Advances in Applied Probability (March 1999)

ON THE SUPREMUM DISTRIBUTION OF INTEGRATED STATIONARY GAUSSIAN PROCESSES WITH NEGATIVE LINEAR DRIFT

JINWOO CHOE,* *Purdue University*

NESS B. SHROFF,** *Purdue University*

Abstract

In this paper we study the supremum distribution of a class of Gaussian processes having stationary increments and negative drift using key results from *Extreme Value Theory.* We focus on deriving an asymptotic upper bound to the tail of the supremum distribution of such processes. Our bound is valid for both discrete- and continuous-time processes. We discuss the importance of the bound, its applicability to queueing problems, and show numerical examples to illustrate its performance.

KEYWORDS: SUPREMUM DISTRIBUTION; GAUSSIAN PROCESS; STATIONARY INCREMENT WITH LINEAR DRIFT; QUEUE LENGTH DISTRIBUTION; EXTREME VALUE THEORY; ASYMPTOTIC UPPER BOUND; TAIL PROBABILITY

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60G15
SECONDARY 60G70;60K25

1. Introduction

Consider a continuous-time stochastic process $\{X_t : t \geq 0\}$ or a discrete-time stochastic process $\{X_n : n = 1, 2, \ldots\}$ described by the following equations.

(1) Continuous-time process : $$X_t = \int_0^t \xi_s ds - \kappa t \quad (t \in [0, \infty)),$$

* Postal address: School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907-1285, U.S.A.

** Postal address: School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907-1285, U.S.A. Tel: +1 765 494-3471, Fax: +1 765 494-3358, E-mail: shroff@ecn.purdue.edu. Please address correspondence to this author.

(2) Discrete-time process : $$X_n = \sum_{m=1}^{n} \xi_m - \kappa n \quad (n \in \{0, 1, 2, \ldots\}).$$

Here ξ is a centered (zero-mean) stationary Gaussian process and κ is a positive constant that determines the drift of X. Since ξ is a centered stationary Gaussian process, the stochastic process X is a Gaussian process with stationary increments and negative linear drift. In this paper we are interested in studying the supremum distribution of this process X. Specifically, we will derive an asymptotic upper bound† to the tail of the supremum distribution of X under the following conditions on C_ξ, the autocovariance function of the centered stationary Gaussian process ξ.

(C1) Continuous-time: $C_\xi(\tau) := \mathbb{E}\{\xi_t \xi_{t+\tau}\}$ is absolutely integrable and $\int_{-\infty}^{\infty} C_\xi(\tau) d\tau > 0$.
Discrete-time: $C_\xi(l) := \mathbb{E}\{\xi_n \xi_{n+l}\}$ is absolutely summable and $\sum_{l=-\infty}^{\infty} C_\xi(l) > 0$.

(C2) Continuous-time: $\tau C_\xi(\tau)$ is absolutely integrable.
Discrete-time: $l C_\xi(l)$ is absolutely summable.

(C3) Continuous-time: $\int_0^\infty \tau C_\xi(\tau) > 0$ and $\int_0^t \tau C_\xi(\tau) d\tau + \int_t^\infty t C_\xi(\tau) d\tau > 0$ for all $t \in (0, \infty)$.
Discrete-time: $\sum_{l=1}^{\infty} l C_\xi(l) > 0$ and $\sum_{l=1}^{m} l C_\xi(l) + \sum_{l=m+1}^{\infty} m C_\xi(l) > 0$ for all $m = 1, 2, \ldots$.

For notational simplicity, we define $\langle w \rangle_\Theta := \sup_{\theta \in \Theta} w_\theta$ (we will not specify the index range Θ when it includes the entire domain of w_θ). The study of the tail distribution $\mathbb{P}(\{\langle X \rangle > x\})$ is motivated by its applicability to queueing systems and high-speed telecommunication networks [6, 7, 8]. In particular, when κ and ξ are appropriately defined, one can show that the steady state queue length distribution of a queueing system is equal to the supremum distribution of X [12, 14]. Therefore, similar problems have been studied in the queueing context. For example, using *Large Deviation* techniques it has been shown for very general classes of stationary processes ξ that the limit $\eta := \lim_{x\to\infty} -\frac{1}{x} \log \mathbb{P}(\{\langle X \rangle > x\})$ exists and is finite [12], that is,

(3) $$\log \mathbb{P}(\{\langle X \rangle > x\}) \sim -\eta x,$$

where $f \sim g$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Also, in the discrete-time case the following stronger result has been shown for stationary ergodic Gaussian processes ξ_n [1]:

(4) $$\mathbb{P}(\{\langle X \rangle > x\}) \sim C e^{-\eta x} \quad \text{as} \quad x \to \infty,$$

†In this paper, we say a positive-valued function f asymptotically bounds a positive-valued function g from above, if $\limsup_{x\to\infty} g(x)/f(x) \le 1$ (or from below, if $\liminf_{x\to\infty} g(x)/f(x) \ge 1$).

that is, the tail of the supremum distribution of X is asymptotically exponential. However, the *asymptotic constant* C is in general difficult to obtain and approximations have been suggested to evaluate it. An important result of this paper is the derivation of an asymptotic upper bound, of an exponential form as in (4), for a fairly large class of Gaussian processes ξ given by (C1)–(C3). This result can be stated in the form of the following theorem.

Theorem 1 Under conditions (C1)–(C3)*,*

$$\limsup_{x\to\infty} e^{\frac{2\kappa x}{S}} \mathbb{P}(\{\langle X\rangle > x\}) \leq e^{-\frac{2\kappa^2 D}{S^2}}.$$

In other words, $\mathbb{P}(\{\langle X\rangle > x\})$ *is asymptotically bounded from above by* $e^{-\frac{2\kappa}{S}(x+\frac{\kappa D}{S})}$.

Here, S and D are positive constants that will be defined later in Section 3. Note that from large deviation studies, η in (3) has already been shown to be $2\kappa/S$ under condition (C1) [12]. Hence, this theorem considerably strengthens the large deviation result in (3) under conditions (C1)–(C3). Further, this theorem also provides the upper bound $e^{-2\kappa^2 D/S^2}$ to the asymptotic constant C which is a useful parameter for network dimensioning.

In the continuous-time case, (4) has been shown in a more limited setting (e.g., when ξ is an *Ornstein-Uhlenbeck* process [19], or when X is a *Brownian Motion* process with negative drift [16]). In this paper, for the continuous-time case, our asymptotic upper bound will also be used to show that there exists a constant η such that $c_1 e^{-\eta x} \leq \mathbb{P}(\{\langle X\rangle > x\}) \leq c_2 e^{-\eta x}$ for some constants c_1, c_2, and all large enough x.

The paper is organized as follows. In Section 2, we first introduce fundamental results from the Extreme Value Theory for Gaussian processes; in Section 3, we derive an asymptotic upper bound to $\mathbb{P}(\{\langle X\rangle > x\})$. To avoid redundancy, we derive the bound only for the continuous-time case and refer to [8] for the derivations in discrete-time; in Section 4, we discuss the importance of the bound in analyzing the behavior of a queueing system; finally, in Section 5 we briefly illustrate the performance of the bound through numerical examples.

2. Results from Extreme Value Theory

Our study of the supremum distribution of X is based on the Extreme Value Theory literature. The following two theorems (from [2]) play key roles in our study.

Theorem 2 (Borell's Inequality) *Let $\{\zeta_t : t \in T\}$ be a centered Gaussian process with sample path bounded a.s.; that is $\langle \zeta \rangle < \infty$ a.s. Then $\mathbb{E}\{\langle \zeta \rangle\}$ is finite and for all $x > \mathbb{E}\{\langle \zeta \rangle\}$,*

$$\mathbb{P}(\{\langle \zeta \rangle > x\}) \leq 2e^{-\frac{(x-\mathbb{E}\{\langle \zeta \rangle\})^2}{2\langle \sigma^2 \rangle}},$$

where $\langle \sigma^2 \rangle := \sup_{t \in T} \mathbb{E}\{\zeta_t^2\}$.

Theorem 3 (Slepian's Inequality) *Let $\{\zeta_t : t \in T\}$ and $\{\upsilon_t : t \in T\}$ be two centered Gaussian processes on an index set T with sample path bounded a.s. If $\mathbb{E}\{\zeta_t^2\} = \mathbb{E}\{\upsilon_t^2\}$ and $\mathbb{E}\{(\zeta_s - \zeta_t)^2\} \leq \mathbb{E}\{(\upsilon_s - \upsilon_t)^2\}$ for all $s, t \in T$, then for all x*

$$\mathbb{P}(\{\langle \zeta \rangle > x\}) \leq \mathbb{P}(\{\langle \upsilon \rangle > x\}).$$

In addition to Theorems 2 and 3, we introduce another important result from [2, Corollary 4.15], which provides us a way to bound $\mathbb{E}\{\langle \zeta \rangle\}$ and will be used together with Theorem 2 to derive a bound for the tail probability $\mathbb{P}(\{\langle \zeta \rangle > x\})$.

Theorem 4 *Let $\{\zeta_t : t \in T\}$ be a centered Gaussian process and define a pseudo-metric[‡] d on T as $d(t_1, t_2) := \sqrt{\mathbb{E}\{(\zeta_{t_1} - \zeta_{t_2})^2\}}$. Also, let $N(\epsilon)$ be the minimum number of closed d-balls of radius ϵ needed to cover T, then there exists a universal constant K such that*

$$\mathbb{E}\{\langle \zeta \rangle\} \leq K \int_0^\infty \sqrt{\log N(\epsilon)}\, d\epsilon.$$

3. Asymptotic Upper Bound for $\mathbb{P}(\{\langle X \rangle > x\})$

In this section, we derive an asymptotic upper bound to the tail probability $\mathbb{P}(\{\langle X \rangle > x\})$ for the stationary Gaussian processes ξ that satisfy (C1)–(C3). This section consists of two parts. We first obtain several preliminary results in Section 3.1, and then from these results we derive our main results in Section 3.2. Since the proofs for the discrete-time case are essentially similar to those for the continuous-time case, we provide derivations only for the continuous-time case. The detailed proofs for the discrete-time case can be found in [8].

3.1. *Preliminaries* We assume that ξ_t is a centered stationary Gaussian process with a *continuous* autocovariance function $C_\xi(\tau)$. Also, we assume ξ_t to be a separable and measurable Gaussian

[‡]Note that d is not a metric, since $d(t_1, t_2) = 0$ does not necessarily imply $t_1 = t_2$.

process in order for X_t to be a well defined stochastic process[§].

From (1), the mean and the autocovariance function of X_t can be obtained as

$$\begin{aligned} \mathbb{E}\{X_t\} &= -\kappa t, \quad \text{and} \\ C_X(t_1,t_2) &:= \mathbb{E}\{(X_{t_1}+\kappa t_1)(X_{t_2}+\kappa t_2)\} = \int_0^{t_2}\int_0^{t_1} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2. \end{aligned} \tag{5}$$

We now define a few parameters which will be used extensively throughout the paper.

$$S := \int_{-\infty}^{\infty} C_\xi(\tau)d\tau, \quad D := 2\int_0^\infty \tau C_\xi(\tau)d\tau, \quad \text{and} \quad \tilde{D} := 2\int_0^\infty \tau |C_\xi(\tau)|d\tau. \tag{6}$$

In the following proposition, we show several important properties of the variance and the autocovariance function of X_t, which will later be used in deriving our bounds.

Proposition 1

(a) $\frac{\mathrm{Var}\{X_t\}}{t}$ *is a continuous and differentiable function for* $t>0$. *Further,*

$$\begin{aligned} \frac{d}{dt}\left(\frac{\mathrm{Var}\{X_t\}}{t}\right) &= \frac{2}{t^2}\int_0^t \tau C_\xi(\tau)d\tau \quad \textit{for } t>0, \textit{ and} \\ \lim_{t\downarrow 0}\frac{\mathrm{Var}\{X_t\}}{t} &= 0. \end{aligned}$$

(b) $C_X(t_1,t_2) = \frac{1}{2}\left(\mathrm{Var}\{X_{t_1}\} + \mathrm{Var}\{X_{t_2}\} - \mathrm{Var}\{X_{|t_1-t_2|}\}\right)$.

(c) *Let* $\alpha \geq 1$, *then under condition* (C1),

$$\lim_{t\to\infty}\frac{C_X(\alpha t,t)}{t} = \lim_{t\to\infty}\frac{C_X(t,\alpha t)}{t} = S.$$

In particular, $\lim_{t\to\infty}\frac{\mathrm{Var}\{X_t\}}{t} = S$.

(d) *Under conditions* (C1) *and* (C2),

$$\begin{aligned} \left|\frac{\mathrm{Var}\{X_{t_1}\}}{t_1} - \frac{\mathrm{Var}\{X_{t_2}\}}{t_2}\right| &\leq \frac{\tilde{D}|t_1-t_2|}{t_1 t_2} \quad \textit{for all} \quad t_1,t_2>0, \quad \textit{and} \\ \lim_{t\to\infty} t\left(S - \frac{\mathrm{Var}\{X_t\}}{t}\right) &= D. \end{aligned}$$

(e) *Under conditions* (C1)–(C3), $\frac{\mathrm{Var}\{X_t\}}{t} < S$ *and there exists a* $t_o>0$ *such that*

$$\frac{\mathrm{Var}\{X_t\}}{t} = \sup_{0<s\leq t}\frac{\mathrm{Var}\{X_s\}}{s} \quad \textit{for all } t\geq t_o.$$

[§]Note that, from the continuity of the autocovariance function, the process ξ_t can always be replaced with its separable and measurable version [11].

Proof of Proposition 1. (a) From (5), we have

$$\begin{aligned}\frac{\mathrm{Var}\{X_t\}}{t} &= \frac{1}{t}\int_0^t\int_0^t C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 \\ &= 2\int_0^t\left(1-\frac{\tau}{t}\right)C_\xi(\tau)d\tau \qquad \text{(by setting } \tau=\tau_2-\tau_1\text{)}.\end{aligned} \tag{7}$$

Differentiating both sides of (7), we get

$$\frac{d}{dt}\left(\frac{\mathrm{Var}\{X_t\}}{t}\right) = \frac{2}{t^2}\int_0^t \tau C_\xi(\tau)d\tau. \tag{8}$$

Also, note that $|(1-\frac{\tau}{t})C_\xi(\tau)| \le |C_\xi(\tau)| \le C_\xi(0)$ for $\tau \in [0,t]$. Therefore,

$$\lim_{t\downarrow 0}\left|\frac{\mathrm{Var}\{X_t\}}{t}\right| \le \lim_{t\downarrow 0}\int_0^t C_\xi(0)d\tau = 0.$$

(b) Without loss of generality (*W.L.O.G.*), assume $t_2 > t_1$. Then

$$\begin{aligned}2C_X(t_1,t_2) &= \int_0^{t_2}\int_0^{t_1} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 + \int_0^{t_1}\int_0^{t_2} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 \\ &= \int_0^{t_2}\int_0^{t_2} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 + \int_0^{t_1}\int_0^{t_1} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 \\ &\quad - \int_0^{t_2}\int_{t_1}^{t_2} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 + \int_0^{t_1}\int_{t_1}^{t_2} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 \\ &= \int_0^{t_2}\int_0^{t_2} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 + \int_0^{t_1}\int_0^{t_1} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 \\ &\quad - \int_0^{t_2-t_1}\int_0^{t_2-t_1} C_\xi(\tau_2-\tau_1)d\tau_1 d\tau_2 \\ &= \mathrm{Var}\{X_{t_2}\} + \mathrm{Var}\{X_{t_1}\} - \mathrm{Var}\{X_{t_2-t_1}\}.\end{aligned}$$

(c) From the symmetry of the autocovariance function, it suffices to show that $\lim_{t\to\infty}\frac{C_X(\alpha t,t)}{t} = S$. Let $h_t(\tau)$ be defined as

$$h_t(\tau) = \begin{cases} \left(1+\frac{\tau}{t}\right)C_\xi(\tau) & \text{if } \tau\in[-t,0), \\ C_\xi(\tau) & \text{if } \tau\in[0,(\alpha-1)t], \\ \left(1-\frac{\tau-(\alpha-1)t}{t}\right)C_\xi(\tau) & \text{if } \tau\in((\alpha-1)t,\alpha t], \\ 0 & \text{otherwise.}\end{cases}$$

Then, again, by changing the variables of integration ($\tau=\tau_2-\tau_1$), we obtain

$$\frac{C_X(\alpha t,t)}{t} = \frac{1}{t}\int_0^t\int_0^{\alpha t} C_\xi(\tau_1-\tau_2)d\tau_1 d\tau_2 = \int_{-\infty}^{\infty} h_t(\tau)d\tau.$$

However, from the definition of h_t, we know that $\lim_{t\to\infty} h_t(\tau) = C_\xi(\tau)$ and $|h_t(\tau)| \le |C_\xi(\tau)|$. Therefore, from condition (C1) and the Dominated Convergence Theorem, it follows that

$$\lim_{t\to\infty}\frac{C_X(\alpha t,t)}{t} = \int_{-\infty}^{\infty} C_\xi(\tau)d\tau = S.$$

(d) *W.L.O.G.* assume $t_2 > t_1 > 0$. From (7), we have

$$\begin{aligned}\frac{\mathrm{Var}\{X_{t_2}\}}{t_2} - \frac{\mathrm{Var}\{X_{t_1}\}}{t_1} &= 2\left(\int_0^{t_2}\left(1-\frac{\tau}{t_2}\right)C_\xi(\tau)d\tau - \int_0^{t_1}\left(1-\frac{\tau}{t_1}\right)C_\xi(\tau)d\tau\right)\\ &= \frac{2(t_2-t_1)}{t_1t_2}\left(\int_0^{t_1}\tau C_\xi(\tau)d\tau + \int_{t_1}^{t_2}\frac{t_1(t_2-\tau)}{t_2-t_1}C_\xi(\tau)d\tau\right).\end{aligned}$$

Since $0 \le \frac{t_1(t_2-\tau)}{t_2-t_1} \le \tau$ for $\tau \in [t_1, t_2]$, it follows that

$$\begin{aligned}\left|\frac{\mathrm{Var}\{X_{t_2}\}}{t_2} - \frac{\mathrm{Var}\{X_{t_1}\}}{t_1}\right| &\le \frac{2(t_2-t_1)}{t_1t_2}\left(\int_0^{t_1}\tau|C_\xi(\tau)|d\tau + \int_{t_1}^{t_2}\frac{t_1(t_2-\tau)}{t_2-t_1}|C_\xi(\tau)|d\tau\right)\\ &\le \frac{2(t_2-t_1)}{t_1t_2}\int_0^{t_2}\tau|C_\xi(\tau)|d\tau \le \frac{(t_2-t_1)\tilde{D}}{t_1t_2}.\end{aligned}$$

Now, let $h_t(\tau)$ be defined for $\tau \ge 0$ by

$$h_t(\tau) = \begin{cases}\tau C_\xi(\tau) & \text{if } \tau \in [0,t),\\ tC_\xi(\tau) & \text{if } \tau \in [t,\infty).\end{cases}$$

Then, from (7) and from the definition of S and $h_t(\tau)$, we get

$$\begin{aligned}t\left(S - \frac{\mathrm{Var}\{X_t\}}{t}\right) &= 2t\left(\int_0^\infty C_\xi(\tau)d\tau - \int_0^t\left(1-\frac{\tau}{t}\right)C_\xi(\tau)d\tau\right)\\ &= 2\int_0^\infty h_t(\tau)d\tau.\end{aligned}$$

On the other hand, from the definition of $h_t(\tau)$, we know that $h_t(\tau) \to \tau C_\xi(\tau)$ as $t \to \infty$ and $|h_t(\tau)| \le \tau|C_\xi(\tau)|$. Therefore, from condition (C2) and the Dominated Convergence Theorem,

$$\lim_{t\to\infty} t\left(S - \frac{\mathrm{Var}\{X_t\}}{t}\right) = 2\int_0^\infty \tau C_\xi(\tau)d\tau = D.$$

(e) From (7) and the definition of S,

$$\begin{aligned}S - \frac{\mathrm{Var}\{X_t\}}{t} &= 2\left(\int_0^\infty C_\xi(\tau)d\tau - \int_0^t (1-\frac{\tau}{t})C_\xi(\tau)d\tau\right)\\ &= \frac{2}{t}\left(\int_0^t \tau C_\xi(\tau)d\tau + \int_t^\infty tC_\xi(\tau)d\tau\right)\\ &> 0 \quad \text{for all } t>0 \quad \text{(from condition (C3)).}\end{aligned}$$

Therefore,

$$\frac{\mathrm{Var}\{X_t\}}{t} < S \quad \text{for all } t > 0. \tag{9}$$

Now, from the Dominated Convergence Theorem and conditions (C2) and (C3), it follows that

$$\lim_{t\to\infty}\int_0^t \tau C_\xi(\tau)d\tau = \int_0^\infty \tau C_\xi(\tau)d\tau > 0.$$

The above equation with (8) implies that there exists a $t_1 > 0$ such that $\frac{d}{dt}(\frac{\text{Var}\{X_t\}}{t}) > 0$ for all $t \geq t_1$; that is, $\frac{\text{Var}\{X_t\}}{t}$ is an increasing function for $t \geq t_1$. Let $a := \sup_{t\in(0,t_1]} \frac{\text{Var}\{X_t\}}{t}$. From (9), the continuity of $\frac{\text{Var}\{X_t\}}{t}$ and the fact that $\lim_{t\downarrow 0} \frac{\text{Var}\{X_t\}}{t} = 0$, it then follows that $a < S$. Therefore, since $\frac{\text{Var}\{X_t\}}{t} \to S$ as $t \to \infty$, there exists a $t_o > t_1$ such that $\frac{\text{Var}\{X_{t_o}\}}{t_o} > a$. Let $t \geq t_o$, then for $s \leq t_1$,

$$\begin{aligned} \frac{\text{Var}\{X_s\}}{s} \leq a \quad &< \quad \frac{\text{Var}\{X_{t_o}\}}{t_o} \quad \text{(from the definition of } t_o) \\ &\leq \quad \frac{\text{Var}\{X_t\}}{t} \quad \text{(because } \tfrac{\text{Var}\{X_t\}}{t} \text{ is increasing on } [t_1,\infty)). \end{aligned}$$

Also, since $\frac{\text{Var}\{X_t\}}{t}$ is increasing on $[t_1, \infty)$, $\frac{\text{Var}\{X_s\}}{s} \leq \frac{\text{Var}\{X_t\}}{t}$ for $s \in (t_1, t)$. Therefore, for all $t \geq t_o$, $\frac{\text{Var}\{X_t\}}{t} = \sup_{0<s\leq t} \frac{\text{Var}\{X_s\}}{s}$.

In this paper, we will study the supremum distribution of X_t through the Gaussian process $\{Y_t^{(x)} : t \geq 0\}$ defined for each $x > 0$ by

$$Y_t^{(x)} := \frac{\sqrt{x}(X_t + \kappa t)}{x + \kappa t} = \frac{\sqrt{x}\int_0^t \xi_s ds}{x + \kappa t}.$$

The following relation between X_t and $Y_t^{(x)}$ directly comes from the definition of $Y_t^{(x)}$ and plays a key role in studying the tail probability $\mathbb{P}(\{\langle X\rangle > x\})$.

(10) For any $t \geq 0$ and any $x > 0$, $\quad X_t > x$ if and only if $Y_t^{(x)} > \sqrt{x}$.

It also immediately follows that $Y_t^{(x)}$ is a centered Gaussian process and its autocovariance function $C_Y^{(x)}(t_1, t_2)$ can be obtained in terms of C_X as

$$C_Y^{(x)}(t_1,t_2) := \mathbb{E}\{Y_{t_1}^{(x)} Y_{t_2}^{(x)}\} = \frac{xC_X(t_1,t_2)}{(x+\kappa t_1)(x+\kappa t_2)}. \tag{11}$$

Now, let $\sigma_{x,t}^2$ be the variance of $Y_t^{(x)}$. It can then be expressed in terms of $\text{Var}\{X_t\}$ as

$$\sigma_{x,t}^2 = \frac{x\text{Var}\{X_t\}}{(x+\kappa t)^2}. \tag{12}$$

Hence, from Proposition 1(c), we have $\lim_{t\to\infty} \sigma_{x,t}^2 = 0$. Since $\sigma_{x,t}^2$ is a continuous function of t (from Proposition 1(a)), there is a finite value $t = \hat{t}_x$ at which $\sigma_{x,t}^2$ attains its maximum $\langle \sigma_x^2 \rangle$ (note that $\langle \sigma_x^2 \rangle$ denotes the supremum of $\sigma_{x,t}^2$ over the time index t). In the next proposition (Proposition 2), we show an important property of $\hat{t}_x$. Before we proceed, for notational simplicity, we define a function $g(t)$ for $t \geq 0$ as

$$g(t) := \begin{cases} 0 & \text{if } t = 0, \\ \frac{\text{Var}\{X_t\}}{St} & \text{if } t > 0. \end{cases}$$

Note that from Proposition 1(a), $g(t)$ is a continuous function of $t \in [0,\infty)$, and $\sigma^2_{x,t}$ can be written in terms of S and $g(t)$ as

$$\sigma^2_{x,t} = \frac{Sxt}{(x+\kappa t)^2} g(t). \tag{13}$$

Proposition 2 Under condition (C1),

$$\hat{t}_x \sim \frac{x}{\kappa} \quad \text{as } x \to \infty.$$

Further, under conditions (C1) *and* (C2), *the following stronger result holds.*

$$\lim_{x\to\infty} \frac{\hat{t}_x - \frac{x}{\kappa}}{x^\epsilon} = 0 \quad \text{for all } \epsilon > 0.$$

Proof of Proposition 2. From Proposition 1(c), it follows that $\lim_{t\to\infty} g(t) = 1$. Let $G := \sup_{t\geq 0} g(t)$ (G is finite and not less than 1). Since $\sigma^2_{x,t}$ attains its maximum at $t = \hat{t}_x$, it follows that

$$\frac{Sg(\frac{x}{\kappa})}{4\kappa} = \sigma^2_{x,\frac{x}{\kappa}} \leq \sigma^2_{x,\hat{t}_x} = \frac{Sx\hat{t}_x g(\hat{t}_x)}{(x+\kappa\hat{t}_x)^2} \leq \frac{Sx\hat{t}_x G}{(x+\kappa\hat{t}_x)^2}. \tag{14}$$

By solving (14) for $\hat{t}_x$, we have

$$\left(2\frac{G}{g(\frac{x}{\kappa})} - 1 - 2\sqrt{\frac{G}{g(\frac{x}{\kappa})}\left(\frac{G}{g(\frac{x}{\kappa})} - 1\right)}\right)\frac{x}{\kappa} \leq \hat{t}_x \leq \left(2\frac{G}{g(\frac{x}{\kappa})} - 1 + 2\sqrt{\frac{G}{g(\frac{x}{\kappa})}\left(\frac{G}{g(\frac{x}{\kappa})} - 1\right)}\right)\frac{x}{\kappa}.$$

Since $g\left(\frac{x}{\kappa}\right) \to 1$ as $x \to \infty$, this implies that $\hat{t}_x \to \infty$ (consequently $g(\hat{t}_x) \to 1$) as $x \to \infty$.

Now, since $\frac{Sxt}{(x+\kappa t)^2}$ attains its maximum $\frac{S}{4\kappa}$ at $t = \frac{x}{\kappa}$, we know from (14) that $g(\frac{x}{\kappa}) \leq g(\hat{t}_x)$ and the following relation should hold.

$$\left(2\frac{g(\hat{t}_x)}{g(\frac{x}{\kappa})} - 1 - 2\sqrt{\frac{g(\hat{t}_x)}{g(\frac{x}{\kappa})}\left(\frac{g(\hat{t}_x)}{g(\frac{x}{\kappa})} - 1\right)}\right)\frac{x}{\kappa} \leq \hat{t}_x \leq \left(2\frac{g(\hat{t}_x)}{g(\frac{x}{\kappa})} - 1 + 2\sqrt{\frac{g(\hat{t}_x)}{g(\frac{x}{\kappa})}\left(\frac{g(\hat{t}_x)}{g(\frac{x}{\kappa})} - 1\right)}\right)\frac{x}{\kappa}. \tag{15}$$

Since both $g(\hat{t}_x)$ and $g(\frac{x}{\kappa})$ approach 1 as $x \to \infty$, it follows from (15) that

$$\lim_{x\to\infty} \frac{\kappa\hat{t}_x}{x} = 1.$$

Thus, we have proved the first part of the proposition.

We next prove the second part of the proposition for which the autocovariance function C_ξ satisfies both conditions (C1) and (C2). From Proposition 1(d), note that

$$|g(t_1) - g(t_2)| = \frac{1}{S}\left|\frac{\mathrm{Var}\{X_{t_2}\}}{t_2} - \frac{\mathrm{Var}\{X_{t_1}\}}{t_1}\right| \leq \frac{\tilde{D}|t_2 - t_1|}{St_1t_2}. \tag{16}$$

Since both $g(\frac{x}{\kappa})$ and $\frac{\hat{t}_x\kappa}{x}$ approach 1 as x increases, we know that $g(\frac{x}{\kappa}), \frac{\hat{t}_x\kappa}{x} \in [\frac{1}{2}, 2]$ for all x sufficiently large. Therefore, for sufficiently large x,

$$\begin{aligned}
\left|\hat{t}_x - \frac{x}{\kappa}\right| &\le \frac{2x}{\kappa g(\frac{x}{\kappa})}\left(\left|g(\hat{t}_x) - g(\frac{x}{\kappa})\right| + \sqrt{g(\hat{t}_x)\left|g(\hat{t}_x) - g(\frac{x}{\kappa})\right|}\right) \quad \text{(from (15))} \\
&\le \frac{4x}{\kappa}\left(\frac{\tilde{D}|\hat{t}_x - \frac{x}{\kappa}|}{S\hat{t}_x\frac{x}{\kappa}} + \sqrt{\frac{G\tilde{D}|\hat{t}_x - \frac{x}{\kappa}|}{S\hat{t}_x\frac{x}{\kappa}}}\right) \quad \text{(from (16) and the definition of } G) \\
&= 4\left(\frac{\tilde{D}|\hat{t}_x - \frac{x}{\kappa}|}{S\hat{t}_x} + \sqrt{\frac{G\tilde{D}|\hat{t}_x - \frac{x}{\kappa}|}{S\hat{t}_x\frac{\kappa}{x}}}\right) \\
&\le 4\left(\frac{\tilde{D}}{S} + \sqrt{\frac{2G\tilde{D}|\hat{t}_x - \frac{x}{\kappa}|}{S}}\right). \qquad (17)
\end{aligned}$$

Now assume that $\lim_{x\to\infty}\frac{\hat{t}_x - \frac{x}{\kappa}}{x^{\epsilon}} = 0$ for some $\epsilon > 0$ (from the fact that $\hat{t}_x \sim \frac{x}{\kappa}$ as $x \to \infty$, this holds with any $\epsilon > 1$). Then, from (17),

$$\left|\frac{\hat{t}_x - \frac{x}{\kappa}}{x^{\frac{\epsilon}{2}}}\right| \le 4\left(\frac{\tilde{D}}{Sx^{\frac{\epsilon}{2}}} + \sqrt{\frac{2G\tilde{D}|\hat{t}_x - \frac{x}{\kappa}|}{Sx^{\epsilon}}}\right) \to 0, \quad \text{as } x \to \infty.$$

Therefore, $\lim_{x\to\infty}\frac{\hat{t}_x - \frac{x}{\kappa}}{x^{\frac{\epsilon}{2}}} = 0$. Thus, by induction we have

$$\lim_{x\to\infty}\frac{\hat{t}_x - \frac{x}{\kappa}}{x^{\epsilon}} = 0, \quad \text{for all } \epsilon > 0.$$

The following proposition is a direct result of Propositions 1(c) and 2, and describes the limit of $\langle\sigma_x^2\rangle$ as $x \to \infty$.

Proposition 3 Under condition (C1),

$$\lim_{x\to\infty}\langle\sigma_x^2\rangle = \frac{S}{4\kappa}.$$

Proof of Proposition 3. From (12), we have

$$\langle\sigma_x^2\rangle = \frac{x\mathrm{Var}\{X_{\hat{t}_x}\}}{(x+\kappa\hat{t}_x)^2} = \frac{1}{\kappa}\frac{\mathrm{Var}\{X_{\hat{t}_x}\}}{\hat{t}_x}\frac{\kappa\hat{t}_x}{x}\frac{1}{(1+\frac{\kappa\hat{t}_x}{x})^2}.$$

Since $\frac{\mathrm{Var}\{X_{\hat{t}_x}\}}{\hat{t}_x} \to S$ (Proposition 1(c)), $\frac{\kappa\hat{t}_x}{x} \to 1$ (Proposition 2), as $x \to \infty$, it follows that $\lim_{x\to\infty}\langle\sigma_x^2\rangle = \frac{S}{4\kappa}$.

3.2. *Main Result* In this section although we provide proofs only for the continuous-time case, all the results are also valid for the discrete-time case with the process $\{Y_n^{(x)} : n = 0, 1, \ldots\}$ and the parameters S and D redefined as

$$Y_n^{(x)} := \frac{\sqrt{x}(X_n + \kappa n)}{x + \kappa n}, \quad S := \sum_{l=-\infty}^{\infty} C_\xi(l), \quad \text{and} \quad D := 2\sum_{l=1}^{\infty} l C_\xi(l). \tag{18}$$

Now as mentioned in Section 1, it has been shown for many classes of stationary processes ξ_t, that (3) holds for some η [12]. In particular, for the case when ξ_t is a stationary Gaussian process that satisfies (C1), it has been shown that $\eta = \frac{2\kappa}{S}$, that is,

$$\lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}(\{\langle X \rangle > x\}) = -\frac{2\kappa}{S}. \tag{19}$$

Let us next consider a simple lower bound to the tail probability $\mathbb{P}(\{\langle X \rangle > x\})$ expressed in terms of the maximum variance $\langle \sigma_x^2 \rangle$. From (10), it follows that

$$\mathbb{P}(\{\langle X \rangle > x\}) = \mathbb{P}(\{\langle Y^{(x)} \rangle > \sqrt{x}\}) \geq \mathbb{P}(\{Y_{\hat{t}_x}^{(x)} > \sqrt{x}\}).$$

However, note that $Y_{\hat{t}_x}^{(x)}$ is a centered Gaussian random variable with variance $\langle \sigma_x^2 \rangle$. Therefore,

$$\Psi\left(\frac{\sqrt{x}}{\langle \sigma_x \rangle}\right) \leq \mathbb{P}(\{\langle X \rangle > x\}), \tag{20}$$

where $\Psi(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy$ is the tail of the standard Gaussian distribution. It is important to note that $\Psi\left(\frac{\sqrt{x}}{\langle \sigma_x \rangle}\right)$ is the probability that $Y_t^{(x)}$ is greater than $\sqrt{x}$ at $t = \hat{t}_x$, which is that value of t for which the variance of $Y_t^{(x)}$ attains its maximum $\langle \sigma_x^2 \rangle$. In the Extreme Value Theory for Gaussian processes, it has been frequently emphasized that the maximum variance of a centered Gaussian process with nonconstant variance, is a very important factor in studying the supremum distribution of the Gaussian process (as can be seen in Borell's inequality) [2, 3, 17, 20]. Also, it has been found that if $\{\zeta_t : t \in T\}$ is a centered Gaussian process with nonconstant variance which attains its maximum variance at $t = \hat{t}$, $\mathbb{P}(\{\langle \zeta \rangle > x\})$ the tail of the supremum distribution of ζ_t can often be closely approximated by the tail probability $\mathbb{P}(\{\zeta_{\hat{t}} > x\})$. Therefore, it would not be surprising if the lower bound, given by (20), accurately approximates the tail probability $\mathbb{P}(\{\langle X \rangle > x\})$. In fact, the lower bound has been used to approximate the tail probability in [6, 7] and found to be quite accurate over a wide range of x. Additionally, it has also been shown in [6] that

$$\lim_{x\to\infty} \frac{1}{x} \log \Psi\left(\frac{\sqrt{x}}{\langle \sigma_x \rangle}\right) = -\frac{2\kappa}{S}. \tag{21}$$

Therefore, from (19) and (21), the lower bound is asymptotically similar to the tail probability in the logarithmic sense; that is,

$$\log \Psi\left(\frac{\sqrt{x}}{\langle\sigma_x\rangle}\right) \sim \log \mathbb{P}(\{\langle X\rangle > x\}) \quad \text{as} \quad x \to \infty.$$

Qualitatively, the above observations on the lower bound suggests that the tail probability $\mathbb{P}(\{\langle X\rangle > x\})$ is concentrated on or around the maximum variance index $\hat{t}_x$. However, similarity in the logarithmic sense does not imply that $\Psi\left(\frac{\sqrt{x}}{\langle\sigma_x\rangle}\right) = \mathbb{P}(\{X_{\hat{t}_x} > x\}) \sim \mathbb{P}(\{\langle X\rangle > x\})$ as $x \to \infty$. In fact, this relation does not hold in general [8]. Therefore, a natural question to ask is whether (and how) we can choose some neighborhood F_x around $\hat{t}_x$ for each x such that $\mathbb{P}(\{\langle Y^{(x)}\rangle_{F_x} > \sqrt{x}\}) \sim \mathbb{P}(\{\langle Y^{(x)}\rangle > \sqrt{x}\})$ as $x \to \infty$. The following theorem gives us an answer to this question, and will be used to obtain an asymptotic upper bound to $\mathbb{P}(\{\langle X\rangle > x\})$.

Theorem 5 Under condition (C1)*, for any* $\alpha > 1$*,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\{\langle X\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]} > x\})}{\mathbb{P}(\{\langle X\rangle > x\})} = \lim_{x\to\infty} \frac{\mathbb{P}(\{\langle Y^{(x)}\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]} > \sqrt{x}\})}{\mathbb{P}(\{\langle Y^{(x)}\rangle > \sqrt{x}\})} = 1.$$

Proof of Theorem 5. The first equality directly follows from (10). Now, in order to show the second equality, it suffices to show that

$$\lim_{x\to\infty} \frac{\mathbb{P}(\{\langle Y^{(x)}\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]^c} > \sqrt{x}\})}{\mathbb{P}(\{\langle Y^{(x)}\rangle > \sqrt{x}\})} = 0 \quad \text{for all } \alpha > 1,$$

where A^c denotes the complementary set of A.

Let $\alpha > 1$. Since $g(t) \to 1$ as $t \to \infty$, there exists a t_o such that $g(t) \le \frac{\alpha+1}{2\sqrt{\alpha}}$ for all $t \ge t_o$. Now, let $G := \sup_{t\ge 0} g(t)$, then there exists an $x_o > \alpha\kappa t_o$ such that

$$\frac{Sxt_oG}{(x+\kappa t_o)^2} \le \frac{S\sqrt{\alpha}}{2\kappa(\alpha+1)} \quad \text{for all } x \ge x_o.$$

Since $\frac{SxtG}{(x+\kappa t)^2}$ is an increasing function of t on $[0, \frac{x}{\kappa}]$, this fact in conjunction with (13) implies that

$$\sigma_{x,t}^2 \le \frac{SxtG}{(x+\kappa t)^2} \le \frac{Sxt_oG}{(x+\kappa t_o)^2} \le \frac{S\sqrt{\alpha}}{2\kappa(\alpha+1)} \quad \text{for all } x \ge x_o \text{ and } t \le t_o. \tag{22}$$

Further, it can be easily verified that

$$\frac{Sxt}{(x+\kappa t)^2} \le \frac{S\alpha}{\kappa(\alpha+1)^2} \quad \text{for } t \in [\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c. \tag{23}$$

From the definition of t_o and (23), we have

$$\sigma_{x,t}^2 = \frac{Sxtg(t)}{(x+\kappa t)^2} \le \frac{S\sqrt{\alpha}}{2\kappa(\alpha+1)} \quad \text{for } x \ge x_o \text{ and } t \in (t_o, \tfrac{x}{\alpha\kappa}) \cup (\tfrac{\alpha x}{\kappa}, \infty). \tag{24}$$

Hence, from (22) and (24), it follows that

$$\langle \sigma_x^2 \rangle_{[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c} \leq \frac{S\sqrt{\alpha}}{2\kappa(\alpha+1)} \quad \text{for all } x \geq x_o. \tag{25}$$

We now define a pseudo-metric $d^{(x)}$ on $[0,\infty)$ as $d^{(x)}(t_1,t_2) := \sqrt{\mathbb{E}\{(Y_{t_2}^{(x)} - Y_{t_1}^{(x)})^2\}}$. Also, let $\mathbf{B}_\epsilon^{(x)}(t) := \{s \in [0,\infty) : d^{(x)}(t,s) \leq \epsilon\}$ be a $d^{(x)}$-ball of radius of ϵ centered at t, and let $N^{(x)}(\epsilon)$ be the minimum number of $d^{(x)}$-balls of radius of ϵ needed to cover $[0,\infty)$. Since $\mathrm{Var}\{Y_t^{(x)}\} \leq \frac{SGxt}{(x+\kappa t)^2} \leq \frac{SG}{4\kappa}$ and since $Y_0^{(x)} = 0$, $\mathbf{B}_\epsilon^{(x)}(0)$ cover $[0,\infty)$ when $\epsilon \geq \sqrt{\frac{SG}{4\kappa}}$. Therefore, for all $x > 0$,

$$N^{(x)}(\epsilon) = 1 \quad \text{if } \epsilon \geq \sqrt{\frac{SG}{4\kappa}}. \tag{26}$$

Now, assume that $\epsilon < \sqrt{\frac{SG}{4\kappa}}$ and $t_2 > t_1$. Then,

$$\begin{aligned} d^{(x)}(t_1,t_2) &= \sqrt{\mathbb{E}\left\{\left(\tfrac{\sqrt{x}(X_{t_2}+\kappa t_2)}{x+\kappa t_2} - \tfrac{\sqrt{x}(X_{t_1}+\kappa t_1)}{x+\kappa t_1}\right)^2\right\}} \\ &= \sqrt{\mathbb{E}\left\{\left(\tfrac{\sqrt{x}(X_{t_2}+\kappa t_2)}{x+\kappa t_2} - \tfrac{\sqrt{x}(X_{t_1}+\kappa t_1)}{x+\kappa t_2} + \tfrac{\sqrt{x}(X_{t_1}+\kappa t_1)}{x+\kappa t_2} - \tfrac{\sqrt{x}(X_{t_1}+\kappa t_1)}{x+\kappa t_1}\right)^2\right\}} \\ &\leq \sqrt{\mathbb{E}\left\{\left(\tfrac{\sqrt{x}(X_{t_2}-X_{t_1}+\kappa(t_2-t_1))}{x+\kappa t_2}\right)^2\right\}} + \sqrt{\mathbb{E}\left\{\left(\tfrac{\kappa(t_2-t_1)\sqrt{x}(X_{t_1}+\kappa t_1)}{(x+\kappa t_2)(x+\kappa t_1)}\right)^2\right\}} \\ &= \tfrac{\sqrt{x}}{x+\kappa t_2}\sqrt{\mathrm{Var}\{(X_{t_2}-X_{t_1})\}} + \tfrac{\kappa(t_2-t_1)\sqrt{x}}{(x+\kappa t_2)(x+\kappa t_1)}\sqrt{\mathrm{Var}\{X_{t_1}\}}. \end{aligned} \tag{27}$$

However, since the stationarity of ξ_t implies that $\mathrm{Var}\{(X_{t_2}-X_{t_1})\} = \mathrm{Var}\{X_{t_2-t_1}\}$, $\mathrm{Var}\{(X_{t_2}-X_{t_1})\}$ and $\mathrm{Var}\{X_{t_1}\}$ are bounded by $SG(t_2-t_1)$ and SGt_1, respectively. Therefore, from (27)

$$\begin{aligned} d^{(x)}(t_1,t_2) &\leq \frac{\sqrt{SGx(t_2-t_1)}}{x+\kappa t_2} + \frac{\kappa(t_2-t_1)\sqrt{SGxt_1}}{(x+\kappa t_1)(x+\kappa t_2)} \\ &\leq \left(\frac{\sqrt{SGx}}{x+\kappa t_2} + \frac{\kappa\sqrt{SGxt_1t_2}}{(x+\kappa t_1)(x+\kappa t_2)}\right)\sqrt{t_2-t_1} \\ &\leq \left(\sqrt{\frac{SG}{x}} + \frac{1}{4}\sqrt{\frac{SG}{x}}\right)\sqrt{t_2-t_1} \leq \sqrt{\frac{2SG}{x}}\sqrt{t_2-t_1} \\ &\qquad \text{(from the fact that } \tfrac{\sqrt{x}}{x+\kappa t_2} \leq \tfrac{1}{\sqrt{x}} \text{ and } \tfrac{\sqrt{t}}{(x+\kappa t)} \leq \tfrac{1}{2\sqrt{x\kappa}}\text{).} \end{aligned}$$

This implies that if $|t_2 - t_1| \leq \frac{x}{2SG}\epsilon^2$, then $d^{(x)}(t_1,t_2) \leq \epsilon$. Consequently,

$$[t - \tfrac{x}{2SG}\epsilon^2, t + \tfrac{x}{2SG}\epsilon^2] \subset \mathbf{B}_\epsilon^{(x)}(t). \tag{28}$$

Also, it can easily be shown that $\mathrm{Var}\{Y_t^{(x)}\} \leq \epsilon^2$ for $t \geq \frac{SGx}{\epsilon^2\kappa^2}$. Since $Y_0^{(x)} = 0$, this implies that

$$[\tfrac{SGx}{\epsilon^2\kappa^2}, \infty) \subset \mathbf{B}_\epsilon^{(x)}(0). \tag{29}$$

Therefore, from (28) and (29), $d^{(x)}$-balls of radius of ϵ centered at t_i ($i = 0, 1, \ldots, \lceil \frac{SGx}{\epsilon^2\kappa^2} / \frac{x\epsilon^2}{SG} \rceil$) covers $[0, \infty)$, where $\lceil w \rceil$ is the smallest integer that is larger than or equal to w and

$$t_i = \begin{cases} 0 & \text{if } i = 0, \\ i\frac{x}{SG}\epsilon^2 - \frac{x}{2SG}\epsilon^2 & \text{otherwise.} \end{cases}$$

Hence, for $\epsilon < \sqrt{\frac{SG}{4\kappa}}$, the minimum number of $d^{(x)}$-balls to cover $[0, \infty)$ is bounded by the following inequality:

$$N^{(x)}(\epsilon) \le \left\lceil \frac{SGx}{\epsilon^2\kappa^2} / \frac{x\epsilon^2}{SG} \right\rceil + 1 \le \frac{S^2G^2}{\kappa^2\epsilon^4} + 2. \tag{30}$$

From (26) and (30), $\bar{N}(\epsilon)$ defined by

$$\bar{N}(\epsilon) := \begin{cases} \frac{S^2G^2}{\kappa^2\epsilon^4} + 2 & \text{if } \epsilon < \sqrt{\frac{SG}{4\kappa}}, \\ 1 & \text{otherwise,} \end{cases}$$

bounds $N^{(x)}(\epsilon)$ for all $x, \epsilon > 0$. Now, let $M := K \int_0^\infty \log^{\frac{1}{2}} \bar{N}(\epsilon) d\epsilon$, where K is the universal constant in Theorem 4 (it can easily be shown that the integral is finite). Then, from Theorem 4,

$$\mathbb{E}\{\langle Y^{(x)} \rangle\} \le M \quad \text{for all } x > 0.$$

Hence, by applying Theorem 2 to $Y_t^{(x)}$ on $t \in [\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c$, we get

$$\begin{aligned} \mathbb{P}(\{\langle Y^{(x)} \rangle_{[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c} > \sqrt{x}\}) &\le 2e^{-\frac{\left(\sqrt{x} - \mathbb{E}\left\{\langle Y^{(x)} \rangle_{[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c}\right\}\right)^2}{2\langle \sigma_x^2 \rangle_{[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c}}} \\ &\le 2e^{-\frac{\kappa(\sqrt{x} - \mathbb{E}\{\langle Y^{(x)} \rangle\})^2(\alpha+1)}{S\sqrt{\alpha}}} \\ &\quad \text{(from (25) and the fact that } \langle Y^{(x)} \rangle_{[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c} \le \langle Y^{(x)} \rangle) \\ &\le 2e^{-\frac{\kappa(\sqrt{x} - M)^2(\alpha+1)}{S\sqrt{\alpha}}} \quad \text{for } x \text{ sufficiently large.} \end{aligned}$$

Therefore, it directly follows that

$$\limsup_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\{\langle Y^{(x)} \rangle_{[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c} > \sqrt{x}\}) \le \lim_{x \to \infty} -\frac{\kappa(\sqrt{x} - M)^2(\alpha+1)}{Sx\sqrt{\alpha}} = -\frac{\kappa(\alpha+1)}{S\sqrt{\alpha}}. \tag{31}$$

Since $-\frac{\kappa(\alpha+1)}{S\sqrt{\alpha}} < -\frac{2\kappa}{S}$ for all $\alpha > 1$, (19) and (31) imply that

$$\lim_{x \to \infty} \frac{\mathbb{P}(\{\langle Y^{(x)} \rangle_{[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]^c} > \sqrt{x}\})}{\mathbb{P}(\{\langle Y^{(x)} \rangle > \sqrt{x}\})} = 0.$$

We will now use Theorem 5 and a well known property of Brownian Motion process to obtain an asymptotic upper bound to $\mathbb{P}(\{\langle X\rangle > x\})$. Let $\{B_t : t \geq 0\}$ be the standard Brownian Motion (Wiener) process, and let $\{V_t : t \geq 0\}$ be defined as

$$V_t := aB_t - bt.$$

This process is often called Brownian Motion process with drift¶ and has been studied extensively. In particular, the supremum distribution of V_t has been found in a simple closed form (see, for example, [16, page 199]) as

$$\mathbb{P}(\{\langle V\rangle > x\}) = \mathbb{P}\left(\left\{\text{there exists a } t \geq 0 \text{ such that } B_t > \frac{bt}{a} + \frac{x}{a}\right\}\right) = e^{-\frac{2bx}{a^2}}. \tag{32}$$

We are now ready to prove Theorem 1, which provides a simple single-exponential based asymptotic upper bound to $\mathbb{P}(\{\langle X\rangle > x\})$, when ξ_t satisfies conditions (C1)–(C3). As will soon be evident, this bound is obtained by comparing $\mathbb{P}(\{\langle X\rangle > x\})$ and the tail of the supremum distribution of a Brownian Motion process with drift, through Slepian's inequality. For the reader's convenience we restate Theorem 1.

Theorem 1 Under conditions (C1)–(C3)*,*

$$\limsup_{x\to\infty} e^{\frac{2\kappa x}{S}} \mathbb{P}(\{\langle X\rangle > x\}) \leq e^{-\frac{2\kappa^2 D}{S^2}}.$$

In other words, $\mathbb{P}(\{\langle X\rangle > x\})$ *is asymptotically bounded from above by* $e^{-\frac{2\kappa}{S}(x+\frac{\kappa D}{S})}$.

Proof of Theorem 1. Let $V_t = \sqrt{S}B_t - \kappa t$ and define a centered Gaussian process $\{Z_t^{(x)} : t \geq 0\}$ for each $x > 0$ by

$$Z_t^{(x)} := \frac{\sqrt{xg(t)}(V_t + \kappa t)}{x + \kappa t} = \frac{\sqrt{xg(t)S}B_t}{x + \kappa t}.$$

Using this definition, $C_Z^{(x)}$ the autocovariance function of $Z_t^{(x)}$ can easily be obtained as

$$C_Z^{(x)}(t_1, t_2) = \frac{Sx \min\{t_1, t_2\}\sqrt{g(t_1)g(t_2)}}{(x + \kappa t_1)(x + \kappa t_2)}. \tag{33}$$

From (13) and (33), we can verify that the variance of $Z_t^{(x)}$ is equal to that of $Y_t^{(x)}$ for any $t \geq 0$ and $x > 0$.

¶An interesting fact is that even though V_t cannot be expressed in the form of (1), Proposition 2, Proposition 3, and Theorem 5 hold with X_t, κ, and S replaced by V_t, b, and a^2, respectively. From the simple autocovariance function $C_V(t_1, t_2) = a^2 \min\{t_1, t_2\}$ of V_t, these results can be obtained in almost the same way as (or usually easier than) in the case of X_t.

Now, let $\alpha > 1$ and consider $Y_t^{(x)}$ and $Z_t^{(x)}$ on the interval $[\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]$. From Proposition 1(e), there exists a $t_o > 0$ such that for all $t \geq t_o$,

$$\frac{\mathrm{Var}\{X_s\}}{s} \leq \frac{\mathrm{Var}\{X_t\}}{t} \quad \text{for all } s < t. \tag{34}$$

Hence if we assume that $t_2 > t_1 \geq t_o$, then

$$\begin{aligned}
\frac{C_X(t_1,t_2)}{t_1} &= \frac{1}{2t_1}\left(\mathrm{Var}\{X_{t_1}\} + \mathrm{Var}\{X_{t_2}\} - \mathrm{Var}\{X_{t_2-t_1}\}\right) \quad \text{(from Proposition 1(b))} \\
&= \frac{1}{2}\left(\frac{\mathrm{Var}\{X_{t_1}\}}{t_1} + \frac{\mathrm{Var}\{X_{t_2}\}}{t_2} + \frac{t_2-t_1}{t_1}\left(\frac{\mathrm{Var}\{X_{t_2}\}}{t_2} - \frac{\mathrm{Var}\{X_{t_2-t_1}\}}{t_2-t_1}\right)\right) \\
&\geq \frac{1}{2}\left(\frac{\mathrm{Var}\{X_{t_1}\}}{t_1} + \frac{\mathrm{Var}\{X_{t_2}\}}{t_2}\right) \quad \text{(from (34))} \\
&\geq \sqrt{\frac{\mathrm{Var}\{X_{t_1}\}\mathrm{Var}\{X_{t_2}\}}{t_1 t_2}} \quad (\text{since } \tfrac{\mathrm{Var}\{X_t\}}{t} \geq 0).
\end{aligned}$$

This implies that

$$\begin{aligned}
S\min\{t_1,t_2\}\sqrt{g(t_1)g(t_2)} &= t_1\sqrt{\frac{\mathrm{Var}\{X_{t_1}\}\mathrm{Var}\{X_{t_2}\}}{t_1 t_2}} \quad \text{(from the definition of } g(t)) \\
&\leq C_X(t_1,t_2) \quad \text{if } t_2 > t_1 \geq t_o.
\end{aligned} \tag{35}$$

Therefore, from (11), (33), and (35), and from the fact that $\mathrm{Var}\{Y_t^{(x)}\} = \mathrm{Var}\{Z_t^{(x)}\}$, it follows for any $x \geq \alpha\kappa t_o$ that

$$\mathbb{E}\{(Y_{t_1}^{(x)} - Y_{t_2}^{(x)})^2\} \leq \mathbb{E}\{(Z_{t_1}^{(x)} - Z_{t_2}^{(x)})^2\} \quad \text{for all } t_1, t_2 \in [\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}].$$

Hence, from Theorem 3,

$$\mathbb{P}(\{\langle Y^{(x)}\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]} > \sqrt{x}\}) \leq \mathbb{P}(\{\langle Z^{(x)}\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]} > \sqrt{x}\}) \quad \text{for all } x \geq \alpha\kappa t_o. \tag{36}$$

We now obtain an upper bound to $\mathbb{P}(\{\langle Z^{(x)}\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]} > \sqrt{x}\})$ for $x \geq \alpha\kappa t_o$.

$$\begin{aligned}
\mathbb{P}(\{\langle Z^{(x)}\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]} > \sqrt{x}\}) &= \mathbb{P}(\{Z_t^{(x)} > \sqrt{x} \text{ for some } t \in [\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]\}) \\
&= \mathbb{P}(\{\sqrt{Sg(t)}B_t > x + \kappa t \text{ for some } t \in [\frac{x}{\alpha\kappa}, \frac{\alpha x}{\kappa}]\}) \\
&\qquad \text{(from the definition of } V_t \text{ and } Z_t^{(x)}) \\
&\leq \mathbb{P}(\{\sqrt{Sg(\frac{\alpha x}{\kappa})}B_t > x + \kappa t \text{ for some } t \geq 0\}) \\
&\qquad (\text{since } g(t) \text{ is increasing on } [\tfrac{x}{\alpha\kappa}, \tfrac{\alpha x}{\kappa}]) \\
&= e^{-\frac{2\kappa x}{Sg(\frac{\alpha x}{\kappa})}} \quad \text{(from (32))}.
\end{aligned} \tag{37}$$

Hence, from (36) and (37),

$$\mathbb{P}(\{\langle Y^{(x)}\rangle_{[\frac{x}{\alpha\kappa},\frac{\alpha x}{\kappa}]} > \sqrt{x}\}) \le e^{-\frac{2\kappa x}{Sg(\frac{\alpha x}{\kappa})}} \quad \text{for } x \ge \alpha\kappa t_o. \tag{38}$$

Further, from Proposition 1(d) and the fact that $g(t) \to 1$ as $t \to \infty$, we have

$$\begin{aligned} e^{\frac{2\kappa x}{S}} e^{-\frac{2\kappa x}{Sg(\frac{\alpha x}{\kappa})}} &= e^{-\frac{2\kappa x}{Sg(\frac{\alpha x}{\kappa})}\left(1-\frac{1}{S\frac{x}{\alpha\kappa}}\mathrm{Var}\left\{X_{\frac{x}{\alpha\kappa}}\right\}\right)} \quad \text{(from the definition of } g(t)\text{)} \\ &= e^{-\frac{2\kappa^2}{S^2\alpha g(\frac{\alpha x}{\kappa})}\frac{\alpha x}{\kappa}\left(S-\frac{1}{\frac{\alpha x}{\kappa}}\mathrm{Var}\left\{X_{\frac{\alpha x}{\kappa}}\right\}\right)} \to e^{-\frac{2\kappa^2 D}{\alpha S^2}} \quad \text{as } x \to \infty. \end{aligned} \tag{39}$$

Therefore, from Theorem 5 and from (10), (38) and (39), it follows that

$$\limsup_{x\to\infty} e^{\frac{2\kappa x}{S}}\mathbb{P}(\{\langle X\rangle > x\}) \le e^{-\frac{2\kappa^2 D}{\alpha S^2}}.$$

Since $\alpha > 1$ is arbitrary, the theorem follows.

An interesting observation is that the asymptotic upper bound given in Theorem 1 can also be achieved by a simple expression given in terms of the maximum variance $\langle\sigma_x^2\rangle$.

Proposition 4 Under conditions (C1) *and* (C2)*,*

$$e^{-\frac{x}{2\langle\sigma_x^2\rangle}} \sim e^{-\frac{2\kappa}{S}(x+\frac{\kappa D}{S})} \quad \textit{as } x \to \infty.$$

Proof of Proposition 4. From (12) and the definition of $\hat{t}_x$, we have

$$\langle\sigma_x^2\rangle = \frac{x\mathrm{Var}\{X_{\hat{t}_x}\}}{(x+\kappa\hat{t}_x)^2}.$$

Therefore,

$$\frac{2\kappa x}{S} - \frac{x}{2\langle\sigma_x^2\rangle} = \frac{-4\kappa\frac{x}{\hat{t}_x}\hat{t}_x\left(S-\frac{\mathrm{Var}\{X_{\hat{t}_x}\}}{\hat{t}_x}\right) - \frac{S\kappa^2}{\hat{t}_x}\left(\frac{x}{\kappa}-\hat{t}_x\right)^2}{2S\frac{\mathrm{Var}\{X_{\hat{t}_x}\}}{\hat{t}_x}}. \tag{40}$$

Since $\frac{x}{\hat{t}_x} \to \kappa$, $\frac{\mathrm{Var}\{X_{\hat{t}_x}\}}{\hat{t}_x} \to S$, $\left(S-\frac{\mathrm{Var}\{X_{\hat{t}_x}\}}{\hat{t}_x}\right)\hat{t}_x \to D$, and $\frac{(\frac{x}{\kappa}-\hat{t}_x)^2}{\hat{t}_x} \to 0$ as $x \to \infty$ from Propositions 1 and 2, and from (40) we get

$$\lim_{x\to\infty}\frac{2\kappa x}{S} - \frac{x}{2\langle\sigma_x^2\rangle} = -\frac{2\kappa^2 D}{S^2}.$$

Hence, $\lim_{x\to\infty} e^{\frac{2\kappa x}{S}} e^{-\frac{x}{2\langle\sigma_x^2\rangle}} = e^{-\frac{2\kappa^2 D}{S^2}}$.

Proposition 4 and Theorem 1 tell us that when the process ξ satisfies conditions (C1)–(C3), the tail of the supremum distribution is asymptotically bounded by $e^{-\frac{x}{2\langle\sigma_x^2\rangle}}$. Note that the class

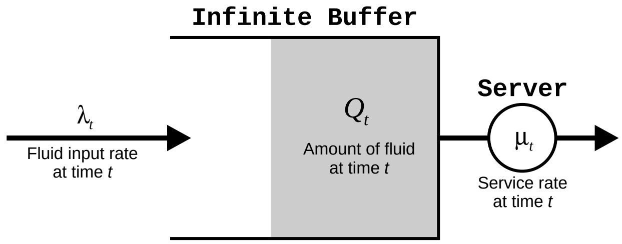

Figure 1. A fluid queueing system with an infinite buffer and a server. λ_t is the instantaneous amount of fluid (work) fed into the system at time t, μ_t is the maximum rate at which fluid can be served at time t, and Q_t is the amount of fluid in the queue at time t.

of stationary Gaussian processes that satisfy conditions (C1)–(C3) is fairly large. For example, any autocovariance function that vanishes faster than $\tau^{-\epsilon}$ ($l^{-\epsilon}$) for some $\epsilon > 2$, satisfy conditions (C1) and (C2) (of course, except for those with $S = 0$). Also, condition (C3) which is somewhat more restrictive, is satisfied by any nonnegative autocovariance function. Hence, the fact that an asymptotic upper bound to $\mathbb{P}(\{\langle X\rangle > x\})$ can be obtained merely from $\langle\sigma_x^2\rangle$, again indicates the importance of the maximum variance in studying the supremum distribution of Gaussian processes.

In the next section, we will discuss the applications and importance of the asymptotic upper bound for the study of queueing systems.

4. Application to Queueing Systems

Consider a queueing system shown in Figure 1. Let Λ_t be an increasing function defined in such a way that $\Lambda_t - \Lambda_s$ is the amount of fluid that arrives into the system during the time interval $(s, t]$. Similarly, we define M_t to be an increasing function such that $\mathrm{M}_t - \mathrm{M}_s$ is the maximum amount of fluid that can be served during the time interval $(s, t]$. Then assuming that the queue is empty at $t = 0$, Q_t the amount of fluid in the system (workload) at time t can be expressed as

$$Q_t = \sup_{0\le s\le t} \left(\mathrm{N}_t - \mathrm{N}_s\right),$$

where $\mathrm{N}_t := \Lambda_t - \mathrm{M}_t$ (see for example [12, 14]).

If we assume that Λ_t and M_t are independent stochastic processes with stationary increments, then

$$\begin{aligned}\mathbb{P}(\{Q_t > x\}) &= \mathbb{P}\left(\left\{\sup_{0\le s\le t}(\mathrm{N}_t - \mathrm{N}_s) > x\right\}\right)\\ &= \mathbb{P}\left(\left\{\sup_{-t\le s\le 0}(\mathrm{N}_0 - \mathrm{N}_s) > x\right\}\right)\end{aligned}$$

$$\rightarrow \quad \mathbb{P}\left(\left\{\sup_{s\leq 0}\left(\mathrm{N}_0-\mathrm{N}_s\right)>x\right\}\right) \quad \text{as } t\rightarrow\infty. \tag{41}$$

Hence, $\mathbb{P}(\{Q>x\}) := \lim_{t\to\infty}\mathbb{P}(\{Q_t>x\}) = \mathbb{P}\left(\left\{\sup_{s\leq 0}\left(\mathrm{N}_0-\mathrm{N}_s\right)>x\right\}\right)$. In other words, the steady state (limiting) queue length distribution coincides with the distribution of $\sup_{s\leq 0}\left(\mathrm{N}_0-\mathrm{N}_s\right)$. The tail of the steady state queue length distribution is an important measure of network congestion and very useful in the design and control of communication networks. Now let λ_t be defined as the instantaneous rate of fluid input and μ_t as the maximum rate at which fluid can be served at time t. Then, $\mathrm{N}_t-\mathrm{N}_s$ can be given by

$$\begin{array}{rl} \text{Continuous-time}: & \mathrm{N}_t-\mathrm{N}_s=\int_s^t \nu_u du, \quad \text{and} \\ \text{Discrete-time}: & \mathrm{N}_t-\mathrm{N}_s=\sum_{m=s+1}^t \nu_m, \end{array} \tag{42}$$

where $\nu_t := \lambda_t-\mu_t$ is the *net input rate* into the queue (note that ν_t can take on both positive and negative values).

Hence, from (41) and (42), it follows under the stationarity of ν_t (or under the stationarity and independence of λ_t and μ_t) that

$$\begin{array}{rl} \text{Continuous-time}: & \mathbb{P}(\{Q>x\})=\mathbb{P}\left(\left\{\sup_{t\geq 0}\int_0^t \nu_{-s}ds>x\right\}\right), \quad \text{and} \\ \text{Discrete-time}: & \mathbb{P}(\{Q>x\})=\mathbb{P}\left(\left\{\sup_{n\geq 0}\sum_{m=0}^n \nu_{-m}>x\right\}\right). \end{array}$$

Fluid queueing models have frequently been employed for the analysis of multiplexers in emerging high-speed communications such as Asynchronous Transfer Mode (ATM) networks [10, 13]. In these applications, the stationary process λ_t models the aggregate traffic input to a multiplexer, and μ_t is often fixed to a constant μ to represent the link capacity of the multiplexer which is usually not time-varying. Since commercial ATM multiplexers and switches are already equipped with very high-capacity links, many traffic sources can be served at a multiplexer. Therefore, the *net input traffic* (the aggregate traffic input minus the link capacity of the multiplexer, which corresponds to ν_t) can usually be accurately characterized by a stationary Gaussian process [6, 7]. Further, it has been found that some important types of individual traffic sources themselves can be modeled as a stationary Gaussian process [15]. Once the net input traffic is characterized by a stationary Gaussian process, as we will discuss next, our asymptotic analysis of $\mathbb{P}(\{\langle X\rangle>x\})$ can be directly applied to study $\mathbb{P}(\{Q>x\})$, the tail of the queue length distribution, in such networks.

Assuming that ν_t is a stationary Gaussian process, it is easy to see that the steady state queue length distribution is equal to the supremum distribution of X (given by (1) or (2)) with ξ and κ defined as

$$\begin{array}{rl} \text{Continuous-time}: & \xi_t=\nu_{-t}-\mathbb{E}\{\nu_0\}, \text{ and } \kappa=-\mathbb{E}\{\nu_0\} \quad \text{or} \\ \text{Discrete-time}: & \xi_n=\nu_{1-n}-\mathbb{E}\{\nu_0\}, \text{ and } \kappa=-\mathbb{E}\{\nu_0\}. \end{array} \tag{43}$$

Therefore, when the net traffic input can be effectively characterized by a stationary Gaussian process that satisfies conditions (C1)–(C3), Theorem 1 provides us an asymptotic upper bound to $\mathbb{P}(\{Q > x\})$, the tail of the queue length distribution. Here it should be noted that while $\mu_t = \mu$ for high-speed ATM networks, it may not be true for other networks; however, all our results are also valid for general time-varying μ_t as long as the net input rate can be effectively modeled as a Gaussian process. Now let us briefly discuss the relevance of our work in the context of the existing literature.

Discrete-Time Case: As mentioned in Section 1, in the discrete-time setting [1], it has been shown for stationary ergodic Gaussian net input processes ν_n that

$$\mathbb{P}(\{Q > x\}) = \mathbb{P}(\{\langle X \rangle > x\}) \sim Ce^{-\frac{2\kappa x}{S}} \quad \text{as } x \to \infty, \tag{44}$$

where ξ_n and κ are given by (43), and S defined by (18). From the above relation, $Ce^{-\frac{2\kappa x}{S}}$ has been suggested as an approximation to $\mathbb{P}(\{Q > x\})$ for large x. This approximation is often called the *asymptotic approximation.* However, since the exact value of the asymptotic constant C cannot be obtained in general, the following simpler approximation (obtained by setting $C = 1$) has also been suggested:

$$\mathbb{P}(\{Q > x\}) \approx e^{-\frac{2\kappa x}{S}}.$$

This approximation is the well known *effective bandwidth* approximation, which can be extended to fairly general classes of net input processes ν_t [12, 13]. In recent papers, however, it has been argued that the effective bandwidth approximation does not account for the advantage of multiplexing and could lead to significant underutilization of the network [9, 18]. Therefore, there is renewed interest in the accurate approximations and bounds for the asymptotic constant C.

It is important to note that the decay rate of the asymptotic upper bound given in Theorem 1 coincides with the decay rate of the tail $\mathbb{P}(\{Q > x\})$ which is equal to $-\frac{2\kappa}{S}$. Therefore, the asymptotic upper bound provides us an upper bound $e^{-\frac{2\kappa^2 D}{S^2}}$ to the asymptotic constant C when ν_n is a stationary Gaussian process that satisfies conditions (C1)–(C3). As previously mentioned, a fairly large class of stationary Gaussian processes satisfy these conditions. Hence, the upper bound to the asymptotic constant is expected to help us to better exploit the advantage of multiplexing when designing these networks.

Continuous-Time Case: In contrast to the discrete-time case, (44) has been shown to be valid in the continuous-time case only for a very limited class of stationary Gaussian processes ν_t. Therefore, obtaining an asymptotic result for the tail probability, which is similar to (44), is very important. In the following part of this section, we show how our asymptotic upper bound can be used to obtain an asymptotic result for $\mathbb{P}(\{Q > x\})$ which is nearly comparable to (44).

Using the results for the discrete-time case, we can show that there exists an asymptotic lower bound to the tail probability $\mathbb{P}(\{\langle X\rangle > x\})$ of the form $Ce^{-\frac{2\kappa x}{S}}$, that is, $\liminf_{x\to\infty} e^{\frac{2\kappa x}{S}}\mathbb{P}(\{\langle X\rangle > x\}) > 0$. Now, consider the continuous-time process X_t expressed by (1). Given a $\Delta > 0$, an asymptotic lower bound to the tail probability $\mathbb{P}(\{\langle X\rangle > x\})$ can be found by looking at the sampled stochastic process $\{\dot{X}_n = X_{n\Delta} : n = 0, 1, 2, \ldots\}$. Note that $\dot{X}_n$ can be expressed as

$$\begin{aligned}\dot{X}_n &= \sum_{m=1}^{n}\int_{(m-1)\Delta}^{m\Delta}\xi_s ds - \kappa n\Delta \\ &= \sum_{m=1}^{n}\dot{\xi}_m - \dot{\kappa}n,\end{aligned}$$

where $\dot{\xi}_m := \int_{(m-1)\Delta}^{m\Delta}\xi_s ds$ and $\dot{\kappa} := \kappa\Delta$. $\dot{\xi}_n$ is a stationary Gaussian process (from its definition) and $C_{\dot{\xi}}(l)$ its autocovariance function can be obtained in terms of $C_\xi(\tau)$ as

$$C_{\dot{\xi}}(l) = \int_{-\Delta}^{\Delta}(\Delta - |\tau|)C_\xi(\tau + l\Delta)d\tau,$$

from which one can verify that

$$\dot{S} := \sum_{-\infty}^{\infty}C_{\dot{\xi}}(l) = \Delta\int_{-\infty}^{\infty}C_\xi(\tau)d\tau = \Delta S.$$

Hence, from (44) there exists a $c_1 > 0$ such that

$$\mathbb{P}(\{\langle\dot{X}\rangle > x\}) \sim c_1 e^{-\frac{2\dot{\kappa}x}{\dot{S}}} = c_1 e^{-\frac{2\kappa x}{S}}.$$

Therefore, we get

$$\begin{aligned}\liminf_{x\to\infty} e^{-\frac{2\kappa x}{S}}\mathbb{P}(\{\langle X\rangle > x\}) &\geq \liminf_{x\to\infty} e^{-\frac{2\kappa x}{S}}\mathbb{P}(\{\langle\dot{X}\rangle > x\}) \\ &\quad (\text{since } \langle X\rangle \geq \langle\dot{X}\rangle = \langle X\rangle_{\{0,\Delta,2\Delta,\ldots\}}) \\ &= c_1 > 0 \quad (\text{from } (4)).\end{aligned} \tag{45}$$

Now, by combining Theorem 1 and (45), it follows that for stationary Gaussian processes ξ_t that satisfy conditions (C1)–(C3),

$$c_1 \leq \liminf_{x\to\infty} e^{-\frac{2\kappa x}{S}}\mathbb{P}(\{\langle X\rangle > x\}) \leq \limsup_{x\to\infty} e^{-\frac{2\kappa x}{S}}\mathbb{P}(\{\langle X\rangle > x\}) \leq e^{-\frac{2\kappa^2 D}{S^2}},$$

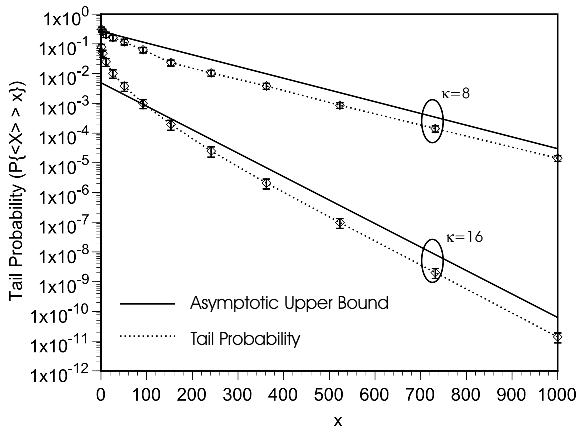

Figure 2. The tail probability $\mathbb{P}(\{\langle X\rangle > x\})$ estimated through simulation and its asymptotic upper bound $e^{-\frac{2\kappa}{S}(x+\frac{\kappa D}{S})}$ for a continuous-time process X_t expressed by (1). In this example, the autocovariance function of ξ_t is given as $C_\xi(\tau) = 80 \times e^{-|\tau|} + 40 \times e^{-|\frac{\tau}{20}|}$ and κ is set to two different values, 8 and 16.

where D is defined by (6). Therefore, if we let $c_2 := e^{-\frac{2\kappa^2 D}{S^2}}$, then the above equation implies that for a fluid queue whose net input rate ν_t $(= \xi_{-t} - \kappa)$ is a stationary Gaussian process that satisfies conditions (C1)–(C3), for any $\epsilon > 1$,

$$(46) \qquad \frac{c_1}{\epsilon} e^{-\frac{2\kappa x}{S}} \le \mathbb{P}(\{Q > x\}) = \mathbb{P}(\{\langle X\rangle > x\}) \le \epsilon c_2 e^{-\frac{2\kappa x}{S}} \quad \text{for all sufficiently large } x.$$

Even though the above relation is not as strong as (44), it tells us that $\mathbb{P}(\{Q > x\})$ is asymptotically enclosed within an exponential envelope when conditions (C1)–(C3) are satisfied by the net input rate ν_t.

5. Numerical Examples

In this section we provide two numerical examples to illustrate the performance of the asymptotic upper bound $\mathbb{P}(\{\langle X\rangle > x\}) \le e^{-\frac{2\kappa}{S}(x+\frac{\kappa D}{S})}$. Our analytical results are compared with simulation results using the *Importance Sampling* technique described in [5], which has been developed to estimate the queue length distribution efficiently. Therefore, to estimate $\mathbb{P}(\{\langle X\rangle > x\})$, we use the fact that the supremum distribution of X is equal to the queue length distribution if ξ and κ are related to ν by (43). Also, in order to show the accuracy of the simulation estimates, 99% confidence intervals are computed by the method of batch mean [4], and displayed as vertical segments around the estimates of the tail probability.

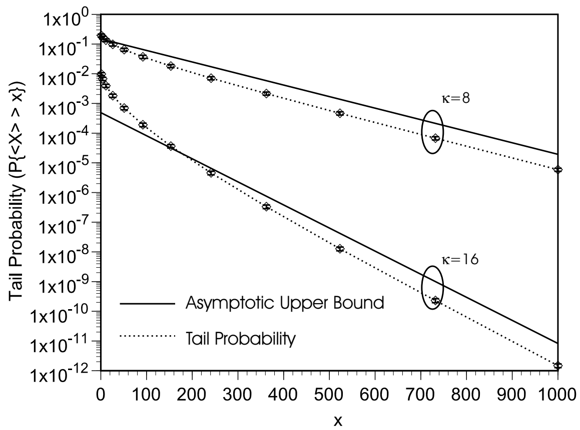

Figure 3. The tail probability $\mathbb{P}(\{\langle X\rangle > x\})$ estimated through simulation and its asymptotic upper bound $e^{-\frac{2\kappa}{S}(x+\frac{\kappa D}{S})}$ for a discrete-time process X_n expressed by (2). In this example, the autocovariance function of ξ_n is given as $C_\xi(l) = 25 \times 0.9^{|l|} + 20 \times 0.97^{|l|}$ and κ is set to two different values, 8 and 16.

In the first example, we consider a continuous-time process X_t given by (1) where ξ_t is a stationary Gaussian process with autocovariance function $C_\xi(\tau) = 80\times e^{-|\tau|}+40\times e^{-\left|\frac{\tau}{20}\right|}$. Since the (queueing) simulation with a Gaussian net input rate cannot be performed in continuous-time, we show the tail probability $\mathbb{P}(\{\langle \dot{X}\rangle > x\})$ instead of $\mathbb{P}(\{\langle X\rangle > x\})$ where $\dot{X}_n$ is the sampled sequence of X_t introduced in the previous section. More precisely, we set Δ to 0.05 to obtain $\dot{X}_n = X_{n\Delta}$ from X_t. In Figure 2, we compare the tail probabilities $\mathbb{P}(\{\langle \dot{X}\rangle > x\})$ estimated via simulation, and the asymptotic upper bounds given in Theorem 1 for $\kappa = 8$ and $\kappa = 16$. Remember that the decay rates of the exact tail probability and the asymptotic upper bound are equal to $-\frac{2\kappa}{S}$. Therefore, as one can see in the figure, the simulation and analytical curves are parallel to each other for large x. Also note that the the asymptotic upper bound is fairly close to the tail probability for sufficiently large x. Although the tail probability $\mathbb{P}(\{\langle X\rangle > x\})$ cannot be directly estimated through simulation, it is bounded by $\mathbb{P}(\{\langle \dot{X}\rangle > x\})$ from below. Hence, for this case, we can conclude that the envelope given by (46) is fairly narrow.

In the second example, we consider a discrete-time process X_n given by (2) where ξ_n is a stationary Gaussian process with its autocovariance function $C_\xi(l) = 25 \times 0.9^{|l|} + 20 \times 0.97^{|l|}$. In Figure 3, we show the tail probability and the asymptotic upper bound again for $\kappa = 8$ and $\kappa = 16$. As in the previous example, the exact tail probability curve estimated by simulation is parallel to the asymptotic upper bound for large values of x. Also, from the figure, we can deduce that the asymptote of the tail probability (as described by (44), there is an exponential asymptote of the tail probability in the discrete-time case) will be quite close to the bound. This suggests that $e^{-\frac{2\kappa^2 D}{S^2}}$ is a tight upper bound to the asymptotic constant C in (44) which can be used as a dimensioning parameter for network design and control. Extensive experimentation with a wide variety of different processes ξ_n has indicated that the upper bound to the asymptotic constant is usually quite tight [8]. It should be noted, however, that the asymptotic constant C is but one important parameter in network design and control. Using a single exponential approximation of the form in (44) may not be enough to accurately predict $\mathbb{P}(\{Q > x\})$ over a large range of x [6, 7, 9]. We are currently developing analytical techniques to address this problem.

References

[1] Addie, R. G. and Zukerman, M. (1994). An Approximation for Performance Evaluation of Stationary Single Server Queues. *IEEE Transactions on Communications* **42,** 3150–3160.

[2] Adler, R. J. (1990). *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes.* Institute of Mathematical Statistics, Hayward, CA.

[3] Berman, S. M. and Kono, N. (1989). The Maximum of a Gaussian Process with Nonconstant Variance: A Sharp Bound for The Distribution Tail. *The Annals of Probability* **17,** 632–650.

[4] Bratley, P., Fox, B. L. and Schrage, L. E. (1987). *A Guide to Simulation* second ed. Springer-Verlag, New York.

[5] Chang, C.-S., Heidelberger, P., Juneja, S. and Shahabuddin, P. (1994). Effective bandwidth and fast simulation of ATM intree networks. *Performance Evaluation* **20,** 45–65.

[6] Choe, J. and Shroff, N. B. (1997). A Central Limit Theorem Based Approach to Analyze Queue Behavior in ATM Networks. In *Proceedings of the 15th International Teletraffic Congress.*

[7] Choe, J. and Shroff, N. B. (1997). A New Method to Determine the Queue Length Distribution at an ATM Multiplexer. In *Proceedings of IEEE INFOCOM.* pp. 550–557.

[8] Choe, J. and Shroff, N. B. (1998). A Central Limit Theorem Based Approach for Analyzing Queue Behavior in High-Speed Networks. *Technical report.* Purdue University, West Lafayette, IN. submitted to *IEEE/ACM Transactions on Networking.*

[9] Choudhury, G. L., Lucantoni, D. M. and Whitt, W. (1996). Squeezing the Most Out of ATM. *IEEE Transactions on Communications* **44,** 203–217.

[10] Elwalid, A. I. (1991). Markov Modulated Rate Processes for Modeling, Analysis and Control of Communication Networks. *PhD thesis.* Graduate School of Arts and Sciences, Columbia University.

[11] Gihman, I. I. and Skorohod, A. V. (1974). *The Theory of Stochastic Processes. I.* Springer-Verlag, New York.

[12] Glynn, P. W. and Whitt, W. (1994). Logarithmic asymptotics for steady-state tail probabilities in a single-server queue. *Journal of Applied Probability* 131–155.

[13] Kesidis, G., Walrand, J., and Chang, C.-S. (1993). Effective Bandwidth for Multiclass Markov Fluid and other ATM Sources. *IEEE/ACM Transactions on Networking* **1,** 424–428.

[14] Loynes, R. M. (1962). The Stability of a Queue with Non-independent Inter-arrival and Service Times. *Proc. Cambridge Philos. Soc.* **58,** 497–520.

[15] Maglaris, B., Anastassiou, D., Sen, P., Karlsson, G. and Robbins, J. D. (1988). Performance Models of Statistical Multiplexing in Packet Video Communication. *IEEE Transactions on Communications* **36,** 834–843.

[16] Ross, S. M. (1983). *Stochastic Processes.* John Wiley & Son, New York.

[17] Samorodnitsky, G. (1991). Probability tails of Gaussian Extrema. *Stochastic Processes and their Applications* **38,** 55–84.

[18] Shroff, N. and Schwartz, M. (1996). Improved Loss Calculations at an ATM Multiplexer. In *Proceedings of IEEE INFOCOM.* vol. 2. pp. 561–568.

[19] Simonian, A. (1991). Stationary Analysis of a Fluid Queue with Input Rate Varying as an Ornstein-Uhlenbeck Process. *SIAM Journal on Applied Mathematics* **51,** 828–842.

[20] Talagrand, M. (1988). Small tails for the supremum of a Gaussian process. *Ann. Inst. Henri Poincare* **24,** 307–315.