The Nonlinear Output Regulation Problem

*Local and Structurally Stable Regulation*

Andrea Serrani

Department of Electrical and Computer Engineering
Collaborative Center for Control Sciences

The Ohio State University
Problem Formulation
Outline

- Problem Formulation
- The Regulator Equations
Outline

- Problem Formulation
- The Regulator Equations
- The Nonlinear Internal Model Principle
Outline

- Problem Formulation
- The Regulator Equations
- The Nonlinear Internal Model Principle
- System Immersion
- The Construction of a Local Regulator
Consider a nonlinear *plant model* of the form

\[
\dot{x} = f(x, u, w, \mu) \\
e = h(x, w, \mu)
\]

with state \(x \in \mathbb{R}^n\), control input \(u \in \mathbb{R}^m\), and error to be regulated \(e \in \mathbb{R}^m\).
Consider a nonlinear plant model of the form

\[
\begin{align*}
\dot{x} &= f(x, u, w, \mu) \\
e &= h(x, w, \mu)
\end{align*}
\]

with state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), and error to be regulated \( e \in \mathbb{R}^m \). The vector \( \mu \in \mathbb{R}^p \) is a vector of constant unknown parameters.
Problem formulation

Consider a nonlinear **plant model** of the form

\[
\dot{x} = f(x, u, w, \mu) \\
e = h(x, w, \mu)
\]

with state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), and error to be regulated \( e \in \mathbb{R}^m \). The vector \( \mu \in \mathbb{R}^p \) is a vector of constant unknown parameters.

The signal \( w \) is generated by a nonlinear **exosystem** of the form

\[
\dot{w} = s(w)
\]

with state \( w \in \mathbb{R}^d \).
Standing Assumptions

The plant model is assumed to satisfy the following assumptions:

- The functions \( f(x, u, w, \mu) \) and \( h(x, w, \mu) \) are smooth.
- The nominal value of the parameter \( \mu \) is \( \mu = 0 \).
- \( f(0, 0, 0, \mu) = 0 \) and \( h(0, 0, \mu) = 0 \) for all \( \mu \) in an open neighborhood \( P \) of \( \mu = 0 \).
- The pair \((A, B)\) is stabilizable and the pair \((C, A)\) is detectable, where

\[
A = \left[ \frac{\partial f}{\partial x} \right]_0, \quad B = \left[ \frac{\partial f}{\partial u} \right]_0, \quad C = \left[ \frac{\partial h}{\partial x} \right]_0.
\]
Standing Assumptions

The exosystem is assumed to be neutrally stable:
- The equilibrium $w = 0$ is stable in the sense of Lyapunov
- Each initial state $w_0 \in \mathcal{W}$ is stable in the sense of Poisson

Note that this implies that

$$S = \left[ \frac{\partial s}{\partial w} \right]_0$$

has all eigenvalues on the imaginary axis.

Caveat: This excludes interesting situations in which $w = s(w)$ generates stable limit cycles. For such a case the theory is still incomplete, although results have started to appear (see Byrnes and Isidori, IEEE Tr-AC 48(10), 2003.)
The problem of **local and structurally stable regulation** is to find a smooth controller of the form

\[
\dot{\xi} = \phi(\xi, e) \\
u = \theta(\xi),
\]

with \( \xi \in \mathbb{R}^\nu \), satisfying \( \phi(0, 0) = 0 \), \( \theta(0, 0) = 0 \), and

\[
F = \left[ \frac{\partial \phi}{\partial \xi} \right]_0, \quad G = \left[ \frac{\partial \phi}{\partial e} \right]_0, \quad H = \left[ \frac{\partial \theta}{\partial \xi} \right]_0,
\]

such that
Problem Formulation

- The origin is a locally exponentially stable equilibrium of the unforced closed loop system

\[
\begin{align*}
\dot{x} &= f(x, \theta(\xi), 0, \mu) \\
\dot{\xi} &= \phi(\xi, h(x, 0, \mu))
\end{align*}
\]

for all \( \mu \) in an open neighborhood \( \mathcal{P} \subset \mathbb{R}^p \) of \( \mu = 0 \).
Problem Formulation

- The trajectories of the closed loop system

\[
\begin{align*}
\dot{w} &= s(w) \\
\dot{x} &= f(x, \theta(\xi), 0, \mu) \\
\dot{\xi} &= \phi(\xi, h(x, w, \mu)) \\
e &= h(x, w, \mu)
\end{align*}
\]

originating within a neighborhood \( \mathcal{W} \times \mathcal{X} \times \Xi \subset \mathbb{R}^{d+n+\nu} \) of the origin are bounded and satisfy

\[
\lim_{t \to \infty} h(x(t), w(t), \mu) = 0
\]

for all \( \mu \) in an open neighborhood \( \mathcal{P} \subset \mathbb{R}^p \) of \( \mu = 0 \).
Solvability of the Problem

Since $\mu$ satisfies

$$\dot{\mu} = 0$$

the role of $w$ and that of $\mu$ need not be kept separate. As a result, we incorporate $\mu$ within the exosystem state $w$. 
Solvability of the Problem

Since $\mu$ satisfies

\[ \dot{\mu} = 0 \]

the role of $w$ and that of $\mu$ need not be kept separate. As a result, we incorporate $\mu$ within the exosystem state $w$.

The closed-loop system can be written as

\[
\begin{align*}
\dot{x} &= Ax + BH\xi + Pw + \varphi(x, \xi, w) \\
\dot{\xi} &= GCx + F\xi + GQw + \chi(x, \xi, w) \\
\dot{w} &= Sw + \psi(w)
\end{align*}
\]

for all $(x, \xi, w) \in \mathcal{X} \times \Xi \times \mathcal{W}$, where $\varphi(x, \xi, w)$, $\chi(x, \xi, w)$, and $\psi(w)$ vanish at the origin with their first derivatives.
Assume that \( \{\phi, \theta\} \) locally exponentially stabilizes the origin of the unforced closed-loop system. Then

\[
A_{cl} = \begin{pmatrix}
A & BH & P \\
GC & F & GQ \\
0 & 0 & S
\end{pmatrix} = \begin{pmatrix}
J & * \\
0 & S
\end{pmatrix}
\]

with

\[
\text{spec}\{J\} \subset \mathbb{C}^{-}, \quad \text{spec}\{S\} \subset \mathbb{C}^{0}.
\]
Assume that $\{\phi, \theta\}$ locally exponentially stabilizes the origin of the unforced closed-loop system. Then

$$A_{cl} = \begin{pmatrix} A & BH \\ GC & F \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P \\ GQ \end{pmatrix} = \begin{pmatrix} J \ast \\ 0 \end{pmatrix}$$

with

$$\text{spec}\{J\} \subset \mathbb{C}^{-}, \quad \text{spec}\{S\} \subset \mathbb{C}^{0}.$$

As a result, the system has a center manifold at the origin, that is, a $d$-dimensional hypersurface

$$\mathcal{M} = \{(x, \xi, w) \in \mathbb{R}^{n+\nu+d} : x = \pi(w), \xi = \sigma(w), \, w \in \mathcal{W}\}.$$
Solvability of the Problem

The center manifold $M$ has the following properties:
- It is invariant w.r.t. the flow of the closed-loop system.
Solvability of the Problem

The center manifold $\mathcal{M}$ has the following properties:

- It is invariant w.r.t. the flow of the closed-loop system.
- The restriction of the flow of the closed-loop system to $\mathcal{M}$ is diffeomorphic to that of the exosystem.
Solvability of the Problem

The center manifold $\mathcal{M}$ has the following properties:

- It is invariant w.r.t. the flow of the closed-loop system.
- The restriction of the flow of the closed-loop system to $\mathcal{M}$ is diffeomorphic to that of the exosystem.
- $\mathcal{M}$ is tangent at the origin to the center subspace $\mathcal{V}^0$:

$$
\pi(0) = 0, \quad \sigma(0) = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial w}(0) = 0, \quad \frac{\partial \sigma}{\partial w}(0) = 0.
$$
Solvability of the Problem

The center manifold \( M \) has the following properties:

- It is invariant w.r.t. the flow of the closed-loop system.
- The restriction of the flow of the closed-loop system to \( M \) is diffeomorphic to that of the exosystem.
- \( M \) is tangent at the origin to the center subspace \( V^0 \):

\[
\pi(0) = 0, \quad \sigma(0) = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial w}(0) = 0, \quad \frac{\partial \sigma}{\partial w}(0) = 0.
\]

- \( M \) is locally exponentially attractive, i.e.,

\[
\lim_{t \to \infty} \| x(t) - \pi(w(t)) \| = 0, \quad \lim_{t \to \infty} \| \xi(t) - \sigma(w(t)) \| = 0
\]

for all \((x(0), \xi(0), w(0)) \in \mathcal{X} \times \Xi, \times \mathcal{W}.\)
The Center Manifold

\[(x_0, \xi_0, w_0)\]

\[(x(t), \xi(t), w(t))\]

\(\pi(w(t)), \sigma(w(t))\)

The condition of invariance of $\mathcal{M}$ is expressed by the homology equations

\[
\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \theta(\sigma(w)), w), \quad \frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), h(\pi(w), w))
\]

which hold for all $w \in \mathcal{W}$.
Solvability of the Problem

The condition of invariance of $\mathcal{M}$ is expressed by the homology equations

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \theta(\sigma(w)), w), \quad \frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), h(\pi(w), w))$$

which hold for all $w \in \mathcal{W}$. The system dynamics reduced to the center manifold is that of the exosystem

$$\dot{w} = s(w), \quad w(0) \in \mathcal{W}$$
Solvability of the Problem

The condition of invariance of $\mathcal{M}$ is expressed by the homology equations

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \theta(\sigma(w)), w), \quad \frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), h(\pi(w), w))$$

which hold for all $w \in \mathcal{W}$. The system dynamics reduced to the center manifold is that of the exosystem

$$\dot{w} = s(w), \quad w(0) \in \mathcal{W}$$

and on $\mathcal{M}$ the error reads as $e(t) = h(\pi(w(t)), w(t))$. 
Solvability of the Problem

The condition of invariance of $\mathcal{M}$ is expressed by the homology equations

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \theta(\sigma(w)), w), \quad \frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), h(\pi(w), w))$$

which hold for all $w \in \mathcal{W}$. The system dynamics reduced to the center manifold is that of the exosystem

$$\dot{w} = s(w), \quad w(0) \in \mathcal{W}$$

and on $\mathcal{M}$ the error reads as $e(t) = h(\pi(w(t)), w(t))$. Since the exosystem is Poisson stable

$$\lim_{t \to \infty} e(t) = 0 \iff h(\pi(w), w) = 0 \quad \forall w \in \mathcal{W}$$
Theorem 1 (Isidori and Byrnes, 1990) A controller which locally exponentially stabilizes the plant achieves regulation if only if there exist mappings \( \pi : \mathcal{W} \rightarrow \mathbb{R}^n \) and \( \sigma : \mathcal{W} \rightarrow \mathbb{R}^\nu \), with \( \pi(0) = 0 \) and \( \sigma(0) = 0 \) such that

\[
\begin{align*}
\frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), \theta(\sigma(w)), w) \\
\frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), 0) \\
0 &= h(\pi(w), w)
\end{align*}
\]

for all \( w \in \mathcal{W} \).
The Regulator Equations

The previous equations can be split into two sets of equations as follows:

\[
\begin{align*}
\frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\
0 &= h(\pi(w), w) \\
\frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), 0, w) \\
c(w) &= \theta(\sigma(w))
\end{align*}
\]

where the mapping \( c : \mathcal{W} \rightarrow \mathbb{R}^m \) satisfies \( c(0) = 0 \).
The Regulator Equations

\[ \frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w) \]
\[ 0 = h(\pi(w), w) \]
\[ \frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), 0) \]
\[ c(w) = \theta(\sigma(w)) \]

- **Regulator Equations**: analogous to

\[ \Pi S = A \Pi + B R + P \]
\[ 0 = C \Pi + Q, \]
The Regulator Equations

\[
\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w) \\
0 = h(\pi(w), w)
\]

\[
\frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), 0) \\
c(w) = \theta(\sigma(w))
\]

**Internal Model Principle**: analogous to

\[
\Sigma S = F \Sigma \\
R = H \Sigma
\]
Necessary Condition

The first equation yields a necessary condition for regulation

**Theorem 2 (Isidori and Byrnes, 1990)**  
*The local output regulation problem is solvable only if there exist mappings* $\pi: \mathcal{W} \rightarrow \mathbb{R}^n$ and $c: \mathcal{W} \rightarrow \mathbb{R}^m$, with $\pi(0) = 0$ and $c(0) = 0$ *such that*

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w)$$

$$0 = h(\pi(w), w)$$

*for all* $w \in \mathcal{W}$,
The first equation yields a necessary condition for regulation.

**Theorem 2 (Isidori and Byrnes, 1990)** The local output regulation problem is solvable only if there exist mappings \( \pi : \mathcal{W} \to \mathbb{R}^n \) and \( c : \mathcal{W} \to \mathbb{R}^m \), with \( \pi(0) = 0 \) and \( c(0) = 0 \) such that

\[
\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w)
\]

\[ 0 = h(\pi(w), w) \]

for all \( w \in \mathcal{W} \), that is, only if there exists a controlled-invariant submanifold \( \mathcal{M}_0 \subset \mathbb{R}^{n+d} \) satisfying

\[
\mathcal{M}_0 \subset \{(x, w) : h(x, w) = 0\}.
\]
Geometric Picture

\[ u = 0 \]

\[ (x(t), w(t)) \]

\( (x_0, w_0) \)

\[ M_0 \]

\[ e(t) \]

Output Regulation of Nonlinear Systems. CeSOS-NTNU 2005 – p.17/28
\[ u(t) = c(w(t)) \]

\( (x_0, w_0) \)

\( (x(t), w(t)) \)

\( \mathcal{M}_0 \)
Any controller must render $\mathcal{M}_0$ invariant and attractive.
Sufficient Conditions

How far is the condition of Theorem 2 from being sufficient?

- Attractivity of $M_0$ is guaranteed by the properties of the center manifold (by local exponential stability of the origin)
Sufficient Conditions

How far is the condition of Theorem 2 from being sufficient?

- Attractivity of $\mathcal{M}_0$ is guaranteed by the properties of the center manifold (by local exponential stability of the origin)
- The capability of the controller to “reconstruct” $c(w)$ is the real issue.
How far is the condition of Theorem 2 from being sufficient?

- Attractivity of $M_0$ is guaranteed by the properties of the center manifold (by local exponential stability of the origin).
- The capability of the controller to “reconstruct” $c(w)$ is the real issue.
- Constructing a controller that satisfies the internal model property is not easy. Further conditions are needed.
Sufficient Conditions

How far is the condition of Theorem 2 from being sufficient?

- Attractivity of $\mathcal{M}_0$ is guaranteed by the properties of the center manifold (by local exponential stability of the origin)
- The capability of the controller to “reconstruct” $c(w)$ is the real issue.
- Constructing a controller that satisfies the internal model property is not easy. Further conditions are needed.
- A crucial role is played by the notion of system immersion.
System Immersion

**Definition 1** Given two systems with same output space

\[
\begin{align*}
\dot{x} &= f(x), \quad x \in \mathcal{X} \\
y &= h(x), \quad y \in \mathbb{R}^m
\end{align*}
\]

\[
\begin{align*}
\dot{X} &= F(X), \quad X \in \mathcal{X} \\
Y &= H(X), \quad Y \in \mathbb{R}^m
\end{align*}
\]

we say that \(\{\mathcal{X}, f, h\}\) is immersed into \(\{\mathcal{X}, F, H\}\) if there exists a smooth mapping \(\tau : \mathcal{X} \to \mathcal{X}\) satisfying \(\tau(0) = 0\) and

\[
\frac{\partial \tau}{\partial x} f(x) = F(\tau(x))
\]

\[
h(x) = H(\tau(x))
\]

for all \(x \in \mathcal{X}\).
System Immersion

**Definition 1** \textit{Given two systems with same output space}

\[
\begin{align*}
\dot{x} &= f(x), \quad x \in \mathcal{X} \\
y &= h(x), \quad y \in \mathbb{R}^m
\end{align*}
\]

we say that \( \{\mathcal{X}, f, h\} \) is \textbf{immersed into} \( \{X, F, H\} \) if there exists a smooth mapping \( \tau : \mathcal{X} \rightarrow X \) satisfying \( \tau(0) = 0 \) and

\[
\frac{\partial \tau}{\partial x} f(x) = F(\tau(x))
\]

\[
h(x) = H(\tau(x))
\]

for all \( x \in \mathcal{X} \).

\textbf{NOTE:} \( \tau \) need not be a diffeomorphism, as \( \dim \mathcal{X} \leq \dim X \).
This means that the flows of the systems are $\tau$-related and

$$h \circ \Phi^f_t(x) = H \circ \tau \circ \Phi^f_t(x) = H \circ \Phi^F_t(\tau(x)).$$

Any output trajectory of $\{X, f, h\}$ is an output trajectory of $\{X, F, H\}$.
Consider the exosystem with output map $y \in \mathbb{R}$

$$\dot{w} = s(w)$$
$$y = c(w).$$
Consider the exosystem with output map $y \in \mathbb{R}$

$$\dot{w} = s(w)$$
$$y = c(w).$$

If there exists $q \in \mathbb{N}$ and a smooth function $\alpha : \mathbb{R}^q \rightarrow \mathbb{R}$ s.t.

$$L_s^q c(w) = \alpha (c(w), L_s c(w), \ldots, L_{s}^{q-1} c(w))$$

for all $w \in \mathcal{W}$, then the exosystem is immersed into $\{\varphi, \gamma\}$

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \xi_3$$
$$\vdots$$
$$\dot{\xi}_q = \alpha (\xi_1, \xi_2, \ldots, \xi_{q-1}) , \quad y = \xi_1.$$
Consider the exosystem with output map \( y \in \mathbb{R} \)

\[
\begin{align*}
\dot{w} &= s(w) \\
y &= c(w).
\end{align*}
\]

If there exists \( q \in \mathbb{N} \) and \( a_i \in \mathbb{R}, i = 0, \ldots, q - 1 \) such that

\[
L_s^q c(w) + a_{q-1} L_s^{q-1} c(w) + \cdots + a_1 L_s c(w) + a_0 c(w) = 0
\]

for all \( w \in \mathcal{W} \), then the exosystem is immersed into \( \{ \Phi, \Gamma \} \)

\[
\Phi = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & \cdots & -a_{q-1}
\end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}
\]
Example

An observable LTI immersion always exists if

- The exosystem is linear, $\dot{w} = Sw$. 

Output Regulation of Nonlinear Systems. CeSOS-NTNU 2005 – p.22/28
An observable LTI immersion always exists if
- The exosystem is linear, $\dot{w} = Sw$.
- The mapping $c(w)$ is a polynomial in the components of $w$. 
An observable LTI immersion always exists if
- The exosystem is **linear**, \( \dot{w} = Sw \).
- The mapping \( c(w) \) is a **polynomial** in the components of \( w \).

Since the set \( \mathbb{P} \) of polynomials is a **linear vector space** over \( \mathbb{R} \), and the mapping \( D_s : \mathbb{P} \rightarrow \mathbb{P} \) given by

\[
c(w) \rightarrow L_sc(w) = \frac{\partial c}{\partial w} s(w)
\]

is **linear**, there exist an integer \( q \) and real numbers \( a_i \in \mathbb{R} \), \( i = 0, \ldots, q - 1 \) such that

\[
D^q_s + a_{q-1} D^{q-1}_s + \cdots + a_1 D_s + a_0 I = 0
\]
Theorem 3  The Error Feedback Output Regulation Problem is solvable if and only if

- There exist mappings $x = \pi(w)$ and $u = c(w)$, with $\tau(0) = 0$ and $c(0) = 0$, satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w), w)$$

$$0 = h(\pi(w), w)$$

for all $w \in \mathcal{W}$. 

**Theorem 3** The Error Feedback Output Regulation Problem is solvable if and only if

- **The autonomous system** $\{\mathcal{W}, s, c\}$ **is immersed into a system**

$$
\begin{align*}
\dot{\xi} &= \varphi(\xi), \quad \xi \in \Xi \subset \mathbb{R}^\nu \\
u &= \gamma(\xi)
\end{align*}
$$

**in which** $\varphi(0) = 0$ **and** $\gamma(0) = 0$, **such that the linear approximation**

$$
\Phi = \left[ \frac{\partial \varphi}{\partial \xi} \right]_0, \quad \Gamma = \left[ \frac{\partial \gamma}{\partial \xi} \right]_0,
$$

**satisfies the following property:**
Theorem 3  The Error Feedback Output Regulation Problem is solvable if and only if

- The pair

\[
\left( \begin{array}{cc}
A & 0 \\
\Theta C & \Phi \\
\end{array} \right), \quad \left( \begin{array}{c}
B \\
0 \\
\end{array} \right)
\]

is stabilizable for some choice of the matrix \( \Theta \in \mathbb{R}^{\nu \times m} \), and

the pair

\[
\left( \begin{array}{cc}
C & 0 \\
0 & \Phi \\
\end{array} \right), \quad \left( \begin{array}{cc}
A & B \Gamma \\
0 & \Phi \\
\end{array} \right)
\]

is detectable.
Proof

*Necessity.* Given a regulator \( \{\phi, \theta\} \), there exist mappings \( x = \pi(w) \) and \( \xi = \sigma(w) \) solving the regulator equations.
Proof

**Necessity.** Given a regulator \( \{ \phi, \theta \} \), there exist mappings \( x = \pi(w) \) and \( \xi = \sigma(w) \) solving the regulator equations. Set

\[
c(w) = \theta(\sigma(w)), \quad \gamma(\xi) = \theta(\xi), \quad \varphi(\xi) = \phi(\xi, 0)
\]

and note that the system \( \{ \mathcal{W}, s, c \} \) is immersed into \( \{ \Xi, \varphi, \gamma \} \), with immersion mapping \( \tau(w) = \sigma(w) \).
Proof

**Necessity.** Given a regulator \( \{ \phi, \theta \} \), there exist mappings \( x = \pi(w) \) and \( \xi = \sigma(w) \) solving the regulator equations. Set

\[
c(w) = \theta(\sigma(w)), \quad \gamma(\xi) = \theta(\xi), \quad \varphi(\xi) = \phi(\xi, 0)
\]

and note that the system \( \{ \mathcal{W}, s, c \} \) is immersed into \( \{ \Xi, \varphi, \gamma \} \), with immersion mapping \( \tau(w) = \sigma(w) \).

Since

\[
\begin{pmatrix}
A & BH \\
GC & F
\end{pmatrix}
= \begin{pmatrix}
A & B\Gamma \\
\Theta C & \Phi
\end{pmatrix}, \quad \Theta = G
\]

is Hurwitz, the given pairs are stabilizable and detectable.
Proof

*Sufficiency.* Since

\[
\begin{pmatrix}
A & B \Gamma \\
\Theta C & \Phi
\end{pmatrix}, \quad \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \begin{pmatrix} C & 0 \end{pmatrix}
\]

is stabilizable and detectable, there exist \( L, M, N \) such that

\[
\begin{pmatrix}
A & B \Gamma & B N \\
\Theta C & \Phi & 0 \\
M C & 0 & L
\end{pmatrix}
\]

is Hurwitz.
Proof

**Sufficiency.** Since

\[
\begin{pmatrix}
A & B \Gamma \\
\Theta C & \Phi
\end{pmatrix}, \quad \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad (C \ 0)
\]

is stabilizable and detectable, there exist \( L, M, N \) such that

\[
\begin{pmatrix}
A & B \Gamma & B N \\
\Theta C & \Phi & 0 \\
M C & 0 & L
\end{pmatrix}
\]

is Hurwitz.

Define the controller

\[
\begin{align*}
\dot{\xi}_0 &= \varphi(\xi_0) + \Theta e \\
\dot{\xi}_1 &= L\xi_1 + Me \\
u &= \gamma(\xi_0) + N\xi_1
\end{align*}
\]
Proof

The controller solves the local output regulation problem:

The Jacobian matrix of the unforced closed-loop system

\[ f_{cl}(x, \xi, 0) = \begin{pmatrix} f(x, \gamma(\xi_0) + N\xi_1, 0) \\ \varphi(\xi_0) + \Theta h(x, 0) \\ L\xi_1 + Mh(x, 0) \end{pmatrix} \]

is precisely

\[ \begin{pmatrix} A & B\Gamma & BN \\ \Theta C & \Phi & 0 \\ MC & 0 & L \end{pmatrix}. \]
Proof

The controller solves the local output regulation problem:

- The mappings

\[ x = \pi(w), \quad u = c(w) \quad \text{(given)} \]

and

\[
\begin{pmatrix}
\xi_0 \\
\xi_1
\end{pmatrix} = \sigma(w) = \begin{pmatrix}
\tau(x) \\
0
\end{pmatrix}
\]

solve the regulator equations.
The regulator is given as the parallel interconnection of an internal model and a stabilizer.

- The internal model provides $u = c(w)$ on the set $\mathcal{M}_0$.
- The stabilizer locally exponentially stabilizes the origin of the closed-loop system, and induces local exponential attractivity of $\mathcal{M}_0$. 

$$\dot{\xi}_0 = \varphi(\xi_0) + \Theta e$$
$$u_{im} = \gamma(\xi_0)$$

$$\dot{\xi}_1 = L\xi_1 + Me$$
$$u_{st} = N\xi_1$$
Conclusions

- The solvability of the local output regulation problem is given in terms of the existence of a controlled-invariant submanifold contained in the kernel of the error map.
Conclusions

- The solvability of the local output regulation problem is given in terms of the existence of a controlled-invariant submanifold contained in the kernel of the error map.
- Any controller must necessarily render the submanifold invariant and attractive.
Conclusions

- The solvability of the local output regulation problem is given in terms of the existence of a controlled-invariant submanifold contained in the kernel of the error map.
- Any controller must necessarily render the submanifold invariant and attractive.
- The regulator equation is a set of PDEs.
Conclusions

- The solvability of the local output regulation problem is given in terms of the existence of a controlled-invariant submanifold contained in the kernel of the error map.
- Any controller must necessarily render the submanifold invariant and attractive.
- The regulator equation is a set of PDEs.
- The concept of system immersion is fundamental in obtaining the internal model property.
Conclusions

- The solvability of the local output regulation problem is given in terms of the existence of a controlled-invariant submanifold contained in the kernel of the error map.
- Any controller must necessarily render the submanifold invariant and attractive.
- The regulator equation is a set of PDEs.
- The concept of system immersion is fundamental in obtaining the internal model property.
- Local regulation is then a byproduct of local exponential stabilization (the error-zeroing submanifold is a center manifold)
Conclusions

- The solvability of the local output regulation problem is given in terms of the existence of a controlled-invariant submanifold contained in the kernel of the error map.
- Any controller must necessarily render the submanifold invariant and attractive.
- The regulator equation is a set of PDEs.
- The concept of system immersion is fundamental in obtaining the internal model property.
- Local regulation is then a byproduct of local exponential stabilization (the error-zeroing submanifold is a center manifold).
- What is required to extend these results beyond local validity?