Anisotropic metamaterial blueprints for cladding control of waveguide modes

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We explore the duality between metric and material properties in the Maxwell’s equations to design metamaterial blueprints for cladding control of waveguide modes. We illustrate designs for circular and rectangular waveguides. In contrast to standard material loading of waveguides, the proposed cladding does not produce hybrid modes. The cutoff frequencies of the propagating modes can be controlled by the material parameters in a simple fashion. © 2010 Optical Society of America

1. INTRODUCTION

It is well known that the ray path followed by an electromagnetic wave is determined according to Fermat’s principle by the bulk constitutive properties of the medium. Equally true is the fact that the ray path of a wave in a vacuum is influenced by the metric of space. According to general relativity, massive objects cause space-time curvature, and the metric or distance measure deviates from the flat Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$. In a curved space-time, propagation paths follow geodesics, leading to such effects as gravitational lensing and a shift in the apparent position of near-occulted stars. It follows from the above that wave propagation paths can be influenced through (1) local electrical interactions governed by the constitutive relationships in the Maxwell’s equations and (2) distant gravitational interactions through the metric of space. Rays that travel along nonstraight paths can in principle be realized either by the gravitational field of some mass distribution or by a medium with a properly chosen set of constitutive parameters. Even though the gravitational fields would need to be too strong to produce substantial effects at laboratory scales, properly designed metamaterials can approximate the necessary constitutive parameters at these scales. The duality between metric and constitutive relations, i.e., the fact that changes in the former can be mimicked by modifications in the latter [1–8], has been recently exploited for a host of possible practical applications such as electromagnetic cloaks [8,9] and masks [10,11], wave splitters [12], polarization rotators [13], field concentrators [14], reflectionless waveguide bends [15,16], and waveguide miniaturization [17] to name just a few. A recent comprehensive survey on these efforts is provided in [18].

In this paper, we explore this duality in a different context: to design metamaterial claddings for waveguide mode control. By using the proposed claddings, better mode uniformity within the core region can be obtained, together with a precise control of the corresponding frequency cutoffs. At the same time, the modal power distribution can be increasingly confined within the cladding region. In contrast to conventional material loading of waveguides, the proposed (meta)material cladding does not produce hybrid modes. Each resulting mode is homotopic to a hollow waveguide mode, in the sense that the former can be produced from a continuous deformation of the latter within the waveguide core. Physically, this means that the mode distribution within the waveguide core is equivalent to that of a waveguide with different dimensions. We illustrate the analysis for circular and rectangular perfectly electrically conducting (PEC) waveguides.

2. METAMATERIAL CLADDINGS FOR WAVEGUIDE MODE CONTROL

A. Circular Waveguide

Throughout this paper, we work in the frequency domain with the $e^{-i\omega t}$ convention. We first illustrate our approach with a circular waveguide. We assume a waveguide with a PEC wall of radius $r$. The metamaterial cladding exists in the region $d \leq \rho \leq R$ for some positive $d$ and where $\rho$ represents the usual radial coordinate, as illustrated in Fig. 1, and $t = R - d$ is hence the thickness of the cladding. The objective of the cladding is to mimic a deformation on the metric of space (“stretching” or “squeezing” of space) for $d \leq \rho \leq R$. The required deformation can be accomplished by the coordinate transformation below:

$$\rho' = \begin{cases} 
\rho & \text{for } 0 \leq \rho \leq d \\
R + \int_d^\rho s(\rho')d\rho & \text{for } d \leq \rho \leq R,
\end{cases} \quad (1)$$

where $s(\rho)$ is a real positive function. The choice $s(\rho) \geq 1$ corresponds to a stretching and the choice $s(\rho) \leq 1$ corre-
sponds to a squeezing of space in the region \( d \leq \rho \leq R \). For simplicity, we assume \( s(\rho) \) to be constant in what follows, although the analysis below can be readily adapted to variable \( s(\rho) \).

In terms of the transformed variable \( \rho' \), the fields inside the waveguide are governed by the Maxwell's equations with constitutive parameters \( \varepsilon_0 \) and \( \mu_0 \). When the equations are rewritten in terms of the variable \( \rho' \), however, they become equivalent to the Maxwell's equations of some transformed fields in an anisotropic medium with constitutive parameters \( [\varepsilon] = \varepsilon_0[\Lambda] \) and \( [\mu] = \mu_0[\Lambda] \). The derivation of the tensor \( [\Lambda] \) can be found (in a different context) in [5, 19] and is summarized below for convenience.

In terms of \( \rho' \), the three components of Faraday's law are given by

\[
\begin{align*}
\nu_0 \partial E^c_{\phi} &= -\frac{1}{\rho'} \frac{\partial}{\partial \phi} \frac{1}{s} \frac{\partial E^{c\phi}}{\partial z}, \\
\nu_0 \partial E^c_{\rho} &= -\frac{1}{\rho'} \frac{\partial}{\partial \phi} \frac{1}{s} \frac{\partial E^{c\phi}}{\partial z}, \\
\nu_0 \partial E^c_z &= -\frac{1}{\rho'} \frac{\partial}{\partial \phi} \frac{1}{s} \frac{\partial E^{c\phi}}{\partial z}.
\end{align*}
\]

Similar equations can be written for Ampere’s law. Since \( \partial \rho' = (1/s) \partial \rho \), we can rewrite the above as

\[
\begin{align*}
\nu_0 \partial E^c_{\phi} &= -\frac{1}{\rho'} \frac{\partial}{\partial \phi} \frac{1}{s} \frac{\partial E^{c\phi}}{\partial z}, \\
\nu_0 \partial E^c_{\rho} &= -\frac{1}{\rho'} \frac{\partial}{\partial \phi} \frac{1}{s} \frac{\partial E^{c\phi}}{\partial z}, \\
\nu_0 \partial E^c_z &= -\frac{1}{\rho'} \frac{\partial}{\partial \phi} \frac{1}{s} \frac{\partial E^{c\phi}}{\partial z}.
\end{align*}
\]

It is clear that due to the extra factors \( 1/s \) and \( \rho/\rho' \), the system in Eqs. (3) does not retain the form of the Maxwell's equations. This is indicated by using the superscript \( c \) for the corresponding field solutions. However, after some simple algebraic manipulations Eqs. (3) can be rewritten as

\[
\begin{align*}
i \nu_0 \partial E^c_{\phi} &= -\frac{1}{s} \frac{\partial}{\partial \phi} \left(E^c_{\phi} \right) - \frac{1}{s} \frac{\partial}{\partial z} \left(E^c_{\phi} \right), \\
i \nu_0 \partial E^c_{\rho} &= -\frac{1}{s} \frac{\partial}{\partial \phi} \left(E^c_{\phi} \right) - \frac{1}{s} \frac{\partial}{\partial z} \left(E^c_{\phi} \right), \\
i \nu_0 \partial E^c_z &= -\frac{1}{s} \frac{\partial}{\partial \phi} \left(E^c_{\phi} \right) - \frac{1}{s} \frac{\partial}{\partial z} \left(E^c_{\phi} \right).
\end{align*}
\]

where we used the fact that \( s \) is a function of \( \rho \), but not \( \phi \) or \( z \). From the above equations (and from a similarly transformed Ampere’s law), it is clear that the transformed fields,

\[
\begin{align*}
E^a_{\rho} &= sE^c_{\rho}, \\
E^a_{\phi} &= (\rho'/\rho)E^c_{\phi}, \\
E^a_z &= E^c_z
\end{align*}
\]

and similarly for \( H^a_{\rho}, H^a_{\phi}, H^a_z \), obey the Maxwell’s equations in an anisotropic medium with material tensors \( [\varepsilon] = \varepsilon_0[\Lambda] \) and \( [\mu] = \mu_0[\Lambda] \), where

\[
[\Lambda] = \text{diag} \left\{ \rho' \rho, \rho'/\rho, 1 \right\}.
\]

The tensor \( [\Lambda] \) can alternatively be written as [5]

\[
[\Lambda] = \begin{cases} 
[I] & \text{for } 0 \leq \rho \leq d \\
(\det[S])^{-1}[S][S]^T & \text{for } d \leq \rho \leq R,
\end{cases}
\]

with

\[
[S] = \text{diag} \left\{ \frac{1}{s}, \frac{1}{s}, \frac{1}{s} \right\}.
\]

The matrix \( [S] \) represents the Jacobian of the transformation \( (d\rho', d\phi', dz) \rightarrow (d\rho, d\phi, dz) \). In terms of \( \rho' \), the field distribution corresponds to that of a hollow waveguide (i.e., with constitutive parameters \( \varepsilon_0 \) and \( \mu_0 \)), which has modal solutions of the form

\[
\begin{align*}
E^c_{\rho} &= \left( \frac{\nu_0 \gamma_{mn}}{k_{mp}^2} \right) J_m(k_{mp}\rho') \left( \cos(n\phi) - \sin(n\phi) \right) e^{ik' z}, \\
E^c_{\phi} &= \left( \frac{\nu_0 \gamma_{mn}}{k_{mp}} \right) J_m(k_{mp}\rho') \left( \sin(n\phi) \right) e^{ik' z}, \\
E^c_z &= 0,
\end{align*}
\]

where \( k_{mp} = \chi_{mp}/R_e \), \( R_e = d + s(R - d) \) (note again that \( s \) is assumed uniform in the cladding), and \( \chi_{mp} \) is the \( p \)th root of \( J_m(\cdot) \), the Bessel function of order \( m \). Expressions for the magnetic field follow similarly. \( E^a_{\rho}, E^a_{\phi}, E^a_z \) in the circular waveguide with cladding are found by straightforward combination of Eqs. (5) and (9).

**B. Rectangular Waveguide**

Now consider a rectangular waveguide with PEC walls as depicted in Fig. 1. The waveguide has cross-section dimensions \( a \times b \) and is backed by a metamaterial cladding with thickness \( t \), as indicated. Assuming the origin of the coordinate system is at the center of the waveguide, the
The effect of Similar expressions apply for the other three quadrants. The effect of \(s_x(x)\) and \(s_y(y)\) is analogous to the effect of \(s(\rho)\) considered previously. For simplicity, we consider \(s_x(x)\) and \(s_y(y)\) to be uniform and equal to \(s\) in the cladding, although the analysis below can be readily adapted to variable \(s_x(x)\) and \(s_y(y)\). In the case of uniform \(s_x(x)\) and \(s_y(y)\) (i.e., \(s_x=s_y=s\) inside the cladding), the transformation above reduces to

\[
x' = \begin{cases} 
  x & \text{for } 0 \leq x \leq \left(\frac{a}{2} - t\right) 
  \left(\frac{a}{2} - t\right) + \int_{a/2-t}^{x} s_x(x)dx & \text{for } \left(\frac{a}{2} - t\right) \leq x \leq \frac{a}{2}, 
\end{cases}
\]

(10a)

\[
y' = \begin{cases} 
  y & \text{for } 0 \leq y \leq \left(\frac{b}{2} - t\right) 
  \left(\frac{b}{2} - t\right) + \int_{b/2-t}^{y} s_y(y)dy & \text{for } \left(\frac{b}{2} - t\right) \leq y \leq \frac{b}{2}. 
\end{cases}
\]

(10b)

In terms of the coordinates \((x', y', z)\), Faraday’s law becomes

\[
io\mu_0 H_x = \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z},
\]

(12a)

\[
io\mu_0 H_y = \frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial z},
\]

(12b)

\[
io\mu_0 H_z = \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y}.
\]

(12c)

Ampere's law is similar. The fields above satisfy the Maxwell's equations in terms of \((x', y', z)\) but not in terms of \((x, y, z)\). As in the circular waveguide case, this is indicated by the use of the superscript \(c\).

Using the fact that \(s_x\) depends on \(\varsigma\) only, and that \(\partial \partial \varsigma(1/s_x)\partial \partial \varsigma\), where \(\varsigma=x, y\), the above can be rewritten as

\[
i\omega\mu_0 H_x = \frac{1}{s_x} \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z},
\]

(13a)

\[
i\omega\mu_0 H_y = \frac{\partial E_x}{\partial x} - \frac{1}{s_x} \frac{\partial E_z}{\partial y},
\]

(13b)

\[
i\omega\mu_0 H_z = \frac{\partial E_y}{\partial y} - \frac{1}{s_x} \frac{\partial E_x}{\partial x},
\]

(13c)

and, after some simple algebraic manipulations, as

\[
i\omega\mu_0 \left( \frac{s_x}{s_y} \right) (s_x H_y) = \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z},
\]

(14a)

\[
i\omega\mu_0 \left( \frac{s_x}{s_y} \right) (s_y H_x) = \frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial y},
\]

(14b)

\[
i\omega\mu_0 (s_x s_y) (H_z) = \frac{\partial E_y}{\partial y} - \frac{\partial E_x}{\partial x},
\]

(14c)

and similarly for Ampere’s law. Note that the last set of equations retains the form of the Maxwell’s equations, with field solutions

\[
E_x^0 = s_x E_x, \quad E_y^0 = s_y E_y, \quad E_z^0 = E_z
\]

(15)

and similarly for \(H_x^0, H_y^0, H_z^0\) in a medium with material tensors given by \([\varepsilon] = \varepsilon_0 \Lambda\) and \([\mu] = \mu_0 \Lambda\), where \(\Lambda\) is obtained again using Eq. (7) with \([S]\) now given by

\[
[S] = \text{diag}(s_x^{-1}, s_y^{-1}, 1),
\]

(16)

which evaluates to

\[
[S] = \begin{cases} 
  \text{diag}(s^{-1}, 1, 1) & \text{at the lateral sidewalls} 
  \text{diag}(1, s^{-1}, 1) & \text{at the top and bottom walls} 
  \text{diag}(s^{-1}, s^{-1}, 1) & \text{at the four corners.} 
\end{cases}
\]

(17)

This matrix corresponds to the Jacobian of the transformation \((dx', dy', dz') \rightarrow (dx, dy, dz)\).

From the above, it is seen that three different sets of constitutive parameters are to be used in the rectangular waveguide cladding,

\[
[\Lambda] = \begin{cases} 
  \text{diag}(s^{-1}, s, s) & \text{at the lateral sidewalls} 
  \text{diag}(s, s^{-1}, s) & \text{at the top and bottom walls} 
  \text{diag}(1, 1, s^2) & \text{at the four corners.} 
\end{cases}
\]

(18)

In terms of \(x'\) and \(y'\), the field distribution corresponds to that of a hollow waveguide. Therefore, the transverse-magnetic mode solutions can be written as

\[
E_x^c = \left( -\frac{i\omega}{k_x k_z} \right) \cos(k_x x') \sin(k_y y') e^{ik_z z},
\]

(19a)

\[
E_y^c = \left( \frac{\omega}{k_x k_z} \right) \sin(k_x x') \cos(k_y y') e^{ik_z z},
\]

(19b)
with \( k_x = m \pi/a_x \), \( k_y = n \pi/b_y \), \( a_x = 2t(s-1) + a \), and \( b_y = 2t(s-1) + b \), and where \( m \) and \( n \) are the mode indices. Similar expressions can be written for transverse-electric (TE) modes and for the associated magnetic fields. Finally, the expressions for the fields \( E_x^n, E_y^n, E_z^n \) in the rectangular waveguide with cladding can be found by straightforward combination of Eqs. (15) and (19).

3. EXAMPLES

In this section, we show results for circular and rectangular PEC waveguides backed by the metamaterial claddings discussed above. For the cylindrical waveguide, we assume a cladding thickness of \( t = 0.2R \). For the rectangular waveguide, we assume an aspect ratio of \( a = 1.4b \) and a cladding thickness of \( t = 0.1a \). All plots below show field distributions normalized to their respective peaks.

Figure 2 depicts the power density distribution \( \vec{E} \times \vec{H} \) of the dominant mode in the circular waveguide with \( s = 1 \) (no cladding), \( s = 2 \), \( s = 3 \), and \( s = 5 \). From Figs. 2(a)–2(d), it is seen that inside the core, the power density becomes progressively more uniform for higher \( s \). Moreover, Figs. 2(a)–2(d) show that the power density becomes gradually more confined inside the cladding.

An example of a rectangular waveguide with cladding inserted on the two lateral walls is shown in Fig. 3, where a plot of the TE\(_{10}\) (dominant) mode for various values of \( s \) is presented. The TE\(_{10}\) power distribution is uniform along \( y \) and only the distribution along \( x \) is affected in this case. An example of a rectangular waveguide with cladding inserted on all four sidewalls is shown next in Fig. 4, with a plot of the TE\(_{11}\) mode whose distribution changes in both \( x \) and \( y \) directions as \( s \) increases. The general observations made before regarding the mode uniformity in the core and mode confinement in the cladding also apply to the rectangular case.

![Fig. 2.](image)

![Fig. 3.](image)

![Fig. 4.](image)

As noted, the decrease in the cutoff frequencies for higher \( s \) is related to the increase in the “dual-hollow” waveguide. If the cladding has uniform \( s \), a dual-hollow circular waveguide with radius \( R_c = d + s(R - d) = d + s \) and a dual-hollow rectangular waveguide with width \( a_x = 2t(s-1) + a \) and height \( b_y = 2t(s-1) + b \) result. In particular, assuming a circular waveguide with \( R = 1 \) cm, and a rectangular waveguide with \( b = 1 \) cm, the cutoff frequencies for the modes shown in Figs. 2 and 3 are given in Table 1, for different \( s \).

We next present three-dimensional (3-D) numerical results based on the finite-difference time-domain (FDTD) method. A time-harmonic electric dipole source is placed inside the rectangular waveguide and is excited by a frequency such that only the dominant mode propagates and all other higher modes are evanescent. Since one of the effects of a higher \( s \) is to decrease the cutoff frequency, this implies that for higher \( s \), progressively lower frequencies are required to produce the propagating mode. The FDTD simulations employ a 3-D grid with \( N_x \times N_y \times N_z = 56 \times 56 \times 56 \) and a time step of \( \Delta t = 0.5 \times 10^{-12} \) s.
×40 \times 360 nodes, a unit cell size of \Delta x = 0.025 \text{ cm}, and a time step of \Delta t = 0.385 \text{ ps}. Assuming that the origin is at one corner of the waveguide, the dipole source is positioned at \( (x, y, z) = (28, 10, 180) \). A perfectly matched layer (PML) is used to suppress reflections from the waveguide ends along the \( z \) direction.

Figure 5 shows snapshots of the steady-state electric and magnetic field distributions along the waveguide, for the different values of the cladding contrast parameter \( s \). The cladding is inserted on the two lateral walls only. The horizontal-cut views shown are located at the midpoint of the waveguide along the \( y \) direction. As noted, the frequencies shown for each \( s \) are chosen above the cutoff frequencies indicated in Table 1 and below the cutoff frequency of the next higher-order mode. Again, the gradual spreading of the electric field distribution along the transversal direction for higher \( s \), which makes the field distribution more uniform in the core, is clearly visible in this figure. The magnetic field of the dominant mode, on the other hand, becomes increasingly confined inside the lateral cladding for higher \( s \).

4. CONCLUSIONS AND FURTHER REMARKS

We have explored the known duality between metric and constitutive parameters in the Maxwell's equations to derive metamaterial blueprints for waveguide claddings that provide homotopic control of waveguide modes. The resulting metamaterial-loaded waveguides exhibit no hybrid modes. The obtained metamaterial blueprints can be viewed as isoimpedance materials with intrinsic impedance matched to free-space, regardless of frequency or propagation angle. The constitutive parameters are specified by a contrast parameter “\( s \)” that provides control over the mode uniformity (in the core region), the degree of field confinement in the cladding, and the resulting waveguide cutoff frequencies. The isoimpedance property is closely related to the PML concept [7,20] but devoid of the absorptive properties of the latter.

It should be emphasized that the type of waveguide geometries considered here is inherently narrowband. This has nothing to do with the proposed cladding \textit{per se}, but follows more fundamentally from the strong chromatic dispersion at frequencies near the cutoff (note that if one tries instead to operate the waveguide far from the cutoff, an even more detrimental type of dispersion ensues: modal dispersion caused by higher-order modes). Narrowband operation facilitates the design and eventual fabrication of cladding structures that approximate the consti-
tutive parameters derived here (as the latter need to be realized only over a narrow range of frequencies). It is also important to stress that even though the metamaterial blueprints derived imply real-valued constitutive tensors, the practical realization of the proposed claddings inevitably entails losses arising from nonzero imaginary parts, as predicted by Kramers–Kronig relations (and assuming a passive media realization). Narrowband operation also facilitates the minimization of such losses (in the frequency range of operation).

It is beyond the objectives of this paper to dwell further into fabrication issues or potential applications. Suffice it to say here that the actual fabrication of the proposed claddings obfuscates the underlying metric/constitutive duality of the Maxwell’s equations. The language of differential forms (exterior calculus) overcomes this issue [1–7] because the only spatial operator then present in the Maxwell’s equations is the exterior derivative operator, which is independent of any metric coefficient. Indeed, in this framework, all metric and constitutive properties are conveniently lumped together into (a pair of) Hodge star operators [6,7,23,24]. It is of note that, in work that precedes papers on cloaking and other “transformation optics” methods, the metric/constitutive duality was explored for the design of PMLs [5,7,25] and for the implementation of finite-difference schemes in irregular meshes [6,26,27], for example.

When written in terms of differential forms [1–3], the Maxwell’s equations factor into a metric-free (or “topological” part) and a metric-dependent part. The metric-free part corresponds to dynamic equations involving time derivatives and the (metric-free) exterior derivative $d$, and is written as [6,27]

$$i\omega B = dE, \quad (A1)$$

$$i\omega D = J - dH, \quad (A2)$$

$$dD = \rho, \quad (A3)$$

$$dB = 0. \quad (A4)$$

In the above, $E$ and $H$ are one-forms; $D$, $B$, and $J$ are two-forms; and $\rho$ is a three-form. Regardless of the metric, the exterior derivative operator always has the form

$$d = \left( \frac{\partial}{\partial u} du + \frac{\partial}{\partial v} dv + \frac{\partial}{\partial w} dw \right) \wedge, \quad (A5)$$

where $u, v, w$ are arbitrary spatial coordinates, with no metric coefficients present.

The metric-dependent part of the Maxwell’s equations corresponds precisely to the constitutive equations, in this case represented by a pair of Hodge star operators relating the electric and magnetic fields [6,23,27],

$$D = *E, \quad (A6)$$

$$H = *\mu^{-1}B. \quad (A7)$$

In general, a Hodge star operator is an isomorphism between $p$-forms and $(n-p)$-forms, where $n$ is the dimension of space. In free-space with no gravitational curvature, the Hodge star operators are diagonal in an orthogonal coordinate system in the sense that a basis $p$-form maps to a single basis $(n-p)$-form. For example, in a Cartesian system $(x,y,z)$, the star operator acts on basis one-forms according to $*dx = e_y dy \wedge dz$, $*dy = e_z dz \wedge dx$, $*dz = e_x dx \wedge dy$. In an anisotropic medium, or in a gravitationally curved region of space, the Hodge star operators in Eqs. (A6) and (A7) become in general nondiagonal [23]. Because metric and constitutive properties are lumped together in Eqs. (A6) and (A7), their duality is explicitly manifest [5,27].

REFERENCES


