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ON THE ASYMPTOTIC RELATIONSHIP BETWEEN THE OVERFLOW PROBABILITY AND THE LOSS RATIO

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Abstract

In this paper we study the asymptotic relationship between the loss ratio in a finite buffer system and the overflow probability (the tail of the queue length distribution) in the corresponding infinite buffer system. We model the system by a fluid queue which consists of a server with constant rate c and a fluid input. We provide asymptotic upper and lower bounds on the difference between $\log \mathbb{P}\{Q > x\}$ and $\log P_L(x)$ under different conditions. The conditions for the upper bound are simple and are satisfied by a very large class of input processes. The conditions on the lower bound are more complex but we show that various classes of processes such as Markov modulated and ARMA type Gaussian input processes satisfy them.

KEYWORDS: OVERFLOW PROBABILITY; LOSS RATIO; ASYMPTOTIC RELATIONSHIP

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1. Introduction

In this paper we study the asymptotic relationship between the overflow probability, $\mathbb{P}\{Q > x\}$, in an infinite buffer system and the loss ratio, $P_L(x)$, in the corresponding finite buffer system. An application of this result is in telecommunications and networking problems where the loss ratio is an important measure of the quality of an application that the network is carrying. The study of

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these two quantities is also important in providing theoretical support for approximating the loss ratio in finite buffer queues through their infinite buffer counterparts[16].

While overflow probability in an infinite buffer system has been extensively studied [1, 2, 9, 11], there have been relatively few studies on the loss probability in finite buffer systems [17, 19]. There have been works on the relationship between the two quantities. It has been shown in [21] that for certain types of systems such as M/Subexponential/1 and GI/Regularly-varying/1 with i.i.d. interarrival and service times, the ratio $\mathbb{P}\{Q > x\}/P_L(x)$ converges to a constant, as $x \to \infty$. It can also be inferred from the result in [13] that for GI/Subexponential/1 with i.i.d. interarrival and service times, this ratio converges to a constant. These works assume i.i.d. interarrival and service times so that the queue content in the finite buffer system converges to a stationary process. We study the long time average loss ratio without invoking this assumption.

We model the system by a fluid queue which consists of a server with constant rate c and a fluid input. We consider two types of queues. Let us call the queue with a finite buffer the *finite queue* and the queue with the infinite buffer the *infinite queue*. Both queues are fed with the same input. Let $\hat{Q}_t$ and Q_t denote the queue length at time t in the finite and infinite queues, respectively. Depending on the index set from which the time index t takes its value, a fluid queue is classified as either a continuous-time fluid queue or a discrete-time fluid queue. In this paper, we focus on the discrete-time fluid queue and use n as time index[†]. Let λ_n be the amount of input that arrives into the system at time n. We assume that λ_n is stationary and ergodic, and that $c > \mathbb{E}\{\lambda_n\}$. Then, it has been shown that Q_n converges to a stationary and ergodic process [18]. In this paper, we assume that Q_n started at $n = -\infty$ so that it is in the stationary regime. The time index n is often omitted to represent the stationary distribution, i.e., $\mathbb{P}\{Q > x\} = \mathbb{P}\{Q_n > x\}$ and $\mathbb{E}\{\lambda\} = \mathbb{E}\{\lambda_n\}$. It has also been shown that $\hat{Q}_n$ converges to a stationary process when the system is a GI/GI/m/x type of queue [7, 12] and when the system is a G/M/m/x type of queue [4]. Although it is likely that $\hat{Q}_n$ converges to a stationary and ergodic process in more generality (counter-examples are generally pathological and shown under $c < \mathbb{E}\{\lambda_n\}$), we do not assume it here.

Since Q_n on the other hand converges to a stationary and ergodic process, the overflow probability in the infinite buffer case, can be expressed as the amount of time the fluid in the infinite buffer

[†]It should be noted that the results derived for the discrete-time queue case can be extended to the continuous time case [15].

system spends above level x divided by the total time:

$$\mathbb{P}\{Q > x\} \quad = \quad \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} I(Q_k > x), \tag{1}$$

where $I(A) = 1$ if A is true; $I(A) = 0$ otherwise. The *time average* loss ratio $P_L(x)^{(N)}$ during an interval of time N, for buffer size x is defined as the ratio of the amount of loss to the total amount of input (or fraction of work lost):

$$P_L(x)^{(N)} = \frac{\sum_{k=1}^{N} (\hat{Q}_{k-1} + \lambda_k - c - x)^+}{\sum_{k=1}^{N} \lambda_k}, \tag{2}$$

where $(x)^+$ denotes $\max\{x, 0\}$. If we assumed ergodicity of $\hat{Q}_n$, the long time average loss ratio, $P_L(x)$, would be

$$P_L(x) = \lim_{N\to\infty} P_L(x)^{(N)} = \frac{\mathbb{E}\{(\hat{Q}_{n-1} + \lambda_n - c - x)^+\}}{\mathbb{E}\{\lambda_n\}}. \tag{3}$$

In a practical sense, "time average" is important because measurements of stochastic processes are based on a time average method. Note that since $\mathbb{P}\{Q > x\}$ is a ratio of "time" to "time", and $P_L(x)$ is a ratio of "fluid" to "fluid", in general there is no exact relationship between these two quantities.

The main results of the paper are expressed as two theorems (Theorem 1 and 2). They provide the following relationships:

$$\liminf_{x\to\infty} \frac{1}{\log x} \left[\log \liminf_{N\to\infty} P_L(x)^{(N)} - \log \mathbb{P}\{Q > x\}\right] \quad > \quad -\infty, \tag{4}$$

$$\limsup_{x\to\infty} \frac{1}{\log x} \left[\log \limsup_{N\to\infty} P_L(x)^{(N)} - \log \mathbb{P}\{Q > x\}\right] \quad < \quad \infty. \tag{5}$$

The *liminf* results and the *limsup* results are presented separately since the conditions on them are quite different. These theorems can be interpreted as asymptotically the loss ratio and the overflow probability curves are quite similar, and if they diverge, they do so slowly. The loss ratio and the overflow probability can differ at most by a polynomial factor of x^a. Note that while this allows for asymptotic divergence, it is still sharper than other well known asymptotic techniques, e.g., large deviations, where results of the form $\lim_{x\to\infty}(1/x)\log \mathbb{P}\{Q > x\} = -\theta^*$ are provided, thus allowing divergence to be as fast as $e^{x^{1-\delta}}$.

The notations $f = o(g), f = O(g)$ and $f \sim g$ mean that $\lim(f/g) = 0, \limsup |f/g| < \infty$ and $\lim(f/g) = 1$, respectively. To avoid ambiguity, we may also write $f \overset{x\to\infty}{\sim} g$. For a nonnegative function of multiple variables, $f(y, z) = O(g(z))$ means that there exist K and z_0 such that $f(y, z) \leq Kg(z)$ for all $z \geq z_0$ and for all y.

This paper is organized as follows. In Section 2, we define some notation and assumptions and provide some intermediate result. In Section 3, we first prove the *limsup* part, i.e., equation (5), and the *liminf* part, i.e., equation (4). We then provide some examples in Section 4 for which the results hold. In the main text we provide proofs only of the theorems. All intermediate proofs are provided in the appendix.

2. Preliminary Results

The queue content Q_n of the infinite queue is expressed by Lindley's equation:

$$Q_n = (Q_{n-1} + \lambda_n - c)^+. \tag{6}$$

Similarly, the queue content $\hat{Q}_n$ of the finite queue is expressed by:

$$\hat{Q}_n = \min\{x, (\hat{Q}_{n-1} + \lambda_n - c)^+\}, \tag{7}$$

where x is the buffer size. Since both Q_n and $\hat{Q}_n$ have the same input λ_n, and since both processes start at the same time, $n = -\infty$, it follows from (6) and (7) that $\hat{Q}_n \leq Q_n$, for all n. Suppose that $Q_n = 0$ at time n_1, then $\hat{Q}_n = Q_n$ until Q_n becomes greater than x at some time $n_2 > n_1$. After time n_2, $\hat{Q}_n < Q_n$ until Q_n becomes zero at time n_3. The amount of loss in this period is equal to $\max_{n_1 \leq n \leq n_3}\{Q_n - \hat{Q}_n\}$. We define a *cycle* as this period, i.e., an interval between times when Q_n becomes zero. We let S_n^x denote the duration for which Q_n stays in the overflow state with threshold x in a cycle to which n belongs. Formally, we define:

- $U_n := \sup\{k \leq n : Q_{k-1} > 0, Q_k = 0\}$ (start time of the current cycle to which n belongs).
- $V_n := \inf\{k > n : Q_{k-1} > 0, Q_k = 0\}$ (start time of the next cycle).
- $W_n := V_n - U_n$ (duration of a cycle to which n belongs).
- $Z_n := V_n - n$ (residual time to reach the end of the cycle).
- $S_n^x := \sum_{k=U_n}^{V_n - 1} 1_{\{Q_k > x\}}$ (duration for which $Q_k > x$ in a cycle containing n).

Note that if $Q_n > 0$, Z_n corresponds to the time elapsed to return to the empty-buffer (or zero) state. Since Q_n is stationary and ergodic, so are the above. Hence, their expectations are equal to time averages. For example,

$$\mathbb{E}\{W_n\} = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{i=-k}^{k} W_i, \tag{8}$$

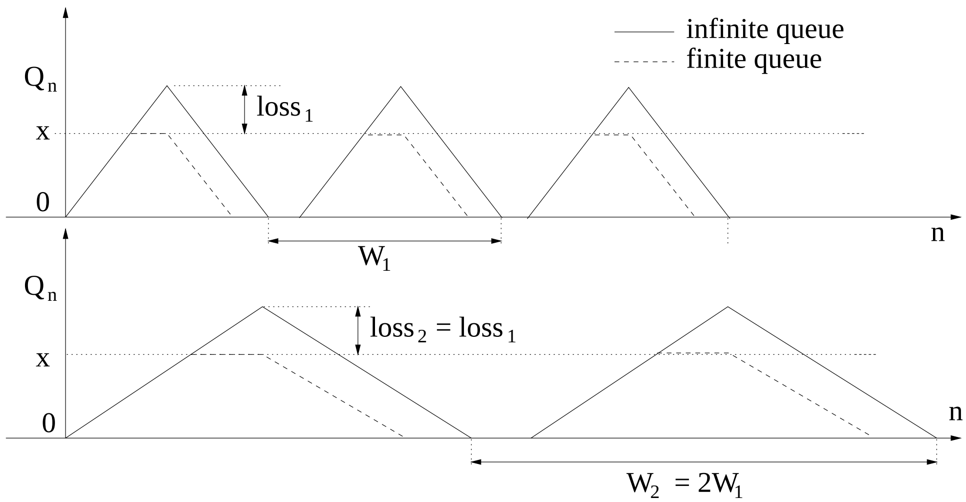

Figure 1. Illustration of "same $\mathbb{P}\{Q > x\}$ but different $P_L(x)$".

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For the purpose of illustration, consider two systems whose sample paths look like those in Figure 1. Both systems have the same $\mathbb{P}\{Q > x\}$ but different $P_L(x)$. The upper system has a larger loss ratio than the lower one. We can infer from this that the loss ratio is closely related to the length of the period in which Q_n is greater than the buffer size x. Since the relationship that we are trying to show is not equality but inequality, we use Z_n which gives us an indication as to how fast Q_n returns to the zero state. Using Z_n, we will find a lower bound type of equation.

Since loss happens only when Q_n is greater than the buffer size x, the quantities (such as S_n and Z_n) given $\{Q_n > x\}$ are of interest. Due to the conditioning on $\{Q_n > x\}$, the problem is quite difficult because Q_n will be a result of the entire history of the input process λ_n. In Lemma 1 presented next, we provide an expression for the upper and lower bound of $P_L(x)$ in terms of $\mathbb{P}\{Q > x\}$ and $\mathbb{E}\{Z|Q > x\}$.

Lemma 1 Assume that $\mathbb{P}\{Q > x\} > 0$ *for all* x. *Then,*

$$\begin{aligned}\int_x^\infty \frac{1}{2\mathbb{E}\{Z|Q>y\}}\mathbb{P}\{Q>y\}dy &\leq \mathbb{E}\{\lambda\}\liminf_{N\to\infty} P_L(x)^{(N)} \\ &\leq \mathbb{E}\{\lambda\}\limsup_{N\to\infty} P_L(x)^{(N)} \leq \int_x^\infty \mathbb{P}\{Q>y\}dy \end{aligned} \tag{9}$$

This lemma tells us that the long time average loss ratio (regardless of the existence of its limit) is between the two values obtained from integrals of $\mathbb{P}\{Q > y\}$.

Let $\phi(x) := -\log \mathbb{P}\{Q > x\}$. Since $\mathbb{P}\{Q > x\}$ is a monotonically decreasing function of x,[‡] if $\mathbb{P}\{Q > x\} = 0$ for some x, there is nothing interesting to say about its asymptotic behavior. So, throughout the paper we assume that $\phi(x)$ is well defined, i.e., $\mathbb{P}\{Q > x\} > 0$ for all x, and that $\phi(x)$ is twice differentiable. In fact, the last inequality of (9) immediately follows if we assume ergodicity of $\hat{Q}_n$. Recall (3) and (6), now since $\hat{Q}_n \le Q_n$,

$$\begin{aligned} \mathbb{E}\{\lambda\}P_L(x) &= \mathbb{E}\{(\hat{Q}_{n-1} + \lambda_n - c - x)^+\} \\ &\le \mathbb{E}\{(Q_{n-1} + \lambda_n - c - x)^+\} = \mathbb{E}\{(Q_n - x)^+\} = \int_x^\infty \mathbb{P}\{Q > y\}dy. \end{aligned} \tag{10}$$

We now list some conditions on $\phi(x)$ which are satisfied by most tail distribution functions such as $\mathbb{P}\{Q > x\} \sim \alpha e^{-\beta x}, \alpha x^{-\beta}, \alpha e^{-\sqrt{x}\log x + \frac{1}{x}\sin x}$, etc.

(C1) $\quad C_0 := \lim_{x\to\infty} \dfrac{\log x}{\phi(x)} < 1$

(C2) $\quad \lim_{x\to\infty} \dfrac{1}{x\phi'(x)} = C_0$

(C3) $\quad \lim_{x\to\infty} \dfrac{-\phi''(x)}{\phi'(x)^2} = C_0$

(C4) $\quad \limsup_{x\to\infty} \dfrac{\phi'(x)}{x^\theta} < \infty \quad$ for some $\theta < \infty$.

Clearly $C_0 \ge 0$ because $\phi(x)$ is nonnegative. Since we assume that $C_0 < 1$, there exists $\delta > 0$ such that $\phi(x) \ge (1+\delta)\log x$ for large x. So $\mathbb{P}\{Q > x\} = O(e^{-(1+\delta)\log x}) = O(x^{-1-\delta})$, hence, the condition $C_0 < 1$ implies that $\mathbb{P}\{Q > x\}$ is integrable, i.e., $\mathbb{E}\{Q\} < \infty$. As long as *L'Hospital's* rule can be applied, (C2) and (C3) follow from (C1). Assuming these conditions, we now state Lemma 2. Since $e^{-\phi(y)} = \mathbb{P}\{Q > x\}$, this lemma will be useful in proving our results because we will use an integral form of $\mathbb{P}\{Q > x\}$ as a bound on $P_L(x)$.

Lemma 2 Under conditions (C1)-(C3),

$$\int_x^\infty e^{-\phi(y)}dy \overset{x\to\infty}{\sim} \frac{1}{(1-C_0)\phi'(x)} e^{-\phi(x)}. \tag{11}$$

Now, let us focus on condition (C4) for a moment. This condition rules out extremely fast decaying overflow probabilities such as $\mathbb{P}\{Q > x\} = e^{-e^x}$. Assuming ergodicity of $\hat{Q}_n$ for now (for simplicity), from (10) and Lemma 2, we have

$$\mathbb{E}\{\lambda_n\}P_L(x) \le \int_x^\infty \mathbb{P}\{Q > y\}dy \overset{x\to\infty}{\sim} \frac{1}{(1-C_0)\phi'(x)}\mathbb{P}\{Q > x\}. \tag{12}$$

[‡]It is obvious that $\mathbb{P}\{Q > x\} \ge \mathbb{P}\{Q > y\}$ since $\{Q > x\} \supset \{Q > y\}$ for $x \le y$.

So if, for example, $\phi(x) = e^x$ which violates (C4),

$$\begin{aligned}\limsup_{x\to\infty} \frac{1}{\log x}\left[\log P_L(x) - \log \mathbb{P}\{Q > x\}\right] &\leq \limsup_{x\to\infty} \frac{\log \frac{2}{1-C_0} - \log \phi'(x)}{\log x} \\ &= \limsup_{x\to\infty} \frac{\log \frac{2}{1-C_0} - x}{\log x} = -\infty\end{aligned}$$

and, hence, the *liminf* is $-\infty$, as well. Through this example, it should be clear that if (C4) were violated then in fact the asymptotic difference between $\log P_L(x)$ and $\log \mathbb{P}\{Q > x\}$ (normalized by $\log x$) would be $-\infty$. This tells us that for very fast decaying queue distributions (e.g. double exponential), the tail and the loss ratio can be quite different.

3. Main Result

We state our main results as two theorems, the *limsup* part and the *liminf* part.

Theorem 1 Under conditions (C1)-(C3),

$$\limsup_{x\to\infty} \frac{1}{\log x}\left(\log \limsup_{N\to\infty} P_L(x)^{(N)} - \log \mathbb{P}\{Q > x\}\right) < \infty. \tag{13}$$

Proof of Theorem 1. From Lemma 1, we have

$$\mathbb{E}\{\lambda\} \limsup_{N\to\infty} P_L(x)^{(N)} \leq \int_x^\infty \mathbb{P}\{Q > y\}dy. \tag{14}$$

Applying Lemma 2,

$$\int_x^\infty \mathbb{P}\{Q > y\}dy = \int_x^\infty e^{-\phi(y)}dy \overset{x\to\infty}{\sim} \frac{1}{(1-C_0)\phi'(x)}e^{-\phi(x)} = \frac{1}{(1-C_0)\phi'(x)}\mathbb{P}\{Q > x\}. \tag{15}$$

From (C1) and (C2), $\frac{1}{\phi'(x)} < x$ for large x. Hence, there exists x_0 such that

$$\int_x^\infty \mathbb{P}\{Q > y\}dy \leq \frac{1}{(1-C_0)} x\mathbb{P}\{Q > x\}, \quad \forall x \geq x_0.$$

Thus,

$$\limsup_{N\to\infty} P_L(x)^{(N)} \leq \frac{1}{(1-C_0)\mathbb{E}\{\lambda\}} x\mathbb{P}\{Q > x\}, \quad \forall x \geq x_0.$$

Taking logs and subtracting $\log \mathbb{P}\{Q > x\}$ from both sides, we get

$$\begin{aligned}\log \limsup_{N\to\infty} P_L(x)^{(N)} - \log \mathbb{P}\{Q > x\} &\leq \log\left(\frac{1}{(1-C_0)\mathbb{E}\{\lambda\}} x\mathbb{P}\{Q > x\}\right) - \log \mathbb{P}\{Q > x\} \\ &= \log \frac{1}{(1-C_0)\mathbb{E}\{\lambda\}} + \log x, \quad \forall x \geq x_0.\end{aligned}$$

So we have the result.

Now, consider (15). If $\mathbb{P}\{Q > x\}$ has an exponential tail, $\phi'(x)$ converges to a positive constant and the next proposition immediately follows.

Proposition 1 Assume that $\phi'(x)$ converges to a positive constant. Under conditions (C1)-(C3),

$$\limsup_{x\to\infty}\left(\log \limsup_{N\to\infty} P_L(x)^{(N)} - \mathbb{P}\{Q > x\}\right) < \infty. \tag{16}$$

From this proposition, we conjecture that if the overflow probability is exponential with some asymptotic decay rate η, so is the loss ratio with the same asymptotic decay rate η. This proposition is important because there is a fairly large class of input processes for which $\phi'(x)$ converges to a positive constant, e.g., Markov modulated fluid processes, short-range dependent Gaussian processes, etc [5, 11].

Now, we focus on the development of the main *liminf* result of the paper. As one can see in Lemma 1, the *liminf* part is related to $\mathbb{E}\{Z|Q > x\}$. Since it is difficult to know the distribution of Z, we use a stochastic process X_n defined as

$$X_n := \sum_{k=1}^{n} \lambda_k - cn + Q_0, \qquad n > 0. \tag{17}$$

Here we have chosen 0 as the origin, but the distribution of X_n does not depend on the origin due to stationarity. Note that X_n will be identical to Q_n until the end of the cycle.

Consider

$$\mathbb{P}\{X_n \le 0\} \;=\; \mathbb{P}\{X_n \le 0|Z_0 \le n\}\mathbb{P}\{Z_0 \le n\} + \mathbb{P}\{X_n \le 0|Z_0 > n\}\mathbb{P}\{Z_0 > n\}. \tag{18}$$

From the definition of Z_0, $Z_0 > n$ implies that $X_n > 0$. Thus, $\mathbb{P}\{X_n \le 0|Z_0 > n\} = 0$. Therefore, we have

$$\mathbb{P}\{X_n \le 0\} = \mathbb{P}\{X_n \le 0|Z_0 \le n\}\mathbb{P}\{Z_0 \le n\} \le \mathbb{P}\{Z_0 \le n\},$$

or

$$\mathbb{P}\{X_n > 0\} \ge \mathbb{P}\{Z_0 > n\}, \tag{19}$$

It follows from (19) that

$$\mathbb{E}\{Z_0\} = \sum_{n=0}^{\infty} \mathbb{P}\{Z_0 > n\} \le \sum_{n=0}^{\infty} \mathbb{P}\{X_n > 0\}. \tag{20}$$

By replacing $\mathbb{E}\{\cdot\}$ and $\mathbb{P}\{\cdot\}$ with $\mathbb{E}\{\cdot|Q_0\}$ and $\mathbb{P}\{\cdot|Q_0\}$ respectively, it follows that

$$\mathbb{E}\{Z_0|Q_0\} \overset{a.s.}{=} \sum_{n=0}^{\infty} \mathbb{P}\{Z_0 > n|Q_0\} \overset{a.s.}{\le} \sum_{n=0}^{\infty} \mathbb{P}\{X_n > 0|Q_0\}, \tag{21}$$

and we have the following lemma.

Lemma 3 If there exist $x_0, K, M > 0, \alpha > 0, \delta > 0$ *such that* $\mathbb{P}\{X_n > 0|Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta)}$ *for all* $n \geq MQ_0^\alpha$ *on* $\{Q_0 \geq x_0\}$*, then* $\mathbb{E}\{Z_0|Q_0\} \overset{a.s.}{=} O(Q_0^\alpha)$.

"$\mathbb{P}\{X_n > 0|Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta)}$ for all $n \geq MQ_0^\alpha$ on $\{Q_0 \geq x_0\}$" means that there exists a $Y_n(z)$ such that $\mathbb{P}\{X_n > 0|Q_0\} \overset{a.s.}{=} \mathbb{P}\{Y_n(z) > 0\}|_{z=Q_0}$ and for each $\omega \in \{Q_0 \geq x_0\}$, $\mathbb{P}\{Y_n(z) > 0\}|_{z=Q_0(\omega)} \leq Kn^{-(1+\delta)}$ for all $n \geq MQ_0(\omega)^\alpha$. "$\mathbb{E}\{Z_0|Q_0\} \overset{a.s.}{=} O(Q_0^\alpha)$" means that there exist K and x_0 such that

$$\left|\frac{\mathbb{E}\{Z_0|Q_0\}(\omega)}{Q_0(\omega)^\alpha}\right| < K \qquad \text{for almost all } \omega \in \{Q_0 \geq x_0\}.$$

For notational simplicity, we often omit *a.s.* or *almost all* as long as it does not cause ambiguity. Once we have $\mathbb{E}\{Z_0|Q_0\} \overset{a.s.}{=} O(Q_0^\alpha)$, we also have $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$ (Lemma 5), and can show the liminf part using the following lemma which is similar to Lemma 2.

Lemma 4 Under conditions (C1)-(C3),

$$\int_x^\infty z^\alpha \, dF_Q(z) = O(x^\alpha \mathbb{P}\{Q > x\}), \tag{22}$$

and

$$\int_x^\infty y^{-\alpha} e^{-\phi(y)} dy \overset{x\to\infty}{\sim} \frac{1}{(1 - C_1)(\phi'(x) + \alpha x^{-1})} x^{-\alpha} e^{-\phi(x)}, \tag{23}$$

where $C_1 = \frac{C_0}{1+\alpha C_0}$ *and* α *is any positive constant such that* $\alpha C_0 < 1$.

In the proof of Theorem 2, we will use (23), and for the next lemma we need (22).

Lemma 5 Assume conditions (C1)-(C3). Let α *be a positive constant such that* $\alpha C_0 < 1$. *If* $\mathbb{E}\{Z_0|Q_0\} \overset{a.s.}{=} O(Q_0^\alpha)$*, then* $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$.

Before we show for which classes of input processes one can find x_0, K, $M > 0, \alpha > 0$ and $\delta > 0$ that satisfies the condition of Lemma 3, we state our main *liminf* result here.

Theorem 2 Assume conditions (C1)-(C4). Further assume that there exist x_0*,* K*,* $M > 0, \alpha > 0$ *and* $\delta > 0$ *such that* $\alpha C_0 < 1$ *and* $\mathbb{P}\{X_n > 0|Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta)}$ *for all* $n \geq MQ_0^\alpha$ *on* $\{Q_0 \geq x_0\}$. *Then,*

$$-\infty < \liminf_{x\to\infty} \frac{1}{\log x} \left[\log \liminf_{N\to\infty} P_L(x)^{(N)} - \log \mathbb{P}\{Q > x\}\right]. \tag{24}$$

Proof of Theorem 2. Since we have assumed (C1)-(C4), $\phi(x)$ is well defined, i.e., $\mathbb{P}\{Q > x\} > 0$. Thus, we can apply Lemma 1. Therefore, there exist x_0 and $K_1 > 0$ such that

$$\liminf_{N\to\infty} P_L(x)^{(N)} \geq \frac{1}{\mathbb{E}\{\lambda\}} \int_x^\infty \frac{1}{2\mathbb{E}\{Z|Q > y\}} \mathbb{P}\{Q > y\} dy \geq \frac{1}{\mathbb{E}\{\lambda\}} \int_x^\infty \frac{K_1}{y^\alpha} \mathbb{P}\{Q > y\} dy, \quad \forall x \geq x_0.$$

From (23), there exist $x_1 \geq x_0$, $K_2 > 0$ and $K_3 > 0$ such that

$$\liminf_{N\to\infty} P_L(x)^{(N)} \geq \frac{1}{\mathbb{E}\{\lambda\}} \int_x^\infty \frac{K_2}{y^\alpha} \mathbb{P}\{Q > y\} dy \geq \frac{K_3}{\phi'(x) + \alpha x^{-1}} x^{-\alpha} \mathbb{P}\{Q > x\}, \quad \forall x \geq x_1.$$

Choose $\theta \geq -1$ satisfying (C4). Then, there exist $x_2 \geq x_1$, $K_4 > 0$ such that

$$\frac{K_3}{\phi'(x) + \alpha x^{-1}} x^{-\alpha} \mathbb{P}\{Q > x\} \geq \frac{K_4}{\mathbb{E}\{\lambda\}} x^{-\theta-\alpha} \mathbb{P}\{Q > x\}, \quad \forall x \geq x_2.$$

Hence, we have

$$\liminf_{x\to\infty} \frac{1}{\log x} \left(\log \liminf_{N\to\infty} P_L(x)^{(N)} - \log \mathbb{P}\{Q > x\} \right) \geq -\theta - \alpha > -\infty.$$

We next construct example input processes that satisfy the conditions on the liminf result of Theorem 2.

4. Examples of Processes for which the Liminf Condition Is Satisfied

The *limsup* part is general and satisfies simple conditions. Hence, it is obvious that it will hold for a very large class of input processes. However, the *liminf* part, especially, the condition on $\mathbb{P}\{X_n > 0|Q_0\}$ is complex. In this section, we provide examples for which this condition holds.

Since Q_0 depends on the entire history up to 0, it is very difficult to handle $\mathbb{P}\{X_n > 0|Q_0\}$ directly. For some classes of processes, given Q_0, the distribution of X_n can be determined from a bounded window of the past history. Let $\mathbf{\Delta}$ be an m-dimensional vector defined as

$$\mathbf{\Delta} = (\Delta_1, \Delta_2, \cdots, \Delta_m) := \boldsymbol{F}(\lambda_0, \lambda_{-1}, \cdots, \lambda_{-m+1}),$$

where $\boldsymbol{F}$ is an m-dimensional function.

Lemma 6 If there exist x_0, K, $M > 0, \alpha > 0$ and $\delta > 0$ such that $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta)}$ for all $n \geq MQ_0^\alpha$ on $\{Q_0 \geq x_0\}$, then $\mathbb{E}\{Z_0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} O(Q_0^\alpha)$, and $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$.

Consider the Chernoff bound:

$$\begin{aligned} \mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} &\overset{a.s.}{=} \mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} \\ &\overset{a.s.}{=} \mathbb{P}\{e^{\theta X_n} > 1|\mathbf{\Delta}, Q_0\} \\ &\overset{a.s.}{\leq} \mathbb{E}\{e^{\theta X_n}|\mathbf{\Delta}, Q_0\}, \qquad \forall \theta > 0. \end{aligned} \tag{25}$$

Here, θ need not be constant with respect to n. To satisfy the condition on $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\}$ in Lemma 6, it suffices to find x_0, K, $M > 0, \alpha > 0, \delta > 0$ and $\theta(n) > 0$ such that $\alpha C_0 < 1$ and

$$\mathbb{E}\{e^{\theta(n)X_n}|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta)} \quad \text{for all } n \geq MQ_0^\alpha \quad \text{on } \{Q_0 \geq x_0\},$$

or equivalently,

$$\mathbb{E}\{e^{\theta(Mt^\alpha)X_{Mt^\alpha}}|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} \frac{K}{M}t^{-\alpha(1+\delta)}, \quad \text{for all } t \geq Q_0 \quad \text{on } \{Q_0 \geq x_0\},$$

where $\theta(n)$ is a positive function of n and t takes values such that Mt^α is integer valued.

4.1. *Markov Modulated Fluid (MMF) Processes* In this subsection, we consider stationary and ergodic MMF processes. An MMF process is specified by $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{R})$, where $\boldsymbol{X}$ is the set of the states, $\boldsymbol{A}$ is the state transition probability matrix, and $\boldsymbol{R}$ is the rate vector. When the process is in state $x_i \in \boldsymbol{X}$, it generates fluid at the rate of r_i, the i-th element of $\boldsymbol{R}$, i.e., $\lambda_n = r_i$, and the probability that it transits to state x_j is a_{ij}, the (i, j)-th element of $\boldsymbol{A}$, i.e., $\mathbb{P}\{\lambda_{n+1} = r_j|\lambda_n = r_i\} = a_{ij}$.

Since the tail distribution of the queue $\mathbb{P}\{Q > x\}$ for these processes is asymptotically exponential [3, 10], it satisfies conditions (C1)-(C4) with $C_0 = 0$. Hence, we only need to check whether $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta)}$ for all $n \geq MQ_0^\alpha$ on $\{Q_0 \geq x_0\}$. For simplicity of illustration, we first check the two-state MMF on-off processes (these processes have widely been used to model voice traffic sources in telecommunication systems [8, 20], and are hence important in their own right). Without loss of generality, let $r_1 = 0$ and $r_2 = 1$. Since the process is stationary and ergodic, it has a unique stationary distribution, i.e., for each fixed j, the (i, j)-th element of $\mathbf{A}^n$ converges to the same value for all i. Then, $0 < a_{ii} < 1$ for $i = 1, 2$ in this two-state MMF case. Suppose that N i.i.d MMF processes are being served by a queue. Let $\lambda_n^{(l)}$ and λ_n denote each process l and the aggregate process, respectively. Then, $p := \mathbb{P}\{\lambda_n^{(l)} = 1\} = \frac{a_{12}}{a_{12}+a_{21}}$, $q := \mathbb{P}\{\lambda_n^{(l)} = 0\} = \frac{a_{21}}{a_{12}+a_{21}}$ and $\mathbb{E}\{\lambda_n\} = \sum_{i=1}^N \mathbb{E}\{\lambda_n^{(l)}\} = pN$. Due to the Markov property, the future is independent of the past. So the 1-dimensional $\mathbf{\Delta} := \lambda_0$ has sufficient information to determine the distribution thereafter. Now, we want to find x_0, K, $M > 0, \alpha > 0, \delta > 0$ and $\hat{\theta} > 0$ such that

$$\mathbb{E}\{e^{\hat{\theta}X_{Mt^\alpha}}|\lambda_0, Q_0\} \overset{a.s.}{\leq} Kt^{-\alpha-\delta} \quad \text{for all } t \geq Q_0 \text{ on } \{Q_0 \geq x_0\}.$$

Let $\mathbf{B}_\theta$ be a diagonal matrix given by:

$$\mathbf{B}_\theta = \begin{bmatrix} e^{\theta r_1} & 0 \\ 0 & e^{\theta r_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^\theta \end{bmatrix}.$$

Let $\mathbf{p}^{(l)}(r)$ be a row vector denoting the state distribution of the process l at time 0, i.e., $\mathbf{p}^{(l)}(r) = [\alpha_1, \alpha_2]$ where $\alpha_i = \mathbb{P}\{\lambda_0^{(l)} = r_i | \lambda_0 = r\}$. Then, $\mathbb{E}\{e^{\theta \sum_{i=1}^n \lambda_i^{(l)}} | \lambda_0, Q_0\}$ can be expressed in terms of $\mathbf{A}, \mathbf{B}_\theta$, and $\mathbf{p}^{(l)}$ as:

$$\mathbb{E}\left\{e^{\theta \sum_{i=1}^n \lambda_i^{(l)}} | \lambda_0, Q_0\right\} \overset{a.s.}{=} \mathbf{p}^{(l)}(\lambda_0) \left(\mathbf{A}\mathbf{B}_\theta\right)^n \mathbf{1}, \tag{26}$$

where $\mathbf{1}$ is a column vector of all unity elements. Let $\eta_1(\theta)$ and $\eta_2(\theta)$ be the eigenvalues of $\mathbf{A}\mathbf{B}_\theta$ which are given by:

$$\eta_1(\theta) = \frac{1}{2}\left[(a_{11} + a_{22}e^\theta) + \sqrt{(a_{11} + a_{22}e^\theta)^2 + 4(a_{11}a_{22} - a_{12}a_{21})e^\theta}\right], \tag{27}$$

$$\eta_2(\theta) = \frac{1}{2}\left[(a_{11} + a_{22}e^\theta) - \sqrt{(a_{11} + a_{22}e^\theta)^2 + 4(a_{11}a_{22} - a_{12}a_{21})e^\theta}\right]. \tag{28}$$

Then, $\mathbf{A}\mathbf{B}_\theta$ can be decomposed as

$$\mathbf{A}\mathbf{B}_\theta = \mathbf{T}_\theta \mathbf{\Lambda}_\theta \mathbf{T}_\theta^{-1}, \tag{29}$$

where $\mathbf{\Lambda}_\theta = \mathrm{diag}\{\eta_1(\theta), \eta_2(\theta)\}$ and $\mathbf{T}_\theta$ is an appropriate matrix. Thus, $\mathbb{E}\{e^{\theta \sum_{i=1}^n \lambda_i^{(l)}} | \lambda_0, Q_0\}$ has the following form:

$$\begin{aligned} \mathbb{E}\left\{e^{\theta \sum_{i=1}^n \lambda_i^{(l)}} | \lambda_0, Q_0\right\} &\overset{a.s.}{=} \mathbf{p}^{(l)}(\lambda_0) \mathbf{T}_\theta \mathbf{\Lambda}_\theta^n \mathbf{T}_\theta^{-1} \mathbf{1} \\ &= \sum_{k=1}^2 f_k(\mathbf{p}^{(l)}(\lambda_0), \theta)[\eta_k(\theta)]^n, \end{aligned} \tag{30}$$

where f_k is a function independent of n. Let $\beta \in (0, c - Np)$. Then,

$$\begin{aligned} \mathbb{E}\{e^{\theta X_n} | \lambda_0, Q_0\} &\overset{a.s.}{=} \left(\prod_{l=1}^N \mathbb{E}\left\{e^{\theta \sum_{i=1}^n \lambda_i^{(l)}} | \lambda_0, Q_0\right\}\right) e^{\theta(Q_0 - cn)} \\ &\overset{a.s.}{=} \left\{\prod_{l=1}^N \left(\sum_{k=1}^2 f_k(\mathbf{p}^{(l)}(\lambda_0), \theta)[\eta_k(\theta)]^n\right)\right\} e^{\theta(Q_0 - cn)}, \\ &= \left(\sum_{k=1}^2 f_k(\mathbf{p}^{(l)}(\lambda_0), \theta)[\eta_k(\theta)]^n\right)^N e^{\theta(Q_0 - cn)}, \\ &= \left(\sum_{k=1}^2 f_k(\mathbf{p}^{(l)}(\lambda_0), \theta)\left[\frac{\eta_k(\theta)}{e^{\theta \frac{c-\beta}{N}}}\right]^n\right)^N e^{\theta(Q_0 - \beta n)}. \end{aligned}$$

Since we are interested in the asymptotic behavior, we consider only the first eigenvalue $\eta_1(\theta)$ whose absolute value is the maximum. Let $g(\theta) := \log \eta_1(\theta)$ and $h(\theta) := \log e^{\theta \frac{c-\beta}{N}}$. One can check that

$g'(0) = \frac{a_{12}}{a_{12}+a_{21}} = p < \frac{c-\beta}{N} = h'(0)$ and $g(0) = h(0) = 0$. [§] So we can choose $\hat{\theta} > 0$ such that $g(\hat{\theta}) < h(\hat{\theta})$. Hence,

$$\frac{\eta_k(\theta)}{e^{\hat{\theta}\frac{c-\beta}{N}}} \le \frac{e^{g(\hat{\theta})}}{e^{h(\hat{\theta})}} < 1, \qquad k = 1, 2.$$

Let $t \ge Q_0$ and set $n = Mt^\alpha$ with $M = \frac{2}{\beta}$ and $\alpha = 1$. Note that t takes values such that Mt is an integer. Then,

$$\mathbb{E}\{e^{\hat{\theta}X_{Mt}}|\lambda_0, Q_0\} \overset{a.s.}{\le} \left(\sum_{k=1}^{2} f_k(\mathbf{p}^{(l)}(\lambda_0), \hat{\theta}) \left[\frac{\eta_k(\hat{\theta})}{e^{\hat{\theta}\frac{c-\beta}{N}}}\right]^{Mt}\right)^N e^{\hat{\theta}(Q_0-2t)}$$

$$\le \left(\sum_{k=1}^{2} f_k(\mathbf{p}^{(l)}(\lambda_0), \hat{\theta})\right)^N e^{-\hat{\theta}t}. \tag{31}$$

Since in (31) all terms in front of $e^{-\hat{\theta}t}$ are constant with respect to t, we can choose x_0 and K such that

$$\left(\sum_{k=1}^{2} f_k(\mathbf{p}^{(l)}(\lambda_0), \hat{\theta})\right)^N e^{-\hat{\theta}t} \le Kt^{-2} \quad \text{for all } t \ge x_0.$$

Hence,

$$\mathbb{E}\{e^{\hat{\theta}X_{Mt}}|\lambda_0, Q_0\} \le Kt^{-2} \quad \text{for all } t \ge Q_0 \text{ on } \{Q_0 \ge x_0\}.$$

At last, we have found $x_0, K, \alpha = 1, M = \frac{2}{\beta}$, and $\delta = 1$.

The general multi-state case can also be shown by the same manner. Consider an M-state MMF process with transition matrix $\mathbf{A}$ and rate vector $\mathbf{R} = [r_1, \cdots, r_M]^t$. Let $\mathbf{B}_\theta = \text{diag}\{e^{\theta r_1}, \cdots, e^{\theta r_M}\}$. Assume that $\mathbf{A}^n$ converges and that $\mathbf{AB}_\theta$ can be decomposed as Jordan form, i.e., $\mathbf{AB}_\theta = \mathbf{T}_\theta \mathbf{J}_\theta \mathbf{T}_\theta^{-1}$ with $\mathbf{J}_\theta = \text{Jordan}\{\eta_1(\theta), \cdots, \eta_{\tilde{M}}(\theta)\}$, where $\tilde{M}$ is the number of distinct eigenvalues. Note that when $\theta = 0$, even though eigenvalues except 1 may not be distinct, their magnitude is less than 1

[§]It can be directly shown by using (27). More generally, since $\mathbf{A}$ is a state transition matrix and $\mathbf{A}^n$ converges to a nonzero matrix, all the eigenvalues are on $(-1, 1]$ and one of them is 1. Since $\mathbf{AB}_0 = \mathbf{A}$, $\eta_1(0) = 1$. Now

$$\det(\mathbf{AB}_\theta - \eta_1(\theta)\mathbf{I}) = [\eta_1(\theta)]^2 - \eta_1(\theta)\sum_i a_{ii}e^{\theta r_i} - (a_{11}a_{22}e^{\theta(r_1+r_2)} + a_{12}a_{21}e^{\theta(r_1+r_2)}) = 0.$$

Differentiate the above equation with respect to θ and set $\theta = 0$. Since $\eta_1(0) = 1$, after some manipulation,

$$2\eta_1'(0) - \eta_1'(0)\sum_i a_{ii} - \sum_i a_{ii}r_i - (r_1 + r_2)(a_{12}a_{21} - a_{11}a_{22}) = 0.$$

Thus,

$$\eta_1'(0) = \frac{\sum_i a_{ii}r_i + (r_1 + r_2)(a_{12}a_{21} - a_{11}a_{22})}{2 - \sum_i a_{ii}} = \frac{a_{12}}{a_{12} + a_{21}} = \mathbb{E}\{\lambda^{(l)}\}.$$

This can also be extended to the m-state case.

in order for $(\mathbf{AB}_\theta)^n$ to converge. Let $\eta_1(\theta)$ be the eigenvalue such that $\eta_1(0) = 1$. As in (30),

$$\begin{aligned} \mathbb{E}\left\{e^{\theta\sum_{i=1}^n \lambda_i^{(l)}}|\lambda_0, Q_0\right\} &= \mathbf{p}^{(l)}(\lambda_0)\mathbf{T}_\theta\mathbf{J}_\theta^n\mathbf{T}_\theta^{-1}\mathbf{1} \\ &= f_1(\mathbf{p}^{(l)}(\lambda_0),\theta)[\eta_1(\theta)]^n + \sum_{k=2}^{\tilde{M}}[\eta_k(\theta)]^n \sum_{l=0}^{p_k-1} f_{k,l}(\mathbf{p}^{(l)}(\lambda_0),\theta)n^l, \end{aligned} \tag{32}$$

where f_1 and $f_{k,l}$ are functions independent of n, and p_k is the order of the corresponding eigenvalue. Since $n^l\zeta^n$ is bounded when $\zeta < 1$, by following the same steps, we can get the result.

4.2. *Gaussian Processes* In this subsection, assume that λ_n is a stationary and ergodic Gaussian process with autocovariance function $C_\lambda(k)$, and that $\mathbf{\Delta}$ is chosen so that both $\mathbf{\Delta}$ and $\{X_n|\mathbf{\Delta}, Q_0\}$ (X_n for given $\mathbf{\Delta}$ and Q_0) are Gaussian.¶ Gaussian processes are characterized by their mean and autocovariance. The next proposition tells us that Theorem 2 is applicable for Gaussian processes whose mean and variance are characterized by (33) and (34). In (33) and (34), ϵ and δ are related to self-similarity. One can see later that $\epsilon = \delta = 1$ for short-range dependent processes represented by Autoregressive Moving Average (ARMA) processes.

Proposition 2 Assume that there exist x_0, $K_1, K_2, \epsilon > 0$ and $\delta > 0$ such that

$$\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} K_1 n^{1-\epsilon} Q_0 - \kappa n, \tag{33}$$

$$Var\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} K_2 n^{2-\delta}, \tag{34}$$

for all $n \geq Q_0$ on $\{Q_0 \geq x_0\}$. Then, we can find x_1, K, $M > 0, \alpha > 0$ and $\delta' > 0$ such that $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta')}$ for all $n \geq MQ_0^\alpha$ on $\{Q_0 \geq x_1\}$.

Clearly $\mathbb{E}\{X_n\} - cn = -\kappa n + \mathbb{E}\{Q_0\}$, and $\text{Var}\{X_n\} \overset{n\to\infty}{\sim} Sn^{2-\delta}$ for some $S > 0$ and $\delta > 0$ for a large class of stationary Gaussian processes. Then, on $\{\mathbf{\Delta} = \boldsymbol{y}, Q_0 = z\}$ with fixed $\boldsymbol{y}$ and z, $\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\} - cn \overset{n\to\infty}{\sim} -\kappa n$ and $\text{Var}\{X_n|\mathbf{\Delta}, Q_0\} \overset{n\to\infty}{\sim} Sn^{2-\delta}$ a.s. These similarity relations are for $\{\mathbf{\Delta} = \boldsymbol{y}, Q_0 = z\}$ whereas (33) and (34) are for $\{Q_0 \geq x_0\}$. But we believe that (33) and (34) may hold for a large class of stationary Gaussian processes. For some classes of stationary Gaussian processes, we can prove that the order of $-\log\mathbb{P}\{Q_n > x\}$ with respect to x is less than 2.

¶ "$\{X_n|\mathbf{\Delta}, Q_0\}$ (X_n for given $\mathbf{\Delta}$ and Q_0) is Gaussian" means that there exists a Gaussian $Y_n(\boldsymbol{y}, z)$ such that $\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\}(\omega) = \mathbb{E}\{Y_n(\boldsymbol{y}, z)\}|_{\boldsymbol{y}=\mathbf{\Delta}(\omega), z=Q_0(\omega)}$ and $\text{Var}\{X_n|\mathbf{\Delta}, Q_0\}(\omega) = \text{Var}\{Y_n(\boldsymbol{y}, z)\}|_{\boldsymbol{y}=\mathbf{\Delta}(\omega), z=Q_0(\omega)}$ for almost all ω.

Proposition 3 If there exist $S > 0$ and $\beta \in [1, 2)$ such that

$$\sum_{k=-n}^{n} C_\lambda(k) \overset{n\to\infty}{\sim} S\beta n^{\beta-1}, \tag{35}$$

then $-\log \mathbb{P}\{Q_n > x\} = o(x^2)$.

A large class of stationary Gaussian processes satisfy equation (35). Since Δ_i is assumed to be Gaussian, the order of $-\log \mathbb{P}\{|\Delta_i| > x\}$ with respect to x is clearly 2. Then, it directly follows that $\mathbb{P}\{|\Delta_i| > x\} = O(\mathbb{P}\{Q_n > x\})$. The fact that $\mathbb{P}\{|\Delta_i| > x\} = O(\mathbb{P}\{Q_n > x\})$ gives us the following results which are modified versions of Proposition 2 and Theorem 2. These results are helpful for handling the Autoregressive Moving Average (ARMA) case. For notational simplicity, let $v := \max\{Q_0, |\Delta_1|, \cdots, |\Delta_m|\}$. Note that in these modified versions, we use "$n \geq Mv^\alpha$" instead of "$n \geq MQ_0^\alpha$".

Proposition 4 [from Proposition 2] Assume that there exist x_0, $K_1, K_2, \epsilon > 0$ and $\delta > 0$ such that

$$\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} K_1 n^{1-\epsilon} v - \kappa n, \tag{36}$$

$$Var\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} K_2 n^{2-\delta}, \tag{37}$$

for all $n \geq Q_0$ on $\{Q_0 \geq x_0\}$. Then, we can find x_1, K, $M > 0, \alpha > 0$ and $\delta' > 0$ such that $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta')}$ for all $n \geq Mv^\alpha$ on $\{Q_0 \geq x_1\}$.

Theorem 3 [from Theorem 2] Assume conditions (C1)-(C4). Further assume that $\mathbb{P}\{|\Delta_i| > x\} = O(\mathbb{P}\{Q_0 > x\})$ and that there exist x_0, K, $M > 0, \alpha > 0$ and $\delta > 0$ such that $\alpha C_0 < 1$ and $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} Kn^{-(1+\delta)}$ for all $n \geq Mv^\alpha$ on $\{Q_0 \geq x_0\}$. Then,

$$-\infty < \liminf_{x\to\infty} \frac{1}{\log x} \left[\log P_L(x) - \log \mathbb{P}\{Q > x\}\right]. \tag{38}$$

We now check the Autoregressive Moving Average (ARMA) processes. Since the tail distribution of the queue for these Gaussian input processes is asymptotically exponential [2, 6], these processes satisfy conditions (C1)-(C4) with $C_0 = 0$. Hence, for Theorem 3 to hold, we only need to check equations (36) and (37) for these processes. For the purpose of illustration, we first check the simplest form, $C_\lambda(l) = \sigma^2 a^{|l|}$ with $a \in (0, 1)$. These Gaussian processes are implemented by first-order AR processes in the form of

$$\lambda_n = a\lambda_{n-1} + g_n,$$

where g_n is i.i.d. Gaussian with mean μ_g and variance σ_g^2. Then, $\mathbb{E}\{\lambda_n\} = \frac{\mu_g}{1-a}$ and the autocovariance function is given by $C_\lambda(l) = \sigma^2 a^{|l|}$ with $\sigma^2 = \frac{\sigma_g^2}{1-a^2}$. As mentioned before, since we are interested in the asymptotic behavior of $\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\}$ and $\mathrm{Var}\{X_n|\mathbf{\Delta}, Q_0\}$, we now find constants in (36) and (37). X_n can be expressed as

$$
\begin{aligned}
X_n &:= \sum_{k=1}^{n} \lambda_k - cn + Q_0 \\
&= a\frac{1-a^n}{1-a}\lambda_0 + \sum_{k=1}^{n} g_k \frac{1-a^{n-k+1}}{1-a} - cn + Q_0.
\end{aligned}
$$

Note that for all $k \geq 1$, g_k is independent of the event at 0. Let $\mathbf{\Delta} := (\lambda_0)$ and $Y_n(y,z) := a\frac{1-a^n}{1-a}y + \sum_{k=1}^{n} g_k \frac{1-a^{n-k+1}}{1-a} - cn + z$. Then, $Y_n(y,z)$ is Gaussian, $\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} \mathbb{E}\{Y_n(y,z)\}|_{y=\lambda_0, z=Q_0}$, and $\mathrm{Var}\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} \mathrm{Var}\{Y_n(y,z)\}|_{y=\lambda_0, z=Q_0}$. Hence, we have

$$
\begin{aligned}
\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\} &\overset{a.s.}{=} \frac{\mu_g}{1-a}\left(n - \frac{a(1-a^n)}{1-a}\right) - cn + Q_0 + a\lambda_0\frac{1-a^n}{1-a} \\
&= z + yh_0(n) - \kappa n - h_1(n), \qquad (39) \\
\mathrm{Var}\{X_n|\mathbf{\Delta}, Q_0\} &\overset{a.s.}{=} \frac{\sigma_g^2}{(1-a)^2}\left(n - 2\frac{a(1-a^n)}{1-a} + \frac{a^2(1-a^{2n})}{1-a^2}\right) \\
&= Sn + h_2(n), \qquad (40)
\end{aligned}
$$

where

$$
\begin{aligned}
h_0(n) &:= \frac{a(1-a^n)}{1-a}, \\
h_1(n) &:= \mu_g\frac{a(1-a^n)}{(1-a)^2}, \\
S &:= \frac{\sigma_g^2}{(1-a)^2} = \sum_{l=-\infty}^{\infty} C_\lambda(l), \\
h_2(n) &:= S\left(-2\frac{a(1-a^n)}{1-a} + \frac{a^2(1-a^{2n})}{1-a^2}\right).
\end{aligned}
$$

Set $K_1 = 1 + h_0(\infty)$ and $K_2 = S + \sup_{n\geq 1} h_2(n)$. Note that $K_1, K_2 < \infty$ since $a \in (0,1)$. Then, for all $n \geq 1$,

$$
\mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} Q_0 + \lambda_0 h_0(n) - \kappa n - h_1(n) \leq K_1 \max\{Q_0, \lambda_0\} - \kappa n,
$$

and

$$
\mathrm{Var}\{X_n|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} Sn + h_2(n) \leq K_2 n.
$$

Hence, first-order AR Gaussian processes satisfy the conditions in Theorem 3

We now treat the more general case of Gaussian input processes, where the autocovariance function has the form of $C_\lambda(l) = \sum_{i=1}^{N} \sigma_i^2 p_i^{|l|}$, i.e., a Gaussian process in the form of an Autoregressive Moving Average (ARMA). We will show that it satisfies the condition on $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\}$ in Theorem 3. Note that autocovariance functions in the form of $C_\lambda(l) = \sum_{i=1}^{N} \sigma_i^2 p_i^{|l|}$ satisfies equation (35) for $\beta = 1$.

Consider an ARMA process:

$$\lambda_n = \sum_{k=1}^{N} a_k \lambda_{n-k} + \sum_{k=0}^{M} b_k g_{n-k}, \tag{41}$$

where g_n is i.i.d. Gaussian with mean μ_g and variance σ_g^2. Assume for stability that all the poles of the transfer function of this ARMA process are inside the unit circle, and its autocovariance function is summable so that it satisfies (35). If $M = 0$, it is the AR process treated earlier.

Let $m = N + M$ and $\mathbf{\Delta} := (\lambda_0, \cdots, \lambda_{-N-1}, g_0, \cdots, g_{-M-1})$. Then, $\mathbf{\Delta}$ is Gaussian and X_n for given $\mathbf{\Delta}$ and Q_0 is also Gaussian. Hence, we have $\mathbb{P}\{|\Delta_i| > x\} = O(\mathbb{P}\{Q_0 > x\})$ and we can apply Theorem 3 by finding K_1, K_2 and x_0 in (36) and (37). Given $\mathbf{\Delta}$, we can write λ_n as:

$$\lambda_n = \sum_{k=0}^{N-1} f_k(n)\Delta_k + \sum_{k=0}^{M-1} g_k(n)\Delta_{k+N} + \sum_{k=1}^{n} h_k(n) g_k, \tag{42}$$

where $f_k(n), g_k(n)$, and $h_k(n)$ are determined by the coefficients, a_i and b_j. Note that $f_k(n), g_k(n)$, and $h_k(n)$ are summable over n due to the stability assumption. Now, consider the summation of λ_k.

$$\sum_{k=1}^{n} \lambda_k = \sum_{k=0}^{N-1} F_k(n)\Delta_k + \sum_{k=0}^{M-1} G_k(n)\Delta_{k+N} + \sum_{k=1}^{n} H_k(n) g_k, \tag{43}$$

where

$$\begin{aligned} F_k(n) &= \sum\nolimits_{i=1}^{n} f_k(i), \qquad k = 0, 1, \cdots, N-1, \\ G_k(n) &= \sum\nolimits_{i=1}^{n} g_k(i), \qquad k = 0, 1, \cdots, M-1, \\ H_k(n) &= \sum\nolimits_{i=k}^{n} h_k(i), \qquad k = 1, 2, \cdots, n. \end{aligned}$$

Taking expectation,

$$\mathbb{E}\left\{\sum_{k=1}^{n} \lambda_k | \mathbf{\Delta}, Q_0\right\} \stackrel{a.s.}{=} \sum_{k=0}^{N-1} F_k(n)\Delta_k + \sum_{k=0}^{M-1} G_k(n)\Delta_{k+N} + \sum_{k=1}^{n} H_k(n)\mu_g. \tag{44}$$

For stable ARMA processes $f_k(n), g_k(n)$, and $h_k(n)$ are summable so that $F_k(n), G_k(n)$, and $H_k(n)$ converge. Moreover, the last term can be expressed as

$$\sum_{k=1}^{n} H_k(n)\mu_g = \mathbb{E}\{\lambda\} n + H(n),$$

where $H(n)$ is a function which converges. Therefore, we can find K_1 and K_2 such that

$$\mathbb{E}\left\{\sum_{k=1}^{n} \lambda_k | \boldsymbol{\Delta}, Q_0\right\} \overset{a.s.}{\leq} K_1 u + \mathbb{E}\{\lambda\} n + K_2,$$

where $u = \max\{|\Delta_1|, \cdots, |\Delta_m|\}$. Since K_1 and K_2 are independent of $\boldsymbol{y}$ and z,

$$\begin{aligned}\mathbb{E}\{X_n | \boldsymbol{\Delta}, Q_0\} &\overset{a.s.}{=} \mathbb{E}\left\{\sum_{k=1}^{n} \lambda_k | \boldsymbol{\Delta}, Q_0\right\} - cn + Q_0 \\ &\overset{a.s.}{\leq} K_1 u + Q_0 - \kappa n + K_2 \\ &\leq (K_1 + 2)\max\{u, Q_0\} - \kappa n, \quad \text{on } \{Q_0 \geq x_0\}, \quad \forall n \geq 1.\end{aligned}$$

One can easily check that

$$\mathrm{Var}\{X_n | \boldsymbol{\Delta}, Q_0\} \overset{a.s.}{\leq} \sum_{k=1}^{n} H_k(n)^2 \sigma_g^2 \leq (S + K)n, \qquad \forall n \geq 1,$$

where $S = \sum_{l=-\infty}^{\infty} C_\lambda(l)$ and K are finite constants.

5. Appendix

5.1. *Proof of Lemma 1.*

Recall V_n, U_n, W_n, Z_n, and S_n^x defined in Section 2, we rewrite them here for convenience.

- $U_n := \sup\{k \leq n : Q_{k-1} > 0, Q_k = 0\}$ (start time of the current cycle to which n belongs).
- $V_n := \inf\{k > n : Q_{k-1} > 0, Q_k = 0\}$ (start time of the next cycle).
- $W_n := V_n - U_n$ (duration of a cycle to which n belongs).
- $Z_n := V_n - n$ (residual time to reach the end of the cycle).
- $S_n^x := \sum_{k=U_n}^{V_n-1} 1_{\{Q_k > x\}}$ (duration for which $Q_k > x$ in a cycle containing n).

We further define $R_n^x := \sum_{k=n}^{V_n-1} 1_{\{Q_k > x\}}$ (residual duration for which $Q_k > x$ in a cycle containing n). Since Q_n is stationary and ergodic, so are the above. Hence, their expectations are equal to time averages. Since we are interested in the behavior of Q_n after loss happens, we consider the conditional expectations:

$$\mathbb{E}\{Z_n | Q_n > x\} = \lim_{k\to\infty} \frac{1}{\sum_{i=1}^{k} 1_{\{Q_i > x\}}} \sum_{i=1}^{k} Z_i^x 1_{\{Q_i > x\}}, \tag{45}$$

$$\mathbb{E}\{S_n^x|Q_n > x\} = \lim_{k\to\infty} \frac{1}{\sum_{i=1}^k 1_{\{Q_i>x\}}} \sum_{i=1}^k S_i^x 1_{\{Q_i>x\}}, \tag{46}$$

$$\mathbb{E}\{R_n^x|Q_n > x\} = \lim_{k\to\infty} \frac{1}{\sum_{i=1}^k 1_{\{Q_i>x\}}} \sum_{i=1}^k R_i^x 1_{\{Q_i>x\}}. \tag{47}$$

Clearly, $\mathbb{E}\{R_n^x|Q_n > x\} \le \mathbb{E}\{Z_n|Q_n > x\}$. And it can also be easily checked that $2\mathbb{E}\{R_n^x|Q_n > x\} \ge \mathbb{E}\{S_n^x|Q_n > x\}$, where the inequality is because n is discrete.

Since $\mathbb{E}\{\lambda\} < c$, there are infinitely many cycles in a sample path. We index the cycles as follows:

- $V^{(1)} := V_1,\ V^{(0)} := U_1$,
  $V^{(i)} := V_{V^{(i-1)}+1}$ for $i > 1$,
- $A^{(i)} := \{n : V^{(i-1)} \le n < V^{(i)}\}$ for $i > 0$ (set of time instants belonging to cycle i).

Define:

- $S_x^{(i)} := \sum_{k\in A^{(i)}} 1_{\{Q_k>x\}}, i = 1, 2, 3, \cdots$,
- $\hat{S}^x := \limsup_{m\to\infty} \frac{\sum_{i=1}^m S_x^{(i)}}{\sum_{i=1}^m 1_{\{S_x^{(i)}>0\}}}$.

Now, We prove the lemma in two steps:

- 1) Show $2\mathbb{E}\{Z_n^x|Q_n > x\} \ge \hat{S}^x$.
- 2) Derive

$$\begin{aligned}\int_x^\infty \frac{1}{2\mathbb{E}\{Z|Q > y\}}\mathbb{P}\{Q > y\}dy &\le \int_x^\infty \frac{1}{\hat{S}_y}\mathbb{P}\{Q > y\}dy \\ &\le \mathbb{E}\{\lambda\} \liminf_{m\to\infty} P_L(x)^{(m)} \\ &\le \mathbb{E}\{\lambda\} \limsup_{m\to\infty} P_L(x)^{(m)} \le \int_x^\infty \mathbb{P}\{Q > y\}dy.\end{aligned}$$

Step 1) For better understanding, we first show

$$\limsup_{m\to\infty} \frac{1}{m}\sum_{i=1}^m S_x^{(i)} \le \lim_{m\to\infty} \frac{1}{\sum_{i=1}^m S_x^{(i)}} \sum_{i=1}^m (S_x^{(i)})^2.$$

Note that all components are nonnegative. Let $a_m := \frac{1}{m}\sum_{i=1}^m S_x^{(i)}$, $b_m := \frac{1}{\sum_{i=1}^m S_x^{(i)}} \sum_{i=1}^m (S_x^{(i)})^2$, $a^* = \limsup a_m$, and $b^* = \lim b_m$. For any $\epsilon > 0$, we can choose M such that $a_M - a^* < \epsilon$ and $|b^* - b_M| < \epsilon$. Then,

$$b^* - a^* = b_M + (b^* - b_M) - a_M - (a^* - a_M) \ge (b_M - a_M) - 2\epsilon \ge -2\epsilon$$

since

$$b_M - a_M = \frac{\sum (S_x^{(i)})^2}{\sum S_x^{(i)}} - \frac{\sum S_x^{(i)}}{M} = \frac{\sum_{i \neq j} (S_x^{(i)} - S_x^{(j)})^2}{M \sum S_x^{(i)}} \geq 0.$$

Since ϵ is arbitrary, we have $b^* \geq a^*$.

Now, it can be easily verified that

$$\hat{S}^x \leq \lim_{m\to\infty} \frac{1}{\sum_{i=1}^m S_x^{(i)}} \sum_{i=1}^m (S_x^{(i)})^2. \tag{48}$$

Construct a new sequence $\{T_x^{(i)}\}$ by removing zero-valued elements of $\{S_x^{(i)}\}$. Then,

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{i=1}^m T_x^{(i)} \leq \lim_{m\to\infty} \frac{1}{\sum_{i=1}^m T_x^{(i)}} \sum_{i=1}^m (T_x^{(i)})^2.$$

Note that

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{i=1}^m T_x^{(i)} = \limsup_{m\to\infty} \frac{1}{\sum_{i=1}^m 1_{\{S_x^{(i)} > x)\}}} \sum_{i=1}^m S_x^{(i)} = \hat{S}^x.$$

Let $B_x^{(i)} := \{n : U^{(i)} \leq n < V^{(i)}, Q_n > x\}$. Since $S_j^x = S_x^{(i)}$ for all $j \in B_x^{(i)}$ and $|B_x^{(i)}| = S_x^{(i)}$,

$$\begin{aligned}
\mathbb{E}\{S_n^x | Q_n > x\} &= \lim_{k\to\infty} \frac{1}{\sum_{i=1}^k 1_{\{Q_i > x\}}} \sum_{i=1}^k S_i^x 1_{\{Q_i > x\}} \\
&= \lim_{k\to\infty} \frac{1}{\sum_{i=1}^k \sum_{j\in B_x^{(i)}} 1} \sum_{i=1}^k \sum_{j \in B_x^{(i)}} S_j^x \\
&= \lim_{k\to\infty} \frac{1}{\sum_{i=1}^k |B_x^{(i)}|} \sum_{i=1}^k \sum_{j \in B_x^{(i)}} S_x^{(i)} \\
&= \lim_{k\to\infty} \frac{1}{\sum_{i=1}^k S_x^{(i)}} \sum_{i=1}^k (S_x^{(i)})^2 \\
&= \lim_{k\to\infty} \frac{1}{\sum_{i=1}^k T_x^{(i)}} \sum_{i=1}^k (T_x^{(i)})^2.
\end{aligned}$$

Hence, (48) follows.

At last, we have

$$\hat{S}^x \leq \mathbb{E}\{S_n^x | Q_n > x\} \leq 2\mathbb{E}\{R_n^x | Q_n > x\} \leq 2\mathbb{E}\{Z_n | Q_n > x\}. \tag{49}$$

Step 2) Let $L^{(i)}$ be the amount of loss in cycle i, and $\Lambda^{(i)}$ be the total amount of input in cycle i. Then, (1) and (2) can be expressed in terms of $S_x^{(i)}, W^{(i)}, L^{(i)}$, and $\Lambda^{(i)}$ as:

$$\mathbb{P}\{Q > x\} = \lim_{m\to\infty} \frac{\sum_{i=1}^m S_x^{(i)}}{\sum_{i=1}^m W^{(i)}}, \tag{50}$$

and

$$P_L(x)^{(m)} = \frac{\sum_{i=1}^m L^{(i)}}{\sum_{i=1}^m \Lambda^{(i)}} = \frac{\sum_{i=1}^m L^{(i)}}{\left(\sum_{i=1}^m W^{(i)}\right)\left(\frac{\sum_{i=1}^m \Lambda^{(i)}}{\sum_{i=1}^m W^{(i)}}\right)}, \tag{51}$$

Note that even though the above $P_L(x)^{(m)}$ is slightly different from $P_L(x)^{(N)}$ in (2), their limsup and liminf are the same, respectively.

The amount of loss in cycle i is greater than or equal to the difference between the maximum queue level of the infinite buffer queue and the buffer size x of the finite buffer queue in cycle i (See Figure 2). In other words,

$$L^{(i)} \geq \max_{k\in A^{(i)}} (Q_k - x)^+ = \int_x^\infty I(\max_{k\in A^{(i)}} Q_k > y)dy = \int_x^\infty I(S_y^{(i)} > 0)dy.$$

On the other hand, it is obvious that

$$L^{(i)} \leq \sum_{k\in A^{(i)}} (Q_k - x)^+.$$

Combining the above two equations and taking summation over i, we have

$$\int_x^\infty \sum_{i=1}^m I(S_y^{(i)} > 0)dy \leq \sum_{i=1}^m L^{(i)} \leq \sum_{i=1}^m \sum_{k\in A^{(i)}} (Q_k - x)^+.$$

Dividing both sides by the total time, $\sum_{i=1}^m W^{(i)}$, we have

$$\int_x^\infty \frac{\sum_{i=1}^m I(S_y^{(i)} > 0)}{\sum_{i=1}^m W^{(i)}} dy \leq \frac{\sum_{i=1}^m L^{(i)}}{\sum_{i=1}^m W^{(i)}} \leq \frac{\sum_{i=1}^m \sum_{k\in A^{(i)}} (Q_k - x)^+}{\sum_{i=1}^m W^{(i)}}. \tag{52}$$

From the above equation, we can write

$$\begin{aligned} \int_x^\infty \inf_{l\geq m} \frac{\sum_{i=1}^l I(S_y^{(i)} > 0)}{\sum_{i=1}^l W^{(i)}} dy &\leq \inf_{l\geq m} \frac{\sum_{i=1}^l L^{(i)}}{\sum_{i=1}^l W^{(i)}} \\ &\leq \sup_{l\geq m} \frac{\sum_{i=1}^l L^{(i)}}{\sum_{i=1}^l W^{(i)}} \leq \sup_{l\geq m} \frac{\sum_{i=1}^l \sum_{k\in A^{(i)}} (Q_k - x)^+}{\sum_{i=1}^l W^{(i)}}. \end{aligned} \tag{53}$$

Consider the rightmost side of (53). As $m \to \infty$, the time average should be equal to the expectation due to the ergodicity of Q_n, i.e.,

$$\sup_{l\geq m} \frac{\sum_{i=1}^l \sum_{k\in A^{(i)}} (Q_k - x)^+}{\sum_{i=1}^l W^{(i)}} \longrightarrow \mathbb{E}\{(Q_n - x)^+\} = \int_x^\infty \mathbb{P}\{Q > y\}dy.$$

For the third term of (53),

$$\sup_{l\geq m} \frac{\sum_{i=1}^l L^{(i)}}{\sum_{i=1}^l W^{(i)}} \leq \sup_{l\geq m} P_L(x)^{(l)} \sup_{l\geq m} \frac{\sum_{i=1}^l \Lambda^{(i)}}{\sum_{i=1}^l W^{(i)}} \longrightarrow \mathbb{E}\{\lambda\} \limsup P_L(x)^{(m)},$$

where $\limsup \frac{\sum_{i=1}^m \Lambda^{(i)}}{\sum_{i=1}^m W^{(i)}} = \mathbb{E}\{\lambda\}$ since λ_k is ergodic. Similarly, for the second term of (53),

$$\inf_{l\geq m} \frac{\sum_{i=1}^l L^{(i)}}{\sum_{i=1}^l W^{(i)}} \geq \inf_{l\geq m} P_L(x)^{(l)} \inf_{l\geq m} \frac{\sum_{i=1}^m \Lambda^{(i)}}{\sum_{i=1}^m W^{(i)}} \longrightarrow \mathbb{E}\{\lambda\} \liminf P_L(x)^{(m)},$$

Now, consider the leftmost side of (53):

$$\begin{aligned}\int_x^\infty \inf_{l\geq m} \frac{\sum_{i=1}^l I(S_y^{(i)}>0)}{\sum_{i=1}^l W^{(i)}} dy &\geq \int_x^\infty \inf_{l\geq m}\left(\frac{\sum_{i=1}^l I(S_y^{(i)}>0)}{\sum_{i=1}^l S_y^{(i)}}\right) \inf_{l\geq m}\left(\frac{\sum_{i=1}^l S_y^{(i)}}{\sum_{i=1}^l W^{(i)}}\right) dy \\ &= \int_x^\infty \left(\sup_{l\geq m} \frac{\sum_{i=1}^l S_y^{(i)}}{\sum_{i=1}^l I(S_y^{(i)}>0)}\right)^{-1} \inf_{l\geq m}\left(\frac{\sum_{i=1}^l S_y^{(i)}}{\sum_{i=1}^l W^{(i)}}\right) dy.\end{aligned}$$

Since $\sup_{l\geq m} \frac{\sum_{i=1}^l S_y^{(i)}}{\sum_{i=1}^l I(S_y^{(i)}>0)} \to \hat{S}^y$, $\inf_{l\geq m} \frac{\sum_{i=1}^l S_y^{(i)}}{\sum_{i=1}^l W^{(i)}} \to \mathbb{P}\{Q>y\}$, and all are nonnegative, by Fatou's Lemma,

$$\liminf_{m\to\infty} \int_x^\infty \left(\sup_{l\geq m} \frac{\sum_{i=1}^l S_y^{(i)}}{\sum_{i=1}^l I(S_y^{(i)}>0)}\right)^{-1} \inf_{l\geq m}\left(\frac{\sum_{i=1}^l S_y^{(i)}}{\sum_{i=1}^l W^{(i)}}\right) dy \geq \int_x^\infty \frac{1}{\hat{S}^y}\mathbb{P}\{Q>y\}dy.$$

From (49),

$$\int_x^\infty \frac{1}{\hat{S}^y}\mathbb{P}\{Q>y\}dy \geq \int_x^\infty \frac{1}{2\mathbb{E}\{Z|Q>y\}}\mathbb{P}\{Q>y\}dy.$$

Finally, we have

$$\begin{aligned}\int_x^\infty \frac{1}{2\mathbb{E}\{Z|Q>y\}}\mathbb{P}\{Q>y\}dy &\leq \mathbb{E}\{\lambda\} \liminf_{m\to\infty} P_L(x)^{(m)} \\ &\leq \mathbb{E}\{\lambda\} \limsup_{m\to\infty} P_L(x)^{(m)} \leq \int_x^\infty \mathbb{P}\{Q>y\}dy.\end{aligned}$$

5.2. *Proof of Lemma 2.*

Recall (C1), (C2) and (C3):

$$C_0 := \lim_{x\to\infty} \frac{\log x}{\phi(x)} = \lim_{x\to\infty} \frac{1}{x\phi'(x)} = \lim_{x\to\infty} \frac{-\phi''(x)}{\phi'(x)^2} < 1.$$

Since $\lim_{x\to\infty} \frac{1}{x\phi'(x)} < 1$, for large x

$$\frac{1}{\phi'(x)} e^{-\phi(x)} \leq x e^{-\phi(x)} = e^{-\phi(x)+\log x}.$$

Consider $-\phi(x)+\log x$. Let $\delta \in (0, 1-C_0)$. Since $\lim_{x\to\infty} \frac{\log x}{\phi(x)} < 1$, there are x_0 and $\delta>0$ such that

$$\frac{\log x}{\phi(x)} < C_0 + (1-C_0-\delta) = 1-\delta, \qquad \forall x \geq x_0.$$

Thus,

$$-\phi(x) + \log x \le -\frac{1}{1-\delta}\log x + \log x = -\frac{\delta}{1-\delta}\log x, \qquad \forall x \ge x_0.$$

Therefore,

$$\lim_{x\to\infty} \frac{1}{\phi'(x)} e^{-\phi(x)} \le \lim_{x\to\infty} e^{-\frac{\delta}{1-\delta}\log x} = 0, \tag{54}$$

and, since $\phi(x)$ is nondecreasing, i.e., $\phi'(x)$ is nonnegative, $\lim_{x\to\infty} \frac{1}{\phi'(x)} e^{-\phi(x)} = 0$. Now, consider

$$\frac{d}{dy}\left(\frac{1}{\phi'(y)} e^{-\phi(y)}\right) = -e^{-\phi(y)} - \frac{\phi''(y)}{\phi'(y)^2} e^{-\phi(y)}. \tag{55}$$

Integrating both sides from x to ∞, we have from (54) that

$$-\frac{1}{\phi'(x)} e^{-\phi(x)} \quad = \quad -\int_x^\infty e^{-\phi(y)} dy - \int_x^\infty \frac{\phi''(y)}{\phi'(y)^2} e^{-\phi(y)} dy. \tag{56}$$

Let $\epsilon \in (0, 1 - C_0)$. We can pick x_1 large enough such that $\left|\frac{\phi''(y)}{\phi'(y)^2} + C_0\right| < \epsilon$ for all $y \ge x_1$. Then,

$$\begin{aligned}
\frac{1}{\phi'(x)} e^{-\phi(x)} + (C_0 - \epsilon)\int_x^\infty e^{-\phi(y)} dy \quad &\le \quad \int_x^\infty e^{-\phi(y)} dy \\
&= \quad \frac{1}{\phi'(x)} e^{-\phi(x)} - \int_x^\infty \frac{\phi''(y)}{\phi'(y)^2} e^{-\phi(y)} dy \\
&\le \quad \frac{1}{\phi'(x)} e^{-\phi(x)} + (C_0 + \epsilon)\int_x^\infty e^{-\phi(y)} dy, \quad \forall x \ge x_1,
\end{aligned} \tag{57}$$

which means that

$$\frac{1}{1 - C_0 + \epsilon}\left(\frac{1}{\phi'(x)} e^{-\phi(x)}\right) \le \int_x^\infty e^{-\phi(y)} dy \le \frac{1}{1 - C_0 - \epsilon}\left(\frac{1}{\phi'(x)} e^{-\phi(x)}\right), \quad \forall x \ge x_1, \tag{58}$$

and the result follows.

5.3. *Proof of Lemma 3.*

Since $\mathbb{P}\{X_n > 0 | Q_0\} \overset{a.s.}{\le} 1$, and since $\mathbb{P}\{X_n > 0 | Q_0\} \overset{a.s.}{\le} K n^{-(1+\delta)}$ for all $n \ge M Q_0^\alpha$ on $\{Q \ge x_0\}$, it follows from (19) that on $\{Q \ge x_0\}$,

$$\begin{aligned}
\mathbb{E}\{Z_0 | Q_0\} \quad &\overset{a.s.}{=} \quad \sum_{n=0}^\infty \mathbb{P}\{Z_0 > n | Q_0\} \overset{a.s.}{\le} \sum_{n=0}^\infty \mathbb{P}\{X_n > 0 | Q_0\} \\
&= \quad \sum_{n=0}^{\lceil M Q_0^\alpha \rceil} \mathbb{P}\{X_n > 0 | Q_0\} + \sum_{n=\lceil M Q_0^\alpha \rceil + 1}^\infty \mathbb{P}\{X_n > 0 | Q_0\} \\
&\overset{a.s.}{\le} \quad \sum_{n=0}^{\lceil M Q_0^\alpha \rceil} 1 + \sum_{n=\lceil M Q_0^\alpha \rceil + 1}^\infty K n^{-1-\delta} \\
&\overset{Q_0 \to \infty}{\sim} \quad M Q_0^\alpha,
\end{aligned} \tag{59}$$

where $\lceil x \rceil$ denotes the smallest integer which is greater than or equal to x. Since $\mathbb{E}\{Z_0|Q_0\}$ is nonnegative a.s., $\mathbb{E}\{Z_0|Q_0\} \stackrel{a.s.}{=} O(Q_0^\alpha)$.

5.4. *Proof of Lemma 4.*

Note that for a nonnegative random variable X with distribution function F_X,

$$\int_0^\infty z^\alpha dF_X(z) = \mathbb{E}\{X^\alpha\} = \int_0^\infty \alpha z^{\alpha-1}\mathbb{P}\{X > z\}dz.$$

Replace X with $Y := Q1_{\{Q>x\}}$. Then,

$$\begin{aligned}
\int_x^\infty z^\alpha \, dF_Y(z) = \mathbb{E}\{Y\} &= \int_0^\infty \alpha z^{\alpha-1}\mathbb{P}\{Y > z\}dz \\
&= \int_0^x \alpha z^{\alpha-1}\mathbb{P}\{Y > z\}dz + \int_x^\infty \alpha z^{\alpha-1}\mathbb{P}\{Y > z\}dz \\
&= \int_0^x \alpha z^{\alpha-1}\mathbb{P}\{Q > x\}dz + \int_x^\infty \alpha z^{\alpha-1}\mathbb{P}\{Q > z\}dz \\
&= x^\alpha \mathbb{P}\{Q > x\} + \int_x^\infty \alpha z^{\alpha-1} e^{-\phi(z)}dz \\
&= x^\alpha \mathbb{P}\{Q > x\} + \int_x^\infty \alpha e^{-\phi(z)+\beta\log z}dz, \qquad (60)
\end{aligned}$$

where $\beta = \alpha - 1$. To prove the lemma, it suffices to show that the second term, $\int_x^\infty \alpha e^{-\phi(z)+\beta\log z}dz$, is $O(x^\alpha \mathbb{P}\{Q > x\})$. Let $f(x) = \phi(x) - \beta\log x$. Then,

$$\begin{aligned}
f'(x) &= \phi'(x) - \beta x^{-1}, \\
f''(x) &= \phi''(x) + \beta x^{-2}.
\end{aligned}$$

From (C2) and (C3),

$$\begin{aligned}
-\frac{f''(x)}{f'(x)^2} &= -\frac{\phi''(x) + \beta x^{-2}}{\left(\phi'(x) - \beta x^{-1}\right)^2} = \frac{-\frac{\phi''(x)}{\phi'(x)^2} - \beta\left(\frac{1}{x\phi'(x)}\right)^2}{\left(1 - \beta\frac{1}{x\phi'(x)}\right)^2} \\
&\stackrel{x\to\infty}{\to} \frac{C_0 - \beta C_0^2}{(1-\beta C_0)^2} = \frac{C_0}{1-\beta C_0} =: C_2. \qquad (61)
\end{aligned}$$

Since from the premise of the lemma, $\alpha C_0 = (\beta+1)C_0$ is less than 1, $0 \le C_0 < 1 - \beta C_0$. Hence, $C_2 = \frac{C_0}{1-\beta C_0} < 1$. Now, we need to show that $\lim_{x\to\infty} \frac{1}{f'(x)} e^{-f(x)} = 0$.

$$\begin{aligned}
\frac{1}{f'(x)} e^{-f(x)} &= \frac{1}{xf'(x)} e^{-f(x)+\log x} \\
&= \frac{1}{x\phi'(x) - \beta} e^{-\phi(x)+(\beta+1)\log x} \\
&= \frac{1}{1 - \frac{\beta}{x\phi'(x)}} \frac{1}{x\phi'(x)} e^{-\phi(x)+\alpha\log x}.
\end{aligned}$$

From (C2),

$$\lim_{x\to\infty} \frac{1}{1-\frac{\beta}{x\phi'(x)}} \frac{1}{x\phi'(x)} = \frac{1}{1-\beta C_0} C_0 = C_2.$$

Consider $-\phi(x) + \alpha \log x$. Let $\delta \in (0, \frac{1-\alpha C_0}{\alpha})$. Since $\lim_{x\to\infty} \frac{\log x}{\phi(x)} = C_0$, there is an x_0 such that

$$-\phi(x) + \alpha \log x \leq -\phi(x) + \alpha(C_0+\delta)\phi(x) = -(1-\alpha C_0 - \alpha\delta)\phi(x), \qquad \forall x \geq x_0.$$

Since $\delta' := 1-\alpha C_0 - \alpha\delta > 0$, $\lim_{x\to\infty} e^{-\phi(x)+\alpha \log x} \leq \lim_{x\to\infty} e^{-\delta'\phi(x)} = 0$, and, hence, $\lim_{x\to\infty} e^{-\phi(x)+\alpha\log x} = 0$ because $e^{-\phi(x)+\alpha \log x}$ is nonnegative. Thus,

$$\lim_{x\to\infty} \frac{1}{f'(x)} e^{-f(x)} = C_2 \cdot 0 = 0. \tag{62}$$

Since $C_2 < 1$ and $\lim_{x\to\infty} \frac{1}{f'(x)} e^{-f(x)} = 0$, as shown in the proof of Lemma 2, we have

$$\int_x^\infty e^{-f(z)} dz \overset{x\to\infty}{\sim} \frac{1}{(1-C_2)f'(x)} e^{-f(x)}. \tag{63}$$

Hence, there is an x_1 such that

$$\begin{aligned}
\int_x^\infty \alpha e^{-f(z)} dz = \ &\leq \frac{\alpha}{1-C_2} x e^{-f(x)} \qquad \text{(from (63) and } \lim_{x\to\infty} \frac{1}{xf'(x)} = C_0 < 1) \\
&= \frac{\alpha}{1-C_2} x^\alpha e^{-\phi(x)} \qquad \text{(from (62))} \\
&= \frac{\alpha}{1-C_2} x^\alpha \mathbb{P}\{Q > x\}, \qquad \forall x \geq x_1,
\end{aligned}$$

and (22) follows because it is nonnegative.

Now, repeat the above procedure with $f(x) = \phi(x) + \alpha \log x$. Then, C_2 becomes C_1 in (61) and, hence, (63) is equivalent to (23).

5.5. *Proof of Lemma 5.*

The assumption that $\mathbb{E}\{Z_0|Q_0\} \overset{a.s.}{=} O(Q_0^\alpha)$ means that there exist x_0 and K such that

$$\mathbb{E}\{Z_0|Q_0\} \overset{a.s.}{\leq} K Q_0^\alpha, \quad \text{on } \{Q_0 \geq x_0\}.$$

Let $x \geq x_0$. Then,

$$\begin{aligned}
\mathbb{E}\{Z_0|Q_0 > x\} &= \frac{1}{\mathbb{P}\{Q_0 > x\}} \int_{\{Q_0 > x\}} Z_0 d\mathbb{P} \\
&= \frac{1}{\mathbb{P}\{Q_0 > x\}} \int_{\{Q_0 > x\}} \mathbb{E}\{Z_0|Q_0\} d\mathbb{P} \\
&\leq \frac{1}{\mathbb{P}\{Q_0 > x\}} \int_{\{Q_0 > x\}} K Q_0^\alpha d\mathbb{P} \\
&= \frac{K}{\mathbb{P}\{Q_0 > x\}} \int_x^\infty z^\alpha \, dF_Q(z), \qquad (64)
\end{aligned}$$

where F_Q is the distribution function of Q_0. From (22) in Lemma 4, there exist $x_1 \geq x_0$ and K' such that

$$
\begin{aligned}
\mathbb{E}\{Z_0|Q_0 > x\} &\leq \frac{K}{\mathbb{P}\{Q_0 > x\}} \int_x^\infty z^\alpha \, dF_Q(z) \\
&\leq \frac{K}{\mathbb{P}\{Q_0 > x\}} K' x^\alpha \mathbb{P}\{Q_0 > x\} \\
&= KK'x^\alpha, \qquad \forall x \geq x_1, \qquad (65)
\end{aligned}
$$

which implies that $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$ since $\mathbb{E}\{Z_0|Q_0 > x\}$ is nonnegative.

5.6. *Proof of Lemma 6.*

It can be shown by the same steps in the proof of Lemma 3 that $\mathbb{E}\{Z_0|\boldsymbol{\Delta}, Q_0\} \stackrel{a.s.}{=} O(Q_0^\alpha)$. On $\{Q_0 \geq x_0\}$,

$$
\begin{aligned}
\mathbb{E}\{Z_0|\boldsymbol{\Delta}, Q_0\} &\stackrel{a.s.}{=} \sum_{n=0}^{\infty} \mathbb{P}\{Z_0 > n|\boldsymbol{\Delta}, Q_0\} \stackrel{a.s.}{\leq} \sum_{n=0}^{\infty} \mathbb{P}\{X_n > 0|\boldsymbol{\Delta}, Q_0\} \\
&= \sum_{n=0}^{\lceil MQ_0^\alpha \rceil} \mathbb{P}\{X_n > 0|\boldsymbol{\Delta}, Q_0\} + \sum_{n=\lceil MQ_0^\alpha \rceil + 1}^{\infty} \mathbb{P}\{X_n > 0|\boldsymbol{\Delta}, Q_0\} \\
&\stackrel{a.s.}{\leq} \sum_{n=0}^{\lceil MQ_0^\alpha \rceil} 1 + \sum_{n=\lceil MQ_0^\alpha \rceil + 1}^{\infty} Kn^{-1-\delta}, \\
&\stackrel{Q_0 \to \infty}{\sim} MQ_0^\alpha, \qquad (66)
\end{aligned}
$$

where $\lceil x \rceil$ denotes the smallest integer which is greater than or equal to x. Since $\mathbb{E}\{Z_0|\boldsymbol{\Delta}, Q_0\}$ is nonnegative a.s., $\mathbb{E}\{Z_0|\boldsymbol{\Delta}, Q_0\} \stackrel{a.s.}{=} O(Q_0^\alpha)$, which means that there exist x_0 and K such that

$$
\mathbb{E}\{Z_0|\boldsymbol{\Delta}, Q_0\} \stackrel{a.s.}{\leq} KQ_0^\alpha, \text{ on } \{Q_0 \geq x_0\}.
$$

We now show that $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$. Let $x \geq x_0$. Then,

$$
\begin{aligned}
\mathbb{E}\{Z_0|Q_0 > x\} &= \frac{1}{\mathbb{P}\{Q_0 > x\}} \int_{\{Q_0 > x\}} Z_0 d\mathbb{P} \\
&= \frac{1}{\mathbb{P}\{Q_0 > x\}} \int_{\{Q_0 > x\}} \mathbb{E}\{Z_0|\boldsymbol{\Delta}, Q_0\} d\mathbb{P} \\
&\leq \frac{1}{\mathbb{P}\{Q_0 > x\}} \int_{\{Q_0 > x\}} KQ_0^\alpha d\mathbb{P} \\
&= \frac{K}{\mathbb{P}\{Q_0 > x\}} \int_x^\infty z^\alpha \, dF_Q(z), \qquad (67)
\end{aligned}
$$

where F_Q is the distribution function of Q_0. From (22) in Lemma 4, there exist $x_1 \geq x_0$ and K' such that

$$\begin{aligned}
\mathbb{E}\{Z_0|Q_0 > x\} &\leq \frac{K}{\mathbb{P}\{Q_0 > x\}} \int_x^\infty z^\alpha \, dF_Q(z) \\
&\leq \frac{K}{\mathbb{P}\{Q_0 > x\}} K' x^\alpha \mathbb{P}\{Q_0 > x\} \\
&= KK'x^\alpha, \qquad \forall x \geq x_1,
\end{aligned} \tag{68}$$

which implies that $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$ since $\mathbb{E}\{Z_0|Q_0 > x\}$ is nonnegative.

5.7. *Proof of Proposition 2.*

Let $m(n)$ and $v(n)$ be functions such that $m(n) \overset{a.s.}{=} \mathbb{E}\{X_n|\mathbf{\Delta}, Q_0\}$ and $v(n) \overset{a.s.}{=} \mathrm{Var}\{X_n|\mathbf{\Delta}, Q_0\}$. Then, since X_n given $\mathbf{\Delta}$ and Q_0 is Gaussian, $\mathbb{E}\{e^{\theta X_n}|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} e^{\theta m(n) + \frac{1}{2}\theta^2 v(n)}$. From now on, we are on $\{Q_0 \geq x_0\}$. From the premise of the proposition, we have $m(n) \overset{a.s.}{\leq} K_1 n^{1-\epsilon} Q_0 - \kappa n$ on $\{Q_0 \geq x_0\}$, where K_1 is chosen to be positive. Replace n by $\lceil Mt^\alpha \rceil$ with $t \geq Q_0$. We may and do choose t such that $\lceil Mt^\alpha \rceil = Mt^\alpha$. Then, on $\{Q_0 \geq x_0\}$,

$$\begin{aligned}
m(Mt^\alpha) &\leq K_1 (Mt^\alpha)^{1-\epsilon} Q_0 - \kappa M t^\alpha \\
&\leq K_1 M^{1-\epsilon} t^{\alpha(1-\epsilon)+1} - \kappa M t^\alpha.
\end{aligned}$$

Set $\alpha = \frac{1}{\epsilon}$. Then, on $\{Q_0 \geq x_0\}$,

$$m(Mt^\alpha) \leq (K_1 M^{1-\epsilon} - \kappa M) t^\alpha.$$

Pick $M = \left(\frac{2K_1}{\kappa}\right)^{\frac{1}{\epsilon}}$. Then,

$$m(Mt^\alpha) \leq -K_1 M^{1-\epsilon} t^\alpha. \tag{69}$$

Set $x_2 = \max\{x_0, \left(\frac{x_0}{M}\right)^\epsilon\}$. On $\{Q_0 \geq x_2\}$, let $t \geq Q_0$. Then $n = Mt^\alpha \geq x_0$. On $\{Q_0 \geq x_2\}$, $v(n) \leq K_2 n^{2-\delta}$ from the assumption, and we have

$$\begin{aligned}
\mathbb{P}\{X_{Mt^\alpha} > 0|\mathbf{\Delta}, Q_0\} &\overset{a.s.}{\leq} \mathbb{E}\{e^{\theta X_{Mt^\alpha}}|\mathbf{\Delta}, Q_0\} \\
&\overset{a.s.}{=} \exp\{\theta m(Mt^\alpha) + \frac{1}{2}\theta^2 v(Mt^\alpha)\} \\
&\leq \exp\{-\theta K_1 M^{1-\epsilon} t^\alpha + \frac{1}{2}\theta^2 K_2 (Mt^\alpha)^{2-\delta}\}
\end{aligned} \tag{70}$$

Let $\theta = t^{-\beta}$. Then, on $\{Q_0 \geq x_2\}$,

$$\begin{aligned} \mathbb{P}\{X_{Mt^\alpha} > 0|\mathbf{\Delta}, Q_0\} &\overset{a.s.}{\leq} \exp\{-t^{-\beta}K_1 M^{1-\epsilon}t^\alpha + \frac{1}{2}t^{-2\beta}K_2 M^{2-\delta}t^{\alpha(2-\delta)}\} \\ &= \exp\{-K_1 M^{1-\epsilon}t^{\alpha-\beta} + \frac{1}{2}K_2 M^{2-\delta}t^{\alpha(2-\delta)-2\beta}\}. \end{aligned} \tag{71}$$

Choose $\beta \in (\alpha(1-\delta), \alpha)$ so that $\alpha - \beta > 0$ and $\alpha - \beta > \alpha(2-\delta) - 2\beta$. Then, the coefficient of the leading term is negative and its order is greater than 0. Therefore, we can choose $x_1 > x_2$ and K such that $\mathbb{P}\{X_{Mt^\alpha} > 0|\mathbf{\Delta}, Q_0\} \leq \frac{K}{M}t^{-\alpha-1} = Kn^{-1-\delta'}$ for all $t \geq Q_0$ on $\{Q_0 \geq x_1\}$, where $\delta' = \frac{1}{\alpha}$. At last, we have found x_1, K, $M > 0, \alpha > 0$, and $\delta' > 0$.

5.8. *Proof of Proposition 3.*

Let $Y_n := \sum_{k=1}^n \lambda_k - cn$, $h(n) := \sum_{l=-n}^n C_\lambda(l)$, $m(n) := \mathbb{E}\{Y_n\} = -\kappa n$, and

$$v(n) := \mathrm{Var}\{Y_n\} = \sum_{k=1}^n \sum_{l=1}^n C_\lambda(l-k) = nC_\lambda(0) + 2\sum_{l=1}^{n-1}(n-l)C_\lambda(l).$$

Note that $v(n+1) - v(n) = h(n)$.

Since both $v(n)$ and n^β approach ∞, $\lim_{n\to\infty} \frac{v(n)}{n^\beta}$ will equal $\lim_{n\to\infty} \frac{v(n+1)-v(n)}{(n+1)^\beta - n^\beta}$, if it exists (discrete version of L'Hospital's rule). Now,

$$\begin{aligned} \lim_{n\to\infty} \frac{v(n+1)-v(n)}{(n+1)^\beta - n^\beta} &= \frac{v(n+1)-v(n)}{\beta n^{\beta-1}} \frac{\beta n^{\beta-1}}{(n+1)^\beta - n^\beta} \\ &= \frac{h(n)}{\beta n^{\beta-1}} \frac{\beta n^{\beta-1}}{(n+1)^\beta - n^\beta} \\ &\overset{n\to\infty}{\to} S \cdot 1 \qquad \text{(from (35) and the definition of } h(n)). \end{aligned} \tag{72}$$

Hence $\lim_{n\to\infty} \frac{v(n)}{n^\beta} = S$. Since $\frac{h(n)}{v(n)} \sim \frac{\beta}{n}$, $f(n) = o(\frac{h(n)}{v(n)})$ implies $f(n) = o(n^{-1})$.

$$\begin{aligned} n\left[\log v(n+1) - \log v(n)\right] &= n\log\left(\frac{v(n+1)}{v(n)}\right) = n\log\left(\frac{v(n)+h(n)}{v(n)}\right) = n\log\left(1 + \frac{h(n)}{v(n)}\right) \\ &= n\left[\frac{h(n)}{v(n)} + o(\frac{1}{n})\right] \qquad \text{(by Taylor's Expansion)} \\ &= \frac{h(n)}{n^{\beta-1}}\frac{n^\beta}{v(n)} + no(\frac{1}{n}) \\ &\overset{n\to\infty}{\to} \beta S \cdot \frac{1}{S} + 0 = \beta. \end{aligned} \tag{73}$$

Since Y_n is Gaussian, $\mathbb{P}\{Q_n > x\}$ can be expressed in terms of $v(n), m(n)$ and the standard Gaussian tail function $\Psi(z) := \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-\frac{r^2}{2}}dr$ as

$$\mathbb{P}\{Y_n > x\} = \Psi\left(\frac{x - m(n)}{\sqrt{v(n)}}\right) = \Psi\left(\frac{x + \kappa n}{\sqrt{v(n)}}\right).$$

For each x, since $\lim_{n\to\infty} \frac{v(n)}{(x+\kappa n)^2} = 0$, there exists a finite value n_x at which $\frac{v(n)}{(x+\kappa n)^2}$ attains its maximum, i.e., $\mathbb{P}\{Y_n > x\}$ is maximized. Define $m_x := \frac{(x+\kappa n_x)^2}{v(n_x)}$ (the reciprocal of the maximum value of $\frac{v(n)}{(x+\kappa n)^2}$). Since $\mathbb{P}\{Q > x\} = \mathbb{P}\{\sup_{n\geq 1} Y_n > x\}$ [18],

$$\mathbb{P}\{Q > x\} = \mathbb{P}\{\sup_{n\geq 1} Y_n > x\} \geq \sup_{n\geq 1} \mathbb{P}\{Y_n > x\} = \mathbb{P}\{Y_{n_x} > x\} = \Psi(\sqrt{m_x}).$$

Under (72) and (73), it has been shown that $m_x \sim K x^{2-\beta}$ with $K = \frac{4\kappa^\beta}{S\beta^\beta(2-\beta)^{2-\beta}}$ (Proposition 3 in [14]). Since $\Psi(z) \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2} - \log z}$, $-\log \Psi(z) \sim \frac{z^2}{2} + \log z$. Hence, $-\log \Psi(\sqrt{m_x}) \sim \frac{m_x}{2} + \frac{1}{2}\log m_x$, and the proposition follows.

5.9. *Proof of Proposition 4.*

In the proof of Proposition 2, let $t \geq v$ instead of $t \geq Q_0$.

5.10. *Proof of Theorem 3.*

To prove this theorem, we need the following lemmas which are the modified versions of Lemmas 6 and 5 (the proofs of these lemmas follow the proof of the theorem).

Lemma 7 [from Lemma 6] If there exist x_0, K, $M > 0, \alpha > 0$ and $\delta > 0$ such that $\mathbb{P}\{X_n > 0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} K n^{-(1+\delta)}$ for all $n \geq M v^\alpha$ on $\{Q_0 \geq x_0\}$, then $\mathbb{E}\{Z_0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} O(v^\alpha)$.

Lemma 8 [from Lemma 5] Assume conditions (C1)-(C3). Further assume that $\mathbb{P}\{|\Delta_i| > x\} = O(\mathbb{P}\{Q_0 > x\})$. Let α be a positive constant such that $\alpha C_0 < 1$. If $\mathbb{E}\{Z_0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} O(v^\alpha)$, then $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$.

By following exactly same steps in the proof of Theorem 2, except by now appealing to Lemmas 7 and 8 instead of Lemmas 6 and 5, the result follows.

5.11. *Proof of Lemma 7.*

As in (59) in the proof of Lemma 6, $\mathbb{E}\{Z_0|\mathbf{\Delta}, Q_0\} \overset{v\to\infty}{\sim} M v^\alpha$. Hence, $\mathbb{E}\{Z_0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{=} O(v^\alpha)$.

5.12. *Proof of Lemma 8.*

From the assumption, there exist x_0 and K such that

$$\mathbb{E}\{Z_0|\mathbf{\Delta}, Q_0\} \overset{a.s.}{\leq} Kv^\alpha, \text{ on } \{Q_0 \geq x_0\}.$$

Let $x \geq x_0$. Then,

$$\begin{aligned}
\mathbb{E}\{Z_0|Q_0 > x\}\mathbb{P}\{Q_0 > x\} &= \int_{\{Q_0>x\}} Z_0 d\mathbb{P} \\
&= \int_{\{Q_0>x\}} \mathbb{E}\{Z_0|\mathbf{\Delta}, Q_0\} d\mathbb{P} \\
&\leq \int_{\{Q_0>x\}} Kv^\alpha d\mathbb{P} \\
&= \int_{\{Q_0>x\}\cap\{Q_0=v\}} KQ_0^\alpha d\mathbb{P} \\
&\quad + \int_{\{Q_0>x\}\cap\{|\Delta_1|=v\}} K|\Delta_1|^\alpha d\mathbb{P} \cdots + \int_{\{Q_0>x\}\cap\{|\Delta_m|=v\}} K|\Delta_m|^\alpha d\mathbb{P} \\
&\leq \int_{\{Q_0>x\}} KQ_0^\alpha d\mathbb{P} \\
&\quad + \int_{\{|\Delta_1|>x\}} K|\Delta_1|^\alpha d\mathbb{P} \cdots + \int_{\{|\Delta_m|>x\}} K|\Delta_m|^\alpha d\mathbb{P} \\
&= \int_x^\infty Kz^\alpha dF_Q(z) \\
&\quad + \int_{\{|y|>x\}} K|y|^\alpha dF_{\Delta_1}(y) \cdots + \int_{\{|y|>x\}} K|y|^\alpha dF_{\Delta_m}(y) \\
&= \int_x^\infty Kz^\alpha dF_Q(z) \\
&\quad + \int_x^\infty K\alpha t^{\alpha-1}\mathbb{P}\{|\Delta_1| > t\}dt + x^\alpha\mathbb{P}\{|\Delta_1| > x\} \\
&\quad \cdots + \int_x^\infty K\alpha t^{\alpha-1}\mathbb{P}\{|\Delta_m| > t\}dt + x^\alpha\mathbb{P}\{|\Delta_m| > x\},
\end{aligned}$$

where F_{Δ_i} is the distribution function of Δ_i. Since $\mathbb{P}\{|\Delta_i| > x\} = O(\mathbb{P}\{Q_0 > x\})$ from the premise of the lemma, $\int_x^\infty t^\gamma \mathbb{P}\{|\Delta_i| > t\}dt = O(\int_x^\infty t^\gamma \mathbb{P}\{Q_0 > t\}dt)$. So there are $x_1 > x_0$ and K' such that

$$\begin{aligned}
\sum_{i=1}^m \left[\int_x^\infty \alpha t^{\alpha-1}\mathbb{P}\{|\Delta_i| > t\}dt + x^\alpha\mathbb{P}\{|\Delta_i| > x\}\right] &\leq K'\left[\int_x^\infty \alpha t^{\alpha-1}\mathbb{P}\{Q_0 > t\}dt + x^\alpha\mathbb{P}\{Q_0 > x\}\right] \\
&= K'\int_x^\infty t^\alpha\, dF_Q(t), \qquad \forall x > x_1.
\end{aligned}$$

By Lemma 4, there exist $x_2 \geq x_1$ and K'' such that

$$\int_x^\infty t^\alpha\, dF_Q(t) \leq K'' x^\alpha \mathbb{P}\{Q_0 > x\}, \qquad \forall x \geq x_2.$$

Thus,

$$\mathbb{E}\{Z_0|Q_0 > x\}\mathbb{P}\{Q_0 > x\} \leq K(1+K')K''x^\alpha\mathbb{P}\{Q_0 > x\}, \qquad \forall x \geq x_2,$$

which means $\mathbb{E}\{Z_0|Q_0 > x\} = O(x^\alpha)$ since $\mathbb{E}\{Z_0|Q_0 > x\}$ is nonnegative.

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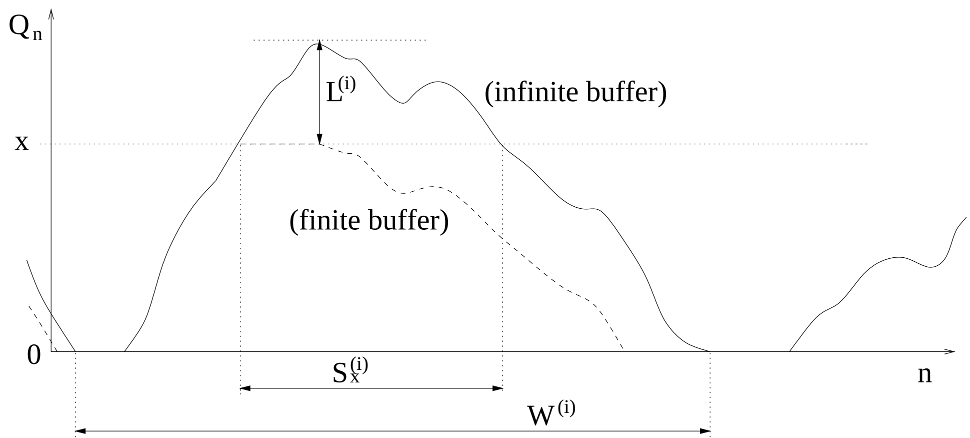
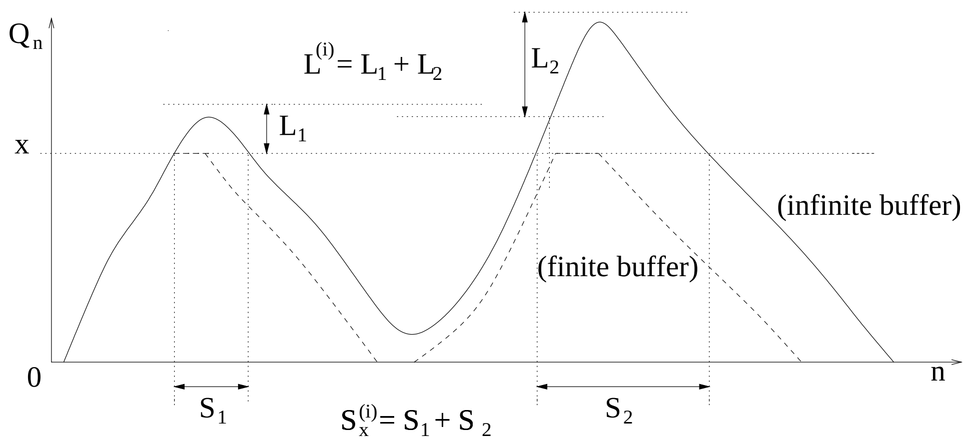


(a) $L^{(i)} = \int_x^\infty I(S_y^{(i)} > 0)dy$



(b) $L^{(i)} > \int_x^\infty I(S_y^{(i)} > 0)dy$

Figure 2. Illustration of $\int_x^\infty I(S_y^{(i)} > 0)dy \le L^{(i)}$