Max-Diversity Affine Precoding for the Noncoherent Doubly Dispersive Channel

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Motivating Questions:

1. What is the maximum achievable diversity order for communication over an *unknown* time/frequency-selective channel?

2. How should the transmitted signal be designed to facilitate maximum diversity reception?
System Model:

\[ r_n = \sum_{l=0}^{N_h-1} h_{n,l} c_{n-l} + w_n \]

\( r := [r_0, \ldots, r_{N-1}]^T \): received samples

\( c := [c_0, \ldots, c_{N-1}]^T \): coded symbols

\( w := [w_0, \ldots, w_{N-1}]^T \): noise samples, \( \mathcal{CN}(0, \sigma^2 I) \)

\( H \): LTV channel matrix

\[
H = \begin{bmatrix}
    h_{0,0} & 0 & \cdots & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
    h_{N_h-1,N_h-1} & \cdots & h_{N_h-1,0} & 0 & \cdots & 0 \\
    0 & h_{N_h,N_h-1} & \cdots & h_{N_h,0} & 0 & \cdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & h_{N-1,N_h-1} & \cdots & h_{N-1,0}
\end{bmatrix}
\]
Karhunen-Loève Basis Expansion Model:

KL-BEM of $l^{th}$-tap trajectory over $N$-sample block:

$$h_l := \begin{bmatrix} h_{0,l} \\ \vdots \\ h_{N-1,l} \end{bmatrix} = B_l \theta_l, \quad \theta_l \in \mathbb{C}^{N_b}$$

$N_b$ : Temporal degrees-of-freedom per tap

WSSUS Rayleigh channel assumption:

$$N_h : \text{Number of taps}$$

$$B = B_l \quad \forall l \in \{0, \ldots, N_h - 1\}$$

$$\theta := \begin{bmatrix} \theta_0 \\ \vdots \\ \theta_{N_h-1} \end{bmatrix} \sim \mathcal{C}\mathcal{N}(0, R_\theta), \quad R_\theta \text{ has full rank } N_h N_b$$
Per-tap DoF $N_b$ vs. normalized Doppler spread $f_D T_s$:

# of eigenvalues within 30dB of largest eigenvalue (Jakes spectrum):

Notice that $N_b \ll N$. 
Noncoherent ML Decoding:

Goal: Estimate $c \in \mathcal{C}$ from $r = Hc + w$ assuming $\{H, w\}$ are unknown but statistics $\{B, R_\theta, \sigma^2\}$ are known.

Writing the received vector as

$$r = C_B \theta + w,$$

where matrix $C_B$ is composed of coded symbols $c$ and basis vectors $B$, the noncoherent ML estimate can be written

$$\hat{c}_{\text{ML}} = \arg \min_{c \in \mathcal{C}} r^H \Phi r - \sigma^2 \log \det(C_B^H C_B + \sigma^2 R_\theta^{-1})$$

$\mathcal{C}$ : set of code vectors

$$\Phi := (C_B R_\theta C_B^H + \sigma^2 I_N)^{-1}$$
Pair-Wise Error Probability:

**Lemma 1**  Say \(\{C_B^{(k)}, C_B^{(l)}\}\) are two possibilities for \(C_B\). If the matrix

\[
M_{kl} := C_B^{(k)}H(I_N - C_B^{(l)}(C_B^{(l)}H C_B^{(l)})^{-1}C_B^{(l)}H)C_B^{(k)}
\]

is full rank, then, at high SNR,

\[
PWEP_{kl} = \left(\frac{1}{\sigma^2}\right)^{-N_hN_b} \det(R_\theta M_{kl})^{-1} \left(\frac{2N_hN_b - 1}{N_hN_b}\right).
\]

Furthermore, \(M_{kl}\) has full rank \(N_hN_b\) if and only if \([C_B^{(k)}, (C_B^{(l)} - C_B^{(k)})]\) has full rank \(2N_hN_b\).

**Main points:**

1. \(N_hN_b\) is the maximum achievable diversity order.
2. Max-diversity requires full-rank \([C_B^{(k)}, (C_B^{(l)} - C_B^{(k)})] \forall k \neq l\) which requires \(N \geq 2N_hN_b\).
Linear Precoding:

Say that the code vectors are generated via

\[ c = Ps \]

\[ s \in S \subset \mathbb{C}^{N_s} \]

where \( P \in \mathbb{C}^{N \times N_s} \) is a linear precoding matrix and \( s \) is a symbol vector.

Lemma 2  Linear precoding does not facilitate maximum-diversity decoding whenever the symbol vector alphabet \( S \) contains elements that differ by no more than a scale factor (e.g., uncoded QAM or PSK).
Affine Precoding:

\[ c = Ps + t \quad s \in S \subset \mathbb{C}^{Ns}. \]

**Lemma 3** If \( N \geq 2N_hN_b \) and if the matrix created from the last \( N - N_h + 1 \) rows of \( B \) is full rank, then choosing \([P, t]\) randomly ensures that \([C_B^{(k)}, (C_B^{(l)} - C_B^{(k)})]\) is full-rank w.p.1.

**Main points:**

1. *Almost any* affine precoder provides maximum diversity!
2. There are no restrictions on the data rate \( N_s/N! \)
3. The rank condition on \( B \) is mild. (It requires that the first \( N_h - 1 \) samples of the \( N \)-sample tap trajectory are non-essential to experiencing the \( N_b \) degrees-of-freedom.)
Numerical Example:

SNR (dB) vs. avg PWEP for different values of $N_s$ (number of branches) and $N_h$ (number of paths). The graph shows the performance of BPSK modulation with Jakes fading model.

- $N = 8$
- $N_h = 2$
- $N_b = 2$
- $S = $ BPSK
**Systematic Affine Precoding:**

\[ c = Ps + t \]  with  \[ P = \begin{bmatrix} I_{Ns} \\ P' \end{bmatrix}, \quad P' \in \mathbb{C}^{N_p \times N_s} \]

**Lemma 4**  If \( N_p \geq N_hN_b - 1 \), if \( N \geq 2N_hN_b \), and if matrix created from the last \( N_p - N_h + 1 \) rows of \( B \) is full-rank, then choosing \([P', t]\) randomly ensures that \([C_B^{(k)}, (C_B^{(l)} - C_B^{(k)})]\) is full-rank w.p.1.

**Main points:**

1. Systematic affine precoding facilitates fast decoding.
2. With \( N_p \geq N_hN_b - 1 \), almost any precoder provides max-diversity!
3. Rate limitation: \( \frac{Ns}{N} \leq 1 - \frac{N_hN_b - 1}{N} \).
4. As before, the rank condition on \( B \) is mild.
Conclusions:

For noncoherent communication over a WSSUS time/frequency-selective channel with \( N_h \) delay taps and \( N_b \) temporal degrees-of-freedom per tap,

1. the maximum diversity order is \( N_h N_b \),

2. block lengths \( N \geq 2N_h N_b \) facilitate max-diversity reception,

3. linear precoding does not facilitate max-diversity reception,

4. almost any affine precoder facilitates max-diversity reception at any rate \( \frac{N_s}{N} \),

5. systematic affine precoding facilitates max-diversity at rates \( \frac{N_s}{N} \leq 1 - \frac{N_h N_b - 1}{N} \) while simplifying the decoding task.