\psi,comb}$) with periods equal to the supercell moment $K_s (K_x = 2\pi/\Lambda_x)$ and $K_z (K_y = 2\pi/\Lambda_y)$ along the $k_x$ and $k_y$ axes, respectively, and centered at $k_0^b = k_0^x x + k_0^y y$, as follows:

$$
\Psi^X_{\psi,comb}(k_0, z) = \sum_{i,m} \Psi^X_{i,m}(k_0, z) \delta(k - k_0^i - ik_i z) \delta(k - k_0^j - mK_j)
$$

$$
= \sum_{i,m} \Psi^X_{i,m}(k_0, z) \delta(k - k_0^i - ik_i z) \delta(k - k_0^j - mK_j),
$$

where $\Psi^X_{i,m}(k_0, z)$ is the $i$th and $m$th element in the $k_x$ and $k_y$ directions of the comb function, respectively, with $k_0^i$ and $k_0^j$ limited to the interval $0 \leq (k_0^i, k_0^j) \leq (K_x, K_y)$ so that no multiple identical combs exist. After solving the $\Psi^X_{\psi,comb}$ for each point of the proposed interval, the total electric field $\Psi^X$ is recalculated as

$$
\Psi^X(k_0, z) = \int \sum_{i,m} \Psi^X_{i,m}(k_0, z) \delta(k - k_0^i - ik_i z) \delta(k - k_0^j - mK_j) dK_x dK_y.
$$

where $\delta(.)$ is the Dirac delta function. For ease of notation, we define the $i$th and $m$th harmonics of $k_0^b(k_0^{bm})$ and $k_{\Delta w}(k_{\Delta w}^{bm})$ in the $k_x$ and $k_y$ directions, respectively, as follows:

$$
k_{i,m}^X(k_0^b) = k_0^x x + k_0^y y = (k_0^i - ik_i z) x + (k_0^j - mK_j) y,
$$

$$
k_{i,m}^{\Delta w}(k_0^{bm}) = \sqrt{(k_0^i)^2 - (k_0^j)^2}.
$$

After defining the electric field as comb functions, $E_{\Delta w, i,m}$ can be calculated as follows:

$$
\left[ E_{\Delta w, i,m}^X \right]_{i,m} = \left[ T_{\nu,p, i,m}^{k_0^x} (k_0^{bm}) \right]_{i,m} \left[ E_{\psi, i,m}^P e^{i\phi_\nu_{\psi,p, i,m}} \right]_{i,m} \left[ T_{\psi, j,m}^{k_0^y} (k_0^{bm}) \right]_{i,m} \left[ E_{\nu, j,m}^C e^{i\phi_\nu_{\nu,j,m}} \right]_{i,m}
$$

$$
\left[ E_{\Delta w, i,m}^X \right]_{i,m} = \left[ T_{\psi, j,m}^{k_0^x} (k_0^{bm}) \right]_{i,m} \left[ E_{\nu, j,m}^P e^{i\phi_\nu_{\psi, j,m}} \right]_{i,m} \left[ T_{\psi, i,m}^{k_0^y} (k_0^{bm}) \right]_{i,m} \left[ E_{\nu, i,m}^C e^{i\phi_\nu_{\nu, i,m}} \right]_{i,m}
$$

where $\phi_\phi$ is the phase difference.
where \([\cdot]_{Y \times U}\) represents a matrix with \(Y\) lines and \(U\) columns, \(T_{CW,\chi,\sigma}^{\kappa,\gamma}\) and \(T_{SW,\chi,\sigma}^{\kappa,\gamma}\) are the transmission coefficients (from cover and substrate to the \(u^\text{th}\) layer, respectively) of the \(\kappa^\text{th}\) and \(\gamma^\text{th}\) diffraction orders along \(x\) and \(y\) directions. The subscripts \(\chi\) and \(\sigma\) indicate the polarization of the incident and transmitted waves, respectively. Differently from 2D systems (nano-pattern along one axis), 3D systems (nano-pattern along two axes) give rise to cross-polarized waves also taken into account by the formalism, i.e., \(T_{CW,\chi,\sigma}^{\kappa,\gamma}\) and \(T_{SW,\chi,\sigma}^{\kappa,\gamma}\) for \(\chi \neq \sigma\) [25]. The transmission coefficients are calculated with the semi-analytical rigorous coupled wave analysis method (RCWA) [40–42] for a total number of diffraction orders \(N = N_xN_y\), with \(N_x\) and \(N_y\) as the number of diffraction orders along the \(x\) and \(y\) axes, respectively. The auxiliary variables used in (8) and (9) \((\alpha, \beta, \kappa\) and \(\gamma\)) are calculated as follows:

\[
\begin{align*}
\alpha &= \text{floor} \left( \frac{\nu-1}{N_x} \right) - \frac{(N_x-1)}{2} \\
\beta &= u - \frac{(N_y-1)}{2} - \text{floor} \left( \frac{\nu-1}{N_y} \right) N_y \\
\kappa &= \alpha - \text{floor} \left( \frac{\nu-1}{N_y} \right) N_y + \frac{(N_y-1)}{2} \\
\gamma &= \beta - \frac{\nu-1}{2} - \frac{(N_x-1)}{2} - \alpha + \frac{(N_x-1)}{2} N_x
\end{align*}
\]

(10)

where, the indices \(u\) and \(v\) represent the matrix’s line and row, respectively.

The electric fields \(\mathbf{A}\) and \(\mathbf{B}\) are obtained with the help of the 2D Fourier transformed electric fields after applying the boundary conditions at \(z = -a\) and \(z = b\). This results in the following linear system:

\[
\mathbf{F} = \mathbf{G}\mathbf{H}
\]

(11)

\[
\mathbf{F} = \begin{bmatrix}
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
\delta_n e^{-j\kappa z L} a \\
0
\end{bmatrix}_{NN} \\
\begin{bmatrix}
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
\delta_n e^{-j\kappa z L} a \\
0
\end{bmatrix}_{NN} \\
\begin{bmatrix}
\delta_n e^{-j\kappa z L} a \\
0
\end{bmatrix}_{NN} \\
\begin{bmatrix}
\delta_n e^{-j\kappa z L} a \\
0
\end{bmatrix}_{NN} \\
\begin{bmatrix}
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
\delta_n e^{-j\kappa z L} a \\
0
\end{bmatrix}_{NN} \\
\begin{bmatrix}
\delta_n e^{-j\kappa z L} a \\
0
\end{bmatrix}_{NN} \\
\begin{bmatrix}
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
-R_{\nu}\kappa^{\gamma} \left( \mathbf{k}^{\alpha,\beta} \right) e^{j\kappa z L} \\
\delta_n e^{-j\kappa z L} a \\
0
\end{bmatrix}_{NN}
\end{bmatrix}
\]

(12)

\[
\mathbf{G} = \begin{bmatrix}
\mathbf{A}_{\nu,\beta}^{\gamma} \\
\mathbf{A}_{\nu,\beta}^{\gamma} \\
\mathbf{B}_{\nu,\beta}^{\gamma} \\
\mathbf{B}_{\nu,\beta}^{\gamma}
\end{bmatrix}_{NN}
\]

(13)
where $\delta_{m}$ is the Kronecker delta, $R_{wC}^{kY}$ and $R_{wS}^{kY}$ are the reflection coefficients (from the $w^{th}$ layer to the cover and to the substrate, respectively) of the $k^{th}$ and $y^{th}$ diffraction orders along $x$ and $y$ directions. The reflection coefficients are calculated with RCWA \cite{40-42} for a total number of diffraction orders $N = N_x N_y$. The accuracy of the proposed method depends on the number of diffracted orders $N$. However, a large $N$ also increases the computational cost required to solve the linear system (11). $A$ and $B$ are obtained by solving (11) for each point in the interval $0 \leq (k_x^0, k_y^0) \leq (K_x, K_y)$, and substituting their harmonics $A_{lm}^{kY}$ and $B_{lm}^{kY}$ into (5).

After calculating $A$ and $B$, the 2D FT of the electric fields transmitted to the cover ($E_{TC}$) and to the substrate ($E_{TS}$) are calculated as follows:

$$E_{TC}(k_x, k_y) = \left[ E_{T}^{\alpha}(k_x, k_y) P_{s}^{\alpha}(k_x) + E_{T}^{\beta}(k_x, k_y) P_{s}^{\beta}(k_x) \right] e^{j[k_x z_L - k_y z_W]}.$$  

$$E_{TC}^{\alpha}(k) = \sum_{l,m=N}^{N_x, N_y} \left[ E_{L}^{\alpha}(k_x, l_m, b) + A^{\alpha}(k_x, l_m, b) + E_{S}^{\alpha}(k_x, l_m, b) \right] T_{wC}^{l_m}(k_x, l_m) +$$

$$E_{TC}^{\alpha}(k) = \sum_{l,m=N}^{N_x, N_y} \left[ E_{S}^{\alpha}(k_x, l_m, k_x, l_m, b) + B^{\alpha}(k_x, l_m, k_x, l_m, b) + E_{C}^{\alpha}(k_x, l_m, k_x, l_m, b) \right] T_{wS}^{l_m}(k_x, l_m) +$$

$$E_{TS}^{\alpha}(k) = \sum_{l,m=N}^{N_x, N_y} \left[ E_{T}^{\alpha}(k_x, l_m, a) + A^{\alpha}(k_x, l_m, a) + E_{S}^{\alpha}(k_x, l_m, a) \right] T_{wC}^{l_m}(k_x, l_m) +$$

$$E_{TS}^{\alpha}(k) = \sum_{l,m=N}^{N_x, N_y} \left[ E_{S}^{\alpha}(k_x, l_m, k_x, l_m, a) + B^{\alpha}(k_x, l_m, k_x, l_m, a) + E_{C}^{\alpha}(k_x, l_m, k_x, l_m, a) \right] T_{wS}^{l_m}(k_x, l_m) +$$

where $R_{wC}^{l_m}$ and $T_{wC}^{l_m}$ are the reflection (from the cover to the $w^{th}$ layer) and transmission (from the $w^{th}$ layer to the cover) coefficients, respectively, of the $p^{th}$ and $m^{th}$ diffraction orders along the $x$ and $y$ directions; $R_{wS}^{l_m}$ and $T_{wS}^{l_m}$ are the reflection (from the substrate to the $w^{th}$ layer) and transmission (from the $w^{th}$ layer to the substrate) coefficients, respectively, of the $p^{th}$ and $m^{th}$ diffraction orders along the $x$ and $y$ directions. After using the stationary phase method to propagate the FT electric field into the far field, the electric field transmitted to the cover ($E_{TC}$) or to the substrate ($E_{TS}$) is obtained as \cite{23}: 

\begin{align*}
H &= \begin{bmatrix}
R_{wC}^{kY}(k_x, k_y) & E_{L}^{\alpha}(k_x, k_y, b) + E_{S}^{\alpha}(k_x, k_y, b) \\
R_{wC}^{kY}(k_x, k_y) & E_{L}^{\alpha}(k_x, k_y, b) + E_{S}^{\alpha}(k_x, k_y, b) \\
R_{wC}^{kY}(k_x, k_y) & E_{L}^{\alpha}(k_x, k_y, b) + E_{S}^{\alpha}(k_x, k_y, b) \\
R_{wC}^{kY}(k_x, k_y) & E_{L}^{\alpha}(k_x, k_y, b) + E_{S}^{\alpha}(k_x, k_y, b)
\end{bmatrix}
+ \begin{bmatrix}
E_{C}^{\alpha}(k_x, k_y, b) \\
E_{C}^{\alpha}(k_x, k_y, b) \\
E_{C}^{\alpha}(k_x, k_y, b) \\
E_{C}^{\alpha}(k_x, k_y, b)
\end{bmatrix}
\end{align*}
\[
E_{\Delta \lambda} = -j k_\Delta \frac{e^{j k_{\Delta} \cdot \mathbf{r}^\prime}}{r} \left\{ \left[ E_{\Delta \lambda}^r \left( k_\lambda \sin \theta \cos \phi, k_\lambda \sin \theta \sin \phi \right) \theta + E_{\Delta \lambda}^\varphi \left( k_\lambda \sin \theta \cos \phi, k_\lambda \sin \theta \sin \phi \right) \varphi \right] \right\},
\]

where \( \theta \) and \( \phi \) are the elevation and azimuthal angles, \( \theta \) and \( \phi \) are unitary vectors in spherical coordinates along \( \theta \) and \( \phi \) directions, respectively. Finally, the total power radiated by all sources in the system (L, S, C located at positions \( z=L, r_S, r_C \), respectively) towards the cover (\( Q_C \)) or the substrate (\( Q_S \)) is calculated as follows:

\[
Q_\Delta = \frac{1}{2I_\Delta} \int_0^{\pi/2} \int_0^{2\pi} r^2 \sin \theta d\theta d\phi.
\]

where \( I_\Delta = \sqrt{\mu_\Delta / \epsilon_\Delta} \) is the cover’s (\( \Delta = C \)) or substrate’s (\( \Delta = S \)) impedance. The formalism used to calculate \( \Gamma \) and \( \eta \) assuming the sources as QEs is described in the next section.

### 2.1 QE source

Here, each QE source is modeled as a dipole source with electric dipole moment \( \mathbf{p}_\Delta \). We assume that the QEs are under weak coupling regime and low excitation power to avoid saturation \([8,19,23,43–45]\). Under these assumptions, the QEs can be modelled as infinitesimal dipoles governed by classical electromagnetic equations. In most applications, the QEs are randomly distributed and oriented inside a layer (either c, s or w) and each QE affects the system’s total decay rate. Therefore, to evaluate the decay curve \( g(t) \) and effective external quantum efficiency \( \eta_{\text{eff}} \) of such a system, we need first to calculate \( \eta_{\text{fp}} \) and \( \Gamma_{\text{fp}} \) at all possible positions (\( r_S, r_L, r_C \)) and orientations (\( \mathbf{p}_\Delta = \mathbf{p}_\Delta x, \mathbf{p}_\Delta y \) or \( \mathbf{p}_\Delta z \)), weighted by the probability of that position being occupied by a QE (\( \Gamma_{\text{fp}} \equiv \Gamma_{\text{fp}} (r_C, r_L, r_S, \mathbf{p}_C, \mathbf{p}_L, \mathbf{p}_S) \) and \( \eta_{\text{fp}} \equiv \eta_{\text{fp}} (r_C, r_L, r_S, \mathbf{p}_C, \mathbf{p}_L, \mathbf{p}_S) \)).

### 2.1.1 Calculating \( \eta_{\text{fp}} \) and \( \Gamma_{\text{fp}} \) for an arbitrary QE configuration

The first step is to write the 2D FT electric field radiated by QE placed at an arbitrary position \( \mathbf{r}_\Delta = x_\Delta \mathbf{x} + y_\Delta \mathbf{y} + z_\Delta \mathbf{z} \) as a comb function, as in (4), where the \( i \)th and \( m \)th harmonics in the \( k_x \) and \( k_y \) directions, respectively, are written as [25]:

\[
E_{\Delta \lambda}^{i,m} (k_{i,x}, k_{i,y}) = \frac{e^{i k_{i,x} (-x_\Delta) + i k_{i,y} (-y_\Delta)}}{4\pi} \begin{bmatrix} k_i \\ k_\Delta \sqrt{(k_i)^2 + (k_m)^2} \\ \pm \sqrt{(k_i)^2 + (k_m)^2} \\ k_\Delta k_i k_m \end{bmatrix} \mathbf{p}_\Delta.
\]
where \( \mu_\lambda \) is the magnetic permeability of the medium in which the source \( \Delta \) is located. The electric fields \( \textbf{A} \) and \( \textbf{B} \) are found by substituting (20) and (21) into (11) and then solving the resulting linear system. The spontaneous emission rate of the source \( \Delta (\Gamma_\Delta) \) with internal quantum efficiency equal to 1 is calculated as:

\[
\frac{\Gamma_\Delta (\textbf{r}_\lambda)}{\Gamma_0} = \frac{\omega}{2Q_0} \text{Im} \left\{ \textbf{p}_\lambda^* \left[ \textbf{A}_{\text{TF}} (\textbf{r}_\lambda) + \textbf{B}_{\text{TF}} (\textbf{r}_\lambda) + \textbf{E}_{\text{L}} (\textbf{r}_\lambda) + \textbf{E}_{\text{SW}} (\textbf{r}_\lambda) + \textbf{E}_{\text{CW}} (\textbf{r}_\lambda) + \textbf{E}_{\text{TS}} (\textbf{r}_\lambda) \right] \right\}, \quad \text{if } \Delta = \text{L}
\]

\[
\frac{\Gamma_\Delta (\textbf{r}_\lambda)}{\Gamma_0} = \frac{\omega}{2Q_0} \text{Im} \left\{ \textbf{p}_\lambda^* \left[ \textbf{E}_{\text{TF}} (\textbf{r}_\lambda) + \textbf{E}_{\text{TF}}^* (\textbf{r}_\lambda) \right] \right\}, \quad \text{if } \Delta = \text{C or S}
\]

where \( \Gamma_0 \) is the QE’s emission rate in vacuum, \( Q_0 \) is the power emitted by a dipole in free-space \( (Q_0 = p_0^2 \omega k_0^2 / 12 \pi \varepsilon_0) \), and \( \textbf{E}_{\text{TF}} (\Psi = \textbf{A}, \textbf{B}, \textbf{E}_\text{L}, \textbf{E}_\text{SW}, \textbf{E}_\text{CW}, \textbf{E}_\text{TS}, \textbf{E}_\text{TC}) \) is the electric field calculated using the 2D inverse FT as follows:

\[
\Psi_{\text{TF}} (\textbf{r}) = \frac{1}{2\pi} \int \Psi (\textbf{k}_\lambda, \varepsilon) e^{i (\textbf{k}_\lambda \cdot \textbf{r})} d\textbf{k}_\lambda.
\]

The calculation of \( \eta^\theta \) requires the knowledge of the total power dissipated by the dipole and of its far field portion radiated towards the substrate and cover. Thus, from (18), the temporal envelop of the electric field transmitted to the cover (\( \textbf{E}_{\text{TC}} \)) or to the substrate (\( \textbf{E}_{\text{TS}} \)) in the far field region is calculated as [23]:

\[
\textbf{E}_{\text{TA}} (t) = -j k_\Delta e^{i (\theta + \phi) / \lambda} \sum_{\lambda=\text{L,S,C}} \left\{ E_{\lambda \Delta}^\text{TA} (k_\lambda \sin \theta \cos \phi, k_\lambda \sin \theta \sin \phi) \right\} e^{-i \lambda / \lambda} + E_{\lambda \Delta}^\text{TA} (k_\lambda \sin \theta \cos \phi, k_\lambda \sin \theta \sin \phi) \Phi e^{i (\theta + \phi) / \lambda},
\]

where \( E_{\lambda \Delta}^\text{TA} \) is the amplitude of the 2D FT electric field transmitted from the source \( \Delta \) to the cover (\( \Delta = \text{C} \)) or to the substrate (\( \Delta = \text{S} \)). \( E_{\lambda \Delta}^\text{TA} \) is calculated with (16) and (17) when only the source \( \Delta \) is embedded in the system. The total power radiated by all the sources in the system (L, S, C located at positions \( \textbf{r} = \textbf{r}_1, \textbf{r}_2, \textbf{r}_3 \)) towards the cover (\( Q_\text{C} \)) or the substrate (\( Q_\text{S} \)) is calculated as follows:

\[
Q_\lambda (t) = \frac{1}{2L_\lambda} \int_0^t \int_0^{2\pi} |\textbf{E}_{\text{TA}} (t)|^2 r^2 \sin \theta d\phi d\theta.
\]

The total dissipated power by the three QEs (\( Q_\lambda \)) is given by

\[
Q_\lambda = Q_\lambda \Gamma^\lambda_\theta.
\]

where \( \Gamma^\lambda_\theta \) is the normalized spontaneous emission rate of all three QEs defined as,

\[
\Gamma^\lambda_\theta = \sum_{\lambda=\text{L,S,C}} \frac{\Gamma_\lambda}{\Gamma_0}.
\]
Finally, the system’s external quantum efficiency ($\eta^{fp}$) with QEs, $C_L$, $S$, and $E_S$ at fixed positions $z = r_x, r_y$, and $r_z$, respectively, is given by the ratio between their total radiated power at $t = 0$ and their total dissipated power, as follows:

$$
\eta^{fp} = \left[100\% \right] \frac{Q_S(0) + Q_C(0)}{Q_L \Gamma^{fp}}.
$$

(28)

Therefore, the mapping of $\eta^{fp}$ and $I^{fp}$ for all possible positions ($r_x, r_y, r_z$) and polarizations $p = (p_x, p_y, p_z)$ is accomplished by repeating this procedure for the desired values of $r_x$ and $p_s$ in (20), (21) and solving (27) and (28) for each case.

2.1.2 Calculating $\eta_{eff}$ and $\Gamma_{eff}$ for QEs randomly distributed and oriented in each layer

The $\eta^{fp}$ and $I^{fp}$ mappings are now used to estimate the system’s effective response due to a random distribution of sources inside each layer. Thus, the system’s effective external quantum efficiency $\eta_{eff}$ is calculated by averaging the efficiency of each QE in all possible configurations as follows:

$$
\eta_{eff} = \int \int \int \sum_{r_x, r_y, r_z, p_c, p_s} F(r_c, r_x, r_y, r_z, p_c, p_s) \eta_{eff} dV_c dV_r dV_z.
$$

(29)

where $V_r$ is the volume where each QEs is found ($[\Delta x_2, -\Delta x_2, \Delta z_{2inf}] \leq [(x, y, z) \leq [\Delta x_2, \Delta y_2, \Delta z_{2sup}])$, $z_{inf}$ and $z_{sup}$ are the lower and upper limits in $z$-axis where the source $\Delta$ can be placed. $F$ is defined as the probability density function that describes the probability of a given position being occupied by a randomly positioned and oriented QE. Similarly, the normalized time-dependent decay curve $g^{\Delta}(t)$ measured in the cover ($\Delta = C$) or the substrate ($\Delta = S$) is calculated as follows [13]:

$$
g^{\Delta}(t) = \frac{1}{\eta_{eff}} \int \int \int \sum_{r_x, r_y, r_z, p_c, p_s} F(r_c, r_x, r_y, r_z, p_c, p_s) \left[ \frac{Q^{\Delta}(0)}{\Gamma^{fp}} \right] dV_c dV_r dV_z.
$$

(30)

In (30), the normalized time-dependent exponential decay ($g^{\Delta}(t)$) is weighted by the QE’s external quantum efficiency at each position. After calculating $g^{\Delta}(t)$, the system’s effective lifetime ($t_{eff}$) is obtained through the condition $g^{\Delta}(t_{eff}) = 1/e$, resulting in the effective enhancement of spontaneous emission rate $P = \Gamma_{eff}/t_0 = t_0/t_{eff}$, where $t_0$ is the QE’s lifetime in free-space.

2.2 Special cases

In this section, we reduce the formalism to the case where the QEs are uniformly distributed and oriented inside each layer. This assumption greatly simplifies the calculation of $\eta_{eff}$ and $g^{\Delta}$, and is normally adopted when polymers embedded with QEs are spin-coated on top of the structure [2,25,27–29,31,33–35]. Thus, we evaluate (29) and (30) for the following scenarios:

1) three fixed QEs, one in the $w^{th}$ layer, one in the substrate and one in the cover layer. This scenario is used to validate the proposed method;

2) sources uniformly distributed and oriented in the NPHM’s $w^{th}$ layer. This scenario is used to demonstrate the optimization procedure.

3) sources uniformly distributed and oriented in the cover, the $w^{th}$ layer and substrate.

2.2.1 Three simultaneous sources at fixed positions (substrate, cover, and $w^{th}$ layer)
The source $Q_{\Delta}$ is fixed at position $\mathbf{r}_s$ with polarization $\mathbf{p}_{\Delta}$. Thus, $F = \delta(\mathbf{r}_c, \mathbf{r}_s, \mathbf{p}_c, \mathbf{p}_l, \mathbf{p}_s)$ and (29) and (30) are rewritten as:

$$\eta_{\text{eff}} = \eta^b = \left[100\%\right] \times \frac{Q_{\Delta}(0) + Q_{\Delta}^c(0)}{Q_0 \sum_{\Delta \in wL, L} \frac{\Gamma_{\Delta}}{\Gamma_0}},$$  \hspace{1cm} (31)

$$g_{\Delta}(t) = \left[ \frac{Q_{\Delta}(t)}{\Gamma_{\Delta}} \right] \frac{Q_{\Delta}(t)}{Q_0 \sum_{\Delta \in wL, L} \frac{\Gamma_{\Delta}}{\Gamma_0}}.$$  \hspace{1cm} (32)

Notice that (31) and (32) require $Q_{\Delta}, \eta^b, \Gamma_3$ to be calculated at a single position and polarization. The method validation is carried out by comparing $Q_{\Delta}, \Gamma_3$ with those directly obtained with the FDTD simulation, as described in section 3.1.

### 2.2.2 QEs uniformly distributed in the $w$th layer (QE$w$)

Here we assume that only the $w$th layer is embedded with QEs (QE$w$). Therefore, the triple integral in (29) and (30) is reduced to a single integral over $V_L$ (defined as the region $|x|, |y|, |z+(a-b)/2| \leq A_x/2, A_y/2, (a+b)/2$). Furthermore, $F$ becomes dependent only of $\mathbf{r}_L$ and $\mathbf{p}_L$ and can be written as follows,

$$F(\mathbf{r}_L, \mathbf{p}_L) = \frac{1}{3A_x A_y h} \mathbf{r}_L, \mathbf{r}_L \in V_L.$$  \hspace{1cm} (33)

The summation in (29) and (30) is carried out only for the three possible QE$w$ orientations ($\mathbf{p}_L = p_L x, p_L y$ or $p_L z$). The following definitions are adopted to relax the notation,

$$\begin{align*}
 Q_{\Delta}^b &= Q_{\Delta} \bigg|_{p_L \rightarrow \xi}, \\
 \Gamma_{\Delta} &= \Gamma_{\Delta} \bigg|_{p_L \rightarrow \xi} , \\
 \eta_{\xi} &= \eta^b \bigg|_{p_L \rightarrow \xi},
\end{align*}$$  \hspace{1cm} (34)

where $\xi$ is an auxiliary vector to designate the QE$w$ orientation ($\xi = x, y$ or $z$) with a corresponding auxiliary variable $\xi$ ($\xi = x, y$ or $z$). With these assumptions, $\eta_{\text{eff}}$ and $g_{\Delta}(t)$ are rewritten as,

$$\eta_{\xi} = \frac{1}{3A_x A_y h} \int_{\Delta} \int_{\Delta} \int_{\Delta} \eta_{\xi} \mathrm{d}z \mathrm{d}x \mathrm{d}y.$$  \hspace{1cm} (35)

$$g_{\xi}(t) = \frac{1}{3A_x A_y h \eta_{\xi}} \int_{\Delta} \int_{\Delta} \int_{\Delta} Q_{\Delta}^b(t) \mathrm{d}z \mathrm{d}x \mathrm{d}y.$$  \hspace{1cm} (36)

Typically, the polymers embedded with QEs are spin-coated on top of the structure [2,26–29]. Consequently, the QEs become uniformly distributed and oriented inside the structure, as adopted in this scenario. This assumption is also used in the optimization procedure described in section 3.2.

### 2.2.3 Sources uniformly distributed in each layer

Finally, assuming QEs uniformly distributed and oriented in each layer ($w$, $s$ and $c$), the following probability density function is obtained,
\[ F(\mathbf{r}_c, \mathbf{r}_L, \mathbf{r}_S, \mathbf{p}_C, \mathbf{p}_L, \mathbf{p}_S) = \frac{1}{27\Omega_c\Omega_L\Omega_S} \mathbf{r}_s \in V, \]  
\tag{37}

where \( \Omega_s \) is the volume of region \( V_s \). The triple integral over \( V_s, V_L \) and \( V_C \) is solved for all possible polarization combinations (3×3×3=27 possibilities). With these assumptions, \( \eta_{\text{eff}} \) and \( g_A(t) \) are calculated as follows:

\[ \eta_{\text{eff}} = \frac{1}{27\Omega_c\Omega_L\Omega_S} \int \int \int \sum_{\mathbf{p}_C, \mathbf{p}_L, \mathbf{p}_S} \eta^0 \mathbf{d}V_c \mathbf{d}V_L \mathbf{d}V_S, \]
\tag{38}

\[ g_A(t) = \frac{1}{27\Omega_c\Omega_L\Omega_S\eta_{\text{eff}}} \int \int \int \sum_{\mathbf{p}_C, \mathbf{p}_L, \mathbf{p}_S} \left[ \frac{Q_{\text{eff}}(t)}{\Gamma^{\text{eff}}} \right] \mathbf{d}V_c \mathbf{d}V_L \mathbf{d}V_S. \]
\tag{39}

This scenario can be used to calculate both the decay curve and the external quantum efficiency when a polymer embedded with QEs is spin-coated in more than one layer of the nano-patterned structure.

3. Results

We divide this section into two parts. In the first part, a validation procedure is carried out between the proposed method (3D-SAM) and Numerical FDTD [46] (used here as benchmark) for a double nano-patterned 11-layer stack excited by two distinct electromagnetic sources. In the second part, we stress our method by carrying out an optimization procedure aiming at maximizing \( \eta_{\text{eff}} \) for a desired value of \( P \) for two nano-patterned geometries (1D- and 2D-NPHM) with the host (polymethylmethacrylate, PMMA) layer embedded with R6G.

3.1 Validation procedure

The multilayer stack adopted in this procedure consists of \( W = 11 \) layers built on top of a SiO\(_2\) substrate, as illustrated in Fig. 2. Layers 1-9 consist of alternating Ag (with \( \varepsilon = -12 + 0.37j \)) and SiO\(_2\) (with \( \varepsilon = 2.132 \)) layers whose thicknesses are \( d_{\text{Ag}} \) and \( d_{\text{SiO}_2} \) (15 and 60 nm), respectively, except for layer 9 whose thickness is \( d_{\text{Ag}}/2 \) as suggested in [24]. The nano-pattern periodicity is \( \Lambda_x = \Lambda_y = 230 \) nm, groove width \( \rho_\Box = 50 \) nm and 3D fill-factor \( f_{\text{3D}} = f_{\text{2D}}^3 = \left( \rho_\Box/\Lambda_x \right)^2 = 0.6125 \), filled with PMMA. Layer 11 consists of a nano-patterned layer of titanium dioxide (TiO\(_2\), \( \varepsilon = 5.95 \)) with thickness \( d_c = 50 \) nm, period \( \Lambda_x = \Lambda_y = 115 \) nm, groove width \( \rho_\Box = 57.5 \) nm \( (f_{\text{3D}} = 0.25) \), filled with air (which is also the cover layer). To stress the method, we assume two QE sources embedded in the system. The first source (QE\(_L\)) is placed inside layer 10 (PMMA host layer) at \( a = 30 \) nm from layer 9 top interface and \( b = 20 \) nm from layer 11 bottom interface. The second source is placed inside the cover layer (QE\(_C\)) at a distance \( b \) from layer 11 top interface. Both sources are oriented parallel to the interfaces and the emission wavelength (\( \lambda_e \)) is 548 nm, which is the peak emission of a R6G [47]. Since both the stack and layer 11 have different nano-pattern periods, we need to define a supercell period along the \( x \) and \( y \) axes, i.e., \( \Lambda_x = \Lambda_y = 230 \) nm. To investigate the influence of QE\(_C\) on the radiation emission parameters of QE\(_C\), we calculate the spontaneous emission enhancement of both sources for scenarios with and without QE\(_L\), using both 3D-SAM (using the special case shown in section 2.2.1) and Numerical FDTD [48].

Figure 3(a) shows $\Gamma_r/\Gamma_0$ (red, circles) and $\Gamma_	ext{C}/\Gamma_0$ (blue, squares) obtained with 3D-SAM (solid lines) and FDTD (symbols) when both sources (QE$_L$ and QE$_C$) are present. This figure also shows $\Gamma_	ext{C}/\Gamma_0$ obtained with 3D-SAM when QE$_L$ is not present (dashed line) for comparison sake. The 3D-SAM results show good agreement with those of FDTD, confirming the accuracy of our model. The small distance between QE$_L$ and the HMM surface (30 nm) contributes to the increase of $\Gamma_r/\Gamma_0$ (up to 15) due to the high coupling between the evanescent modes with the HMM high-$k$ modes [25]. Moreover, the power radiated by QE$_C$ that reaches QE$_L$ is smaller than that dissipated by the high-$k$ modes, making the influence of QE$_C$ on $\Gamma_r/\Gamma_0$ very small and slightly dependent of $b_c$, as seen Fig. 3(a). In contrast, QE$_L$ strongly influences $\Gamma_	ext{C}/\Gamma_0$ because its radiation emission, especially the evanescent waves, constructively interferes with that from QE$_C$, therefore enhancing its dissipated power and, consequently, $\Gamma_r/\Gamma_0$. Furthermore, as $b_c$ increases, the evanescent waves decay exponentially, reducing the influence of QE$_L$ on $\Gamma_	ext{C}/\Gamma_0$, as indicated by the $\Gamma_	ext{C}/\Gamma_0$ convergence for the scenarios with and without QE$_L$.

Another important aspect is the power radiated to the far-field in the cover layer ($Q_C$). Figure 3(b) shows $Q_C/\Omega_0$ calculated as function of $b_c$ with both sources present, QE$_L$ and QE$_C$, (solid line) and without (dashed line) QE$_L$, and the FDTD results (squares) for the sake of comparison. Once again, the results obtained with the 3D-SAM nicely agree with those of the FDTD method. Note the increase of $Q_C/\Omega_0$ due to the presence of the additional electromagnetic source QE$_L$. Furthermore, as $b_c$ increases, $Q_C/\Omega_0$ decreases for both scenarios, since less evanescent waves are being converted into radiated waves by the TiO$_2$ nano-patterning. In summary, in addition to validating the proposed modeling, these results show that it is possible to control the radiation parameters of QE$_C$ ($\Gamma_r/\Gamma_0$ and $Q_C/\Omega_0$) with the addition of a second electromagnetic source (QE$_L$).
3.2 Optimization Procedure

Now we apply the proposed model to a more realistic case, with R6G (assumed as a uniformly distributed source, QE\textsubscript{L}) embedded in a PMMA host layer (\(\varepsilon = 2.22\), thickness \(h\)) on top of a NPHM. As mentioned, the proposed optimization procedure maximizes \(\eta_{\text{eff}}\) for any desired \(P\). However, in this example, we set \(P = 10\) because it represents a good compromise between \(P\) and \(\eta_{\text{eff}}\) (note that according to (35)-(36) \(P\) and \(\eta_{\text{eff}}\) are inversely proportional). Furthermore, since the R6G’s sources are assumed uniformly distributed, both \(P\) and \(\eta_{\text{eff}}\) are calculated following the steps described in section 2.2.2. The wavelength is again 548 nm. The optimization procedure is divided into three steps, as follows: In step (1) we optimize the layer thicknesses of a non-patterned HMM with the 3D-SAM, taking advantage of the structure’s axial symmetry and lack of nano-patterns to reduce the computation cost required to calculate \(\eta_{\text{eff}}\) and \(\Gamma_{\text{eff}}\). In step (2), we use the computationally efficient 2D-SAM to optimize the nanopattern’s period and fill-factor. This step helps hasten the optimization procedure in the more realistic 3D model. In step (3), we use the 2D optimized parameters into the 3D model for final tuning. In this step, the full \(\Gamma_{\xi}\) and \(\eta_{\xi}\) maps are built for a few different scenarios to evaluate the ones capable of achieving the desired \(\Gamma_{\text{eff}}\). Regarding the fine-tuning, we calculate \(P\) and \(\eta_{\text{eff}}\) for different values of \(h\) to determine the one that results in \(P = 10\) with maximum \(\eta_{\text{eff}}\).

Step 1: 3D-SAM optimization of a non-patterned HMM

Initially, we assume a non-patterned HMM with 9 alternating layers of Ag and SiO\textsubscript{2} [24] because this arrangement causes most of the radiated power (\(k \leq k_0\)) to be reflected back from the HMM’s surface into the cover layer. The Ag and SiO\textsubscript{2} layer thicknesses are \(d_{\text{Ag}}\) and \(d_{\text{SiO\textsubscript{2}}}\), respectively, except for the first layer whose thickness is \(d_{\text{Ag}}/2\), as depicted in Fig. 4(a).
Fig. 4. (a) HMM consisting of 9 alternating Ag/SiO$_2$ layers on top of a SiO$_2$ substrate. The PMMA cover layer with thickness $h$ is positioned on top of the HMM. The substrate and air cover layers (above the PMMA) are assumed semi-infinite. The (b) and (c) maps show the values of $h$ and $\eta_{\text{eff}}$, respectively, as function of the metal ($d_{\text{me}}$) and dielectric ($d_{\text{di}}$) thicknesses required to achieve $P=10$. The black lines in (b) and (c) separate the regions where the stack behaves as a metal, a HMM and a dielectric, according to [24].

The optimization procedure starts by adjusting $d_{\text{di}}$, $d_{\text{me}}$ and the host thickness $h$ to maximize $\eta_{\text{eff}}$ while maintaining $P=10$. We assume the unit cell size is infinite ($K_x$, $K_y \to 0$), there is no cross-polarized terms, and only the 0$^{th}$ diffraction order is present ($N_x=N_y=N=1$). With this assumption, the linear system (11) is split into two 2×2 systems, one for TM and another for TE, and each $k_\parallel$ point is decoupled and solved individually, resulting in the following set of equations:

\begin{align}
A'(k_\perp) &= \frac{E_{i}^x(k_\perp) + E_{i}^y(k_\perp)R_{w_{y}}^{0,0} (k_\perp) e^{2j\phi_{z,a}}}{1 - R_{w_{z}}^{0,0} (k_\perp) R_{w_{x}}^{0,0} (k_\perp) e^{2j\phi_{z,a}}}, \\
B'(k_\perp) &= \frac{E_{i}^x(k_\perp) + E_{i}^y(k_\perp)R_{w_{y}}^{0,0} (k_\perp) e^{2j\phi_{z,a}}}{1 - R_{w_{z}}^{0,0} (k_\perp) R_{w_{x}}^{0,0} (k_\perp) e^{2j\phi_{z,a}}},
\end{align}

(40)

(41)

Moreover, since neither $\eta_{\text{eff}}$ nor $P$ depends on the QE’s position along the $x$ and $y$ axes, the triple integral in (35) is reduced to a single integral along the $z$-axis only. The thickness $h$ required to achieve $P=10$ is obtained as function of $d_{\text{di}}$ and $d_{\text{me}}$ by solving (40) and (41) and then substituting the results back into (35). Figure 4(b) shows the resulting map for $0 \text{ nm} \leq d_{\text{di}} \leq 100 \text{ nm}$ and $0 \text{ nm} \leq d_{\text{me}} \leq 40 \text{ nm}$. Note that the total radiated power increases with $h$ because more QEs can be embedded into the PMMA layer in this case. The map corresponding to $\eta_{\text{eff}}$ (shown in Fig. 4(c)) is calculated using the value of $h$ obtained from Fig. 4(b). The black lines in Figs. 4(b) and 4(c) separate the maps into dispersion regions where the stack behaves as a dielectric, an HMM, and a metal according to [24]. The maximum values of $\eta_{\text{eff}}$ and $h$ in both figures occur close to the transition region where the stack changes its behavior from HMM to dielectric. In this region, the permittivity tensor elements $\varepsilon_x$ and $\varepsilon_y$ (obtained with [24]) not only change their signs from negative to positive, but they are also close to zero. According to [19], $P$ is higher in regions where both $\varepsilon_x$ and $\varepsilon_y$ are simultaneously close to zero and negative, which is consistent with the results shown in Figs. 4(b) and 4(c). Note also that, despite the claim that metals outperform HMMs [49–51], in the present configuration the best performance occurs...
when the stack behaves as an HMM, as indicated by the high $\eta_{\text{eff}}$ in this region (see Fig. 4(c)). This optimized performance gives $\eta_{\text{eff}} = 4.7\%$ for $d_{\text{di}} = 60 \text{ nm}$, $d_{\text{me}} = 15 \text{ nm}$ and $h = 40 \text{ nm}$.

**Step 2: NPHM period and fill-factor optimization with the 2D-SAM**

![2D-SAM Diagram](image)

After calculating both the host and HMM layer thicknesses, the next step is to calculate the NPHM optimum period ($\Lambda_x$) and fill factor $f_{2D}$ for the system shown in Fig. 5(a). To do so, we treat the system as 2D so that $d/dy = k_y = K_y = 0$ [25]. This guarantees that only TM modes are excited, as obtained after substituting $k_y = 0$ into (20) and (21). It also guarantees that only in-plane solutions of $R_{\omega k_x}^{\omega m}$ are analyzed since this coefficient now depends only on $k_x$ (rather than on both $k_x$ and $k_y$). $N_y = 1$, and there is no cross-polarization [25]. Consequently, the time required to calculate $R_{\omega k_x}^{\omega m}$ is markedly reduced compared to that of a 3D system when using RCWA. This feature makes the 2D method a good first approximation to hasten the optimization of more realistic 3D systems. Note that a 2D dipole source can be arbitrarily oriented in the $xz$ plane (as shown in Fig. 5(a)) and still behave equivalently to a dipole line along the $y$-axis in 3D. Therefore, we denote it here as 2D-dipole. After these assumptions, (11) is rewritten as follows
Although our formalism can be applied to an arbitrarily oriented 2D-dipole in the xz-plane, in the optimization procedure we assume the 2D-dipole oriented parallel to the HMM surface, i.e., along the x-axis (p = px). This assumption is justified because 2/3 of the QEs are oriented parallel to the HMM surface (x and y axes) in the 3D case [9]. Moreover, we also assume the 2D-dipole at a fixed position since the QEs’ position affects the amplitude and phase of the evanescent fields that interact with the NPHM surface, but not with the modes responsible for outcoupling the high-k waves. These modes are affected by the nanopatterns geometrical parameters. Consequently, optimizing \( \Gamma / T_0 \) and \( \eta_s \) at one position is enough to guarantee that both parameters are optimized for any position inside the host layer. In this sense, we assume the 2D-dipole embedded in a 40 nm thick PMMA host centered at the NPHM ridge at \( \alpha = 10 \) nm from the NPHM surface because \( \Gamma / T_0 \) is higher in this case [26]. Assuming \( 0.5 \leq f_{2D} \leq 0.99 \) and \( 50 \) nm \( \leq \Lambda_x \leq 500 \) nm, we obtain the \( \Gamma / T_0 \) and \( \eta_s \) maps shown in Figs. 5(b) and 5(c), respectively. Figures 5(b) and 5(c) also present \( f_{2D} \) for a groove width \( \rho_g = 50 \) nm (white lines) to facilitate an eventual fabrication process. Observe in these figures that the \( \Gamma / T_0 \) peak values correspond to four resonant modes indicated by the dashed lines for ridge widths \( \rho_g = 42 \) nm (A), 156 nm (B), 278 nm (C) and 400 nm (D). These resonances are not function of \( \rho_g \), but of \( \rho_x \). This behavior suggests that the modes are confined in the NPHM ridge, which is a signature of localized surface plasmon polaritons (LSPP). At the resonance condition, \( \Gamma / T_0 \) is highly enhanced, reaching values as high as 80 according to Fig. 5(c). However, the high ohmic losses associated with LSPPs attenuates most of the coupled energy and reduces the amount of NPHM power radiated into free-space. Consequently, \( \eta_s \) is minimum at the LSPP resonance, as also indicated by equation (35) and Fig. 5(c). This suggests a compromise between \( \Gamma / T_0 \) and \( \eta_s \), which is achieved with \( \rho_g = 50 \) nm, \( \Lambda_x = 230 \) mm and \( f_{2D} = 0.6 \), resulting in \( \Gamma / T_0 = 28 \) and \( \eta_s = 27\% \).

Step 3: NPHM fine tuning with the 3D-SAM

We now use the optimized 2D parameters estimated in the previous section as a first approximation for the 3D model. First, we calculate \( \eta_s \) and \( \Gamma / T_0 \) at each possible position inside the host layer (to be used in the calculation of the system’s \( \eta_{eff} \) and \( P \)). We investigate three different structures to demonstrate the potential of the proposed 3D model for optimization purposes, namely: structure 1 (S1), consisting of a non-patterned HMM with thicknesses \( d_a = 60 \) nm, \( d_{me} = 15 \) nm, see Fig. 6 (a); structure 2 (S2), which is structure S1 patterned along the x-axis (1D-NPHM) with period \( \Lambda_x = 230 \) nm and \( \rho_x = 50 \) nm, see Fig. 6 (e); and structure 3 (S3), which is structure S1 patterned along the x and y axes (2D-NPHM) with periods \( \Lambda_x = \Lambda_y = 230 \) nm and \( \rho_x = 50 \) nm, as seen in Fig 6 (i). For all three cases the host thickness is set to \( h = 40 \) nm. The reflection and transmission coefficients \( (\zeta_{x,y}^{\text{inc}}(k_x, k_y), \zeta = R_{x,c}, R_{y,s} \text{ or } T_{x,c}) \) are calculated using the 3D-RCWA for \( (N_x, N_y) = (1 \times 1), (15 \times 1) \) and \( (15 \times 15) \) relative to S1, S2 and S3, respectively. The dipole is simulated assuming the parallel wavevector components of the incident plane wave \( (k_x, k_y) \) are swept by 1801 points each in the interval \( 0 \leq |k_x| |k_y| |k_x|, k_y| \leq 20 \). After calculating \( \zeta_{x,y}^{\text{inc}}(k_x, k_y), \) \( \Psi_{\text{comb}}^X \) is computed by solving the linear system (11) in the intervals \( 0 \leq k_x, k_y \leq K, \) \( 0 \leq k_x, k_y \leq K. \) Finally, \( \Psi_{\text{comb}}^X \) is used into (5) to obtain \( \Psi^X \), which is then used in (22) to obtain \( \Gamma / T_0 \).
Figure 6 shows $\Gamma / \Gamma_0$ as function of position for structures S1 (Figs. 6(b)-6(d)), S2 (Figs. 6(f)-6(h)) and S3 (Figs. 6(j)-6(l)). The dipole is assumed oriented along $\xi = z$ (Figs. 6(b), 6(f), 6(j)), $x$ (Figs. 6(c), 6(g), 6(k)) and $y$ (Figs. 6(d), 6(h), 6(l)). Note that $\Gamma / \Gamma_0$ for S1 and S2 is plotted in the $xz$ plane because there are no variations along the $y$-axis. In contrast, $\Gamma / \Gamma_0$ for S3 is plotted in three different planes, namely $z=5$, -20 and -35nm because the patterns are along the $x$ and $y$ axes. In all cases, $\Gamma / \Gamma_0$ increases exponentially as the QE approaches the NPHM/HMM interface due to strong evanescent field coupling with the structure. Moreover, when $p = px$ or $py$ the QE only radiates TM polarized waves (see (20) and (21)). In contrast, when $p = pz$ the QE only radiates TM polarized waves. Note that $\Gamma / \Gamma_0$ is higher than $\Gamma / \Gamma_0$ and $\Gamma / \Gamma_0$, as shown in Fig. 6, because the hyperbolic dispersion profile occurs only for TM polarization [24]. Note also that $\Gamma / \Gamma_0$ of S1 is constant (and identical for $\xi = x$ or $y$ due to symmetry) in the $xy$-plane regardless of the polarization (see Figs. 6(b)-6(d)) owed to its uniformity along the $x$ and $y$ directions. In contrast, $\Gamma / \Gamma_0$ of S2 varies along $x$ and is maximum close to the ridge’s edge due to scattering, as seen in Figs. 6(f)-6(h). Moreover, the reduced coupling of evanescent waves due to QEs located far from the groove (for S2 or S3) results in a decrease of $\Gamma / \Gamma_0$. Regarding S3, $\Gamma / \Gamma_0$ is maximum at the center of the NPHM for $\xi = x$ (Fig. 6(k)) or $\xi = y$ (Fig. 6(l)), while it is minimum for $\xi = z$ (Fig. 6(j)). Note the similarity of $\Gamma / \Gamma_0$ (Figs. 6(f) and 6(j)) and $\Gamma / \Gamma_0$ (Figs. 6(h) and 6(l)) relative to S2 and S3, respectively. The same pattern is observed for $\Gamma / \Gamma_0$ if S2 is rotated $90^\circ$ around the $z$-axis.

Next, $\eta_c$ is calculated as function of position inside the host layer using (28) for structures S1 (Figs. 7(a)-7(c)), S2 (Figs. 7(d)-7(f)) and S3 (Figs. 7(g)-7(i)) assuming $\xi = z$ (Figs. 7(a), 7(d), 7(g)), $x$ (Figs. 7(b), 7(e), 7(h)), and $y$ (Figs. 7(c), 7(f), 7(i)), as shown in Fig. 7. According to (28), $\eta_c$ is inversely proportional to $\Gamma / \Gamma_0$ and thus $\eta_c$ increases when the QE is located further away from the NPHM surface, as shown in Fig. 7 for S1, S2, and S3. Furthermore, when $b << \lambda$, the interference at the QE position between the wave radiated by the QE ($|\mathbf{k}_s| \leq \mathbf{k}_0$) and the wave reflected from the NPHM surface is destructive for $p = pz$ and constructive for $p = px$ or $py$. The combination of high $\Gamma / \Gamma_0$ with destructive interference makes $\eta_c$ smaller than $\eta_i$ and $\eta_t$, as observed in Fig. 7. In fact, $\eta_c$ is close to 0% (Fig. 7(a)) for any QE position for S1. In
contrast, $\eta_z$ and $\eta_y$ are as high as 40% (Figs. 7(b) and 7(c)) for QEs far from the NPHM interface. The nano-patterning present in S2 and S3 convert the evanescent waves ($|k_\parallel| \geq k_0$) inside the NPHM into propagating waves ($|k_\parallel| \leq k_0$), therefore increasing $\eta_z$ for QEs located at any position inside the host layer, especially for those close to the NPHM surface. As expected, $\eta_z$ and $\eta_y$ of S2 increase when the QE is located above the NPHM groove but distant from the NPHM surface (see Figs. 7(d)-7(f)), since these are the positions where $\Gamma_\parallel/\Gamma_\parallel$ is minimum.

Similarly, for S3, the position where $\eta_z$ is maximum coincides with the positions where $\Gamma_\parallel/\Gamma_\parallel$ is minimum, as seen in Figs. 7(g)-7(i).

Fig. 7. $\eta_z$ as function of position inside the host layer on top of structures S1 (a,b,c), S2 (d,e,f), and S3 (g,h,i) assuming $\zeta_\parallel$ (a,d,g), $\alpha$ (b,e,h), and $\gamma$ (c,f,i).

Once $\Gamma_\parallel/\Gamma_\parallel$ and $\eta_z$ have been obtained, the next step is to calculate $\eta_{\text{eff}}$ and $P$ using (35) and (36), respectively. However, increasing $P$ is not a trivial task because $\Gamma_\parallel/\Gamma_\parallel$ is weighed by $\eta_z$. Thus, mapping $\eta_z$ and $\Gamma_\parallel/\Gamma_\parallel$ at each position inside the host layer is essential to simplify this process and may also prove to be a useful tool for understanding the system’s overall behavior. Usually, this mapping is carried out numerically with computationally costly methods such as FDTD [13,26]. For example, in a regular desktop computer (Intel Core i7-3960X processor with 64GB of RAM), each 3D-FDTD simulation of structure S2 takes approximately 20 hours. Therefore, approximately 180 days would be required to build the maps for each polarization (each map size is 24x18 pixels, but only 12x18 pixels are solved due to symmetry). This is reduced to approximately seven days for all three polarizations with the proposed model (4 days to calculate all $\zeta_{\lambda,\sigma}^{l,m}$ terms and 20 minutes for each pixel), which corresponds to about a 96% reduction in the processing time.

The map building for structure S3, in contrast, is computationally more intensive since it is nano-patterned along both $x$ and $y$ axes. With the 3D-FDTD method, approximately 906 days would be necessary to obtain the maps of all three polarizations of Figs. 6 and 7 (each map cut is 21x21 pixels, but only 11x11 pixels are solved due to symmetry). However, with our approach $\zeta_{\lambda,\sigma}^{l,m}$ is calculated in 4 days and each pixel is calculated in approximately 3 hours, resulting in a total simulation time of 19 days for all three polarizations. In summary, with the
The proposed method the computational time for simulating 1D and 2D NPHMs is reduced by 96% and 98%, respectively, when compared to FDTD. This allows $P$ and $\eta_{ef}$ to be calculated more efficiently as function of $h$.

The last parameter to be optimized is the host layer thickness $h$ since it is directly related to $P$ and $\eta_{ef}$. The plots of $P$ and $\eta_{ef}$ for 10 nm $\leq h \leq 50$ nm are shown, respectively, in Figs. 8(a) and 8(b) for structures S1 (blue line), S2 (squares) and S3 (circles). The number of simulated $h$ points differs for each structure because of the time required to map $\eta_{l}$ and $\Gamma/\Gamma_0$. According to Fig. 8(a), a high $P$ value is achieved for small $h$ due to the QEs strong evanescent wave coupling to the HMM surface in this situation. This figure also shows that S1’s $P$ is higher than those from S2 and S3, particularly for small $h$. The relatively small $P$ of S2 and S3 is due to the low $\Gamma/\Gamma_0$ and high $\eta_{l}$ regions (see (35) and (36)) for QEs close to the HMM surface but above the grooves of these structures, as shown in Figs. 6 and 7. Note that part of the QEs is located far from the NPHM surface when $h$ is increased, causing the NPHM influence on the QE decay to reduce and, consequently, the $P$ of all structures to converge to the same value, as observed in Fig. 8(b). In contrast, S1’s $\eta_{ef}$ diverges from those of S2 and S3 as $h$ increases, as shown in Fig. 8(b). Moreover, the lack of nano-patterning in S1 not only increases $P$, but also reduces $\eta_{ef}$, as evidenced in Fig. 8(b) since no high-$k$ modes decoupling mechanism exists inside the HMM in this case. However, S3’s nano-patterning decouples the high-$k$ modes that propagate along the $x$ and $y$ axes, whereas S2’s decouples only those propagating along the $x$-axis resulting in $\eta_{ef}$ higher for S3 than for S2. This effect is more evident for $h<40$ nm due to the more efficient coupling of evanescent waves to high-$k$ modes. According to Figs. 8(a) and 8(b), it is possible to achieve a good compromise between $P$ and $\eta_{ef}$ with an appropriate choice of $h$, which is computationally viable with the proposed method.

Finally, the NPHM designs relative to structures S1, S2, and S3 that produce $P=10$ with high $\eta_{ef}$ require $h=40$ nm ($\eta_{ef}=4.3\%$), 30 nm ($\eta_{ef}=7.3\%$), and 40 nm ($\eta_{ef}=11.6\%$), respectively. Therefore, in addition to reducing the computational time, the proposed model and optimization procedure can increase the QEs’ external quantum efficiency by about 69% and 170% using 1D and 2D NPHMs, respectively.

Fig. 8. (a) $P$ and (b) $\eta_{ef}$ calculated for structures S1 (blue line), S2 (squares), and S3 (circles) for 10 nm $\leq h \leq 50$ nm.

4. Conclusion

In this paper, we have proposed a semi-analytical method capable of calculating, both in 2D and 3D, all radiation parameters of any electromagnetic source type (including QEs modeled as a dipole) embedded in any stratified media (nano-patterned or not). This method also allows for multiple electromagnetic sources to be arbitrarily distributed in the cover, substrate, and any inner ($w^{th}$)-layer and used simultaneously. The validation procedure consisted in calculating $\Gamma_3$ and $Q_c$ for a two-source system consisting of QEs (inside a host bounded by nano-patterned layers, $A=$L) and a dipole source (in the cover layer, $A=C$). The influence of the former on the
latter was also investigated and compared with FDTD simulations. We were also able to map $I_\gamma$ and $\eta_\gamma$ as function of the QE and NPHM relative position, which is only feasible because of the method’s low computational cost. This mapping is an important tool for understanding the decay behavior of the whole system due to the random distribution/orientation nature of QEs in NPHMs. We were also able to calculate analytically the decay curve $g(t)$ of a multi-source system, which is fundamentally different from the exponential decay fitting and exponential summation techniques normally adopted in the literature. Besides allowing us to calculate $\eta_{eff}$ and $I_{\gamma eff}$ more efficiently, this feature also allowed us to propose a new optimization procedure to maximize $\eta_{eff}$ for the desired $I_{\gamma eff}$. The optimization procedure was divided into three steps: 1) a non-patterned HMM was used to optimize the metallic and dielectric layer thicknesses with the proposed 3D-SAM. Here, we took advantage of the structure's axial symmetry and lack of nano-patterns to reduce the 3D-SAM computational cost even further and thus calculate $\eta^p$ and $\eta^d$ maps required to calculate $\eta_{eff}$ and $I_{\gamma eff}$; 2) the computationally efficient 2D-SAM model was then employed to optimize the nanopattern's period and fill-factor; 3) finally, the 2D optimized parameters were fed into the 3D-SAM model for final tuning. This procedure was applied to maximize the $\eta_{eff}$ of Rhodamine 6G (R6G) with a tenfold increase in $I_{\gamma eff}$. We showed that an NPHM consisting of 9 alternating layers of Ag and SiO$_2$ could enhance $\eta_{eff}$ from 4.3% (non-patterned) up to 7.3% and 11.6% if patterned along one (1D-NPHM) and two axes (2D-NPHM), respectively. We also showed that the computational time required to build the $I_\gamma$ and $\eta_\gamma$ maps could be reduced by 96% (1D NPHMs) and 98% (2D NPHMs) when compared to FDTD simulations.

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