# Geometrical Interpretations of the MOE Receiver

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## 1 Introduction

The minimum output energy (MOE) receiver [1] is a specification of an FIR linear receiver that may be used to accomplish blind symbol estimation in DS-CDMA applications. Assuming baud frequency and carrier frequency lock, the only parameters assumed known are the code vector and symbol timing of the desired user. (We note that MOE concepts have been applied to the estimation of symbol timing, though these issues are not investigated here.) Notably, the unknown information includes propagation channels as well as other users' codes and timing.

It is well known that, in the absence of multipath propagation, the MOE and MMSE receivers coincide. In the sequel, we analyze the difference between the MOE and MMSE receivers when in the presence of multipath.

We assume the following noiseless DS-CDMA model. The elements of the $k^{th}$ user's symbol sequence $\{s_n^{(k)}\}$ are multiplied by a code vector $\mathbf{c}^{(k)}$ of one symbol duration and of length $N$. The resulting chip-rate output sequence then passes through a linear convolutive channel defined by the length-$L_h$ impulse response vector $\mathbf{h}^{(k)}$. The receiver sees the superposition of $K$ such user signals and forms a linear estimate of the $k^{th}$ user's symbol sequence. Specifically, $s_n^{(k)}$ is estimated from the previous $L_r$ chip-spaced received samples at symbol index $n$, which are collected in the vector $\mathbf{r}(n)$. The received vector can be written

$$\mathbf{r}(n) \;=\; \sum_{k=1}^{K} \mathbf{H}^{(k)}\mathbf{C}^{(k)}\mathbf{s}^{(k)}(n), \tag{1}$$

where $\mathbf{H}^{(k)}$ is a channel convolution matrix constructed from $\mathbf{h}^{(k)}$, $\mathbf{C}^{(k)}$ is a block-diagonal code matrix:

$$\mathbf{H}^{(k)} \triangleq \begin{pmatrix} h_0^{(k)} & h_1^{(k)} & \dots & h_{L_h-1}^{(k)} & & & \\ & h_0^{(k)} & h_1^{(k)} & \dots & h_{L_h-1}^{(k)} & & \\ & & \ddots & & & \ddots & \\ & & & h_0^{(k)} & h_1^{(k)} & \dots & h_{L_h-1}^{(k)} \end{pmatrix}, \quad \mathbf{C}^{(k)} \triangleq \begin{pmatrix} \mathbf{c}^{(k)} & & & \\ & \mathbf{c}^{(k)} & & \\ & & \ddots & \\ & & & \mathbf{c}^{(k)} \end{pmatrix},$$

and $\mathbf{s}^{(k)}(n)$ is a vector comprised of all user-$k$ symbols contributing to $\mathbf{r}(n)$:

$$\mathbf{s}^{(k)}(n) \;\triangleq\; (s_n^{(k)}, \dots, s_{n-L_s+1}^{(k)})^t.$$

Even more compact notation is obtained by concatenating the user quantities as follows:

$$\mathcal{H} \;\triangleq\; \big(\mathbf{H}^{(1)}\mathbf{C}^{(1)}, \mathbf{H}^{(2)}\mathbf{C}^{(2)}, \dots \mathbf{H}^{(K)}\mathbf{C}^{(K)}\big), \tag{2}$$

$$\mathbf{s}(n) \;\triangleq\; \big(\mathbf{s}^{(1)t}(n), \mathbf{s}^{(2)t}(n), \dots, \mathbf{s}^{(K)t}(n)\big)^t. \tag{3}$$

so that

$$\mathbf{r}(n) \;=\; \mathcal{H}\mathbf{s}(n). \tag{4}$$

In the derivations below, we assume that the source sequences $\{s_n^{(k)}\}$ (for $k = 1, \dots, K$) are mutually uncorrelated, zero mean, and unit variance. We assume that $\mathcal{H}$ is full row rank, so that $\mathbf{R} = \mathcal{H}\mathcal{H}'$ is nonsingular. We do not, however, assume that $\mathcal{H}$ is full column rank.

# 2 Derivation of MOE Receiver

The remainder of the report studies the properties of linear receivers $\mathbf{f}$ that form the desired user's symbol estimates $\hat{s}_n^{(k)} = \mathbf{f}'\mathbf{r}(n)$. Henceforth we omit the desired-user superscript notation on $s_n^{(k)}$ and $\mathbf{c}^{(k)}$.

The MOE receiver $\mathbf{f}_{\text{moe}}$ is defined as the coefficient vector $\mathbf{f}$ minimizing output energy subject to the constraint $\mathbf{f}'\mathbf{c} = 1$. Since output energy can be written $\mathrm{E}\{|\hat{s}_n|^2\} = \mathrm{E}\{|\mathbf{f}'\mathbf{r}(n)|^2\} = \mathbf{f}'\mathbf{R}\mathbf{f}$, where $\mathbf{R} = \mathcal{H}\mathcal{H}'$ denotes the received signal autocorrelation matrix,

$$\mathbf{f}_{\text{moe}} \stackrel{\triangle}{=} \arg\min_{\mathbf{f}} \mathbf{f}'\mathbf{R}\mathbf{f}\Big|_{\mathbf{f}'\mathbf{c}=1}. \tag{5}$$

Straightforward use of Lagrange multipliers reveals that

$$\mathbf{f}_{\text{moe}} = \frac{\mathbf{R}^{-1}\mathbf{c}}{\mathbf{c}'\mathbf{R}^{-1}\mathbf{c}}. \tag{6}$$

# 3 Analysis of MOE Receiver

In this section we derive the MSE of the MOE receiver and compare it to that of the MMSE receiver (for a given user/delay combination).

For a parameter space interpretation, we will consider the "overall system response" $\mathbf{q} \stackrel{\triangle}{=} \mathcal{H}'\mathbf{f}$, where $\hat{s}_n = \mathbf{q}'\mathbf{s}(n)$. Using $\delta$ to denote the index of the desired symbol in $\mathbf{s}(n)$ and $\mathbf{e}_\delta$ to denote a vector with a one in the $\delta^{th}$ position and zeros elsewhere, the MSE can be written

$$\mathrm{E}\{|s_n - \hat{s}_n|^2\} = \mathrm{E}\{|\big(\mathbf{e}_\delta - \mathbf{q}\big)'\mathbf{s}(n)|^2\} = \|\mathbf{e}_\delta - \mathbf{q}\|_2^2.$$

For purposes of comparison, it is often convenient to consider the unbiased versions of each estimator. An unbiased linear estimator of $s_\delta$ is characterized by a system response $\mathbf{q}$ with $\delta^{th}$ element equal to 1, i.e., $\mathbf{e}_\delta'\mathbf{q} = 1$.

## 3.1 Unbiased MOE Receiver

When $\mathcal{H}\mathbf{e}_\delta = \mathbf{c}$, which occurs in the time-synchronized no-multipath scenario, the MOE response $\mathbf{q}_{\text{moe}} = \mathcal{H}'\mathbf{f}_{\text{moe}} = \frac{\mathcal{H}'\mathbf{R}^{-1}\mathbf{c}}{\mathbf{c}'\mathbf{R}^{-1}\mathbf{c}}$ is clearly unbiased since $\mathbf{e}'\mathbf{q}_{\text{moe}} = 1$. For general $\mathcal{H}$, the unbiased MOE quantities are

$$\mathbf{f}_{\text{umoe}} = \frac{\mathbf{R}^{-1}\mathbf{c}}{\mathbf{e}_\delta'\mathcal{H}'\mathbf{R}^{-1}\mathbf{c}}, \quad \mathbf{q}_{\text{umoe}} = \frac{\mathcal{H}'\mathbf{R}^{-1}\mathbf{c}}{\mathbf{e}_\delta'\mathcal{H}'\mathbf{R}^{-1}\mathbf{c}}. \tag{7}$$

### 3.2 Parameter Space Interpretation

It is well known that the MMSE receiver is defined by $\mathbf{f}_{\mathrm{m}} = \mathbf{R}^{-1}\mathcal{H}\mathbf{e}_\delta$, and similarly that $\mathbf{q}_{\mathrm{m}} = \mathcal{H}'\mathbf{R}^{-1}\mathcal{H}\mathbf{e}_\delta$. The unbiased MMSE response is then given by

$$\mathbf{q}_{\mathrm{um}} = \frac{\mathcal{H}'\mathbf{R}^{-1}\mathcal{H}\mathbf{e}_\delta}{\mathbf{e}_\delta'\mathcal{H}'\mathbf{R}^{-1}\mathcal{H}\mathbf{e}_\delta} \tag{8}$$

Figure 1 illustrates the MOE and MMSE system responses for the case of general $\mathcal{H}$.

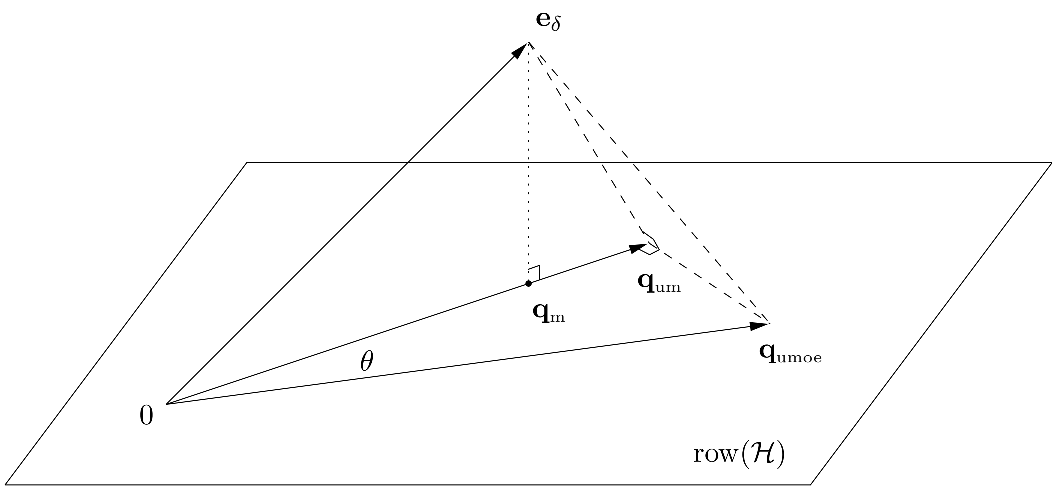


Figure 1: Parameter space interpretation of MOE and MMSE system responses.

### 3.3 Output Space Interpretation

We may also consider the estimation problem in the output space. This is a Hilbert space based on a vector space of random variables with an inner product defined by $< y, s >= \mathrm{E}\{ys\}$.

The linear estimates of $s_n$ depicted in Figure 2 are given by $y_{\mathrm{m}} = \mathbf{f}_{\mathrm{m}}'\mathbf{r}(n)$, $y_{\mathrm{um}} = \mathbf{f}_{\mathrm{um}}'\mathbf{r}(n)$,$y_{\mathrm{moe}} = \mathbf{f}_{\mathrm{moe}}'\mathbf{r}(n)$, and $y_{\mathrm{umoe}} = \mathbf{f}_{\mathrm{umoe}}'\mathbf{r}(n)$. As each of these estimates are linear combinations of the $L_r$ observations in $\mathbf{r}(n)$, the estimates are in the so-called "signal-space" algebraically defined by the span of the elements in $\mathbf{r}(n)$. We denote this by $\mathrm{span}\{r_0, \ldots, r_{L_r-1}\}$.

The orthogonality principle of MMSE estimation implies that the MMSE estimate $y_{\mathrm{m}}$ is the projection of the desired parameter $s_n$ onto the plane of the observations, while the unbiased MMSE estimate $y_{\mathrm{um}}$ is a scaled version of $y_{\mathrm{m}}$. The set of all unbiased estimates is illustrated by the affine space $\mathcal{Y}_u$ constructed in the following manner: for any $y$ in $\mathcal{Y}_u$, $\mathrm{E}\{ys_n\} = 1$.

The MOE estimate appears as the minimum energy estimate that lives in the contraint space $\mathcal{Y}_c$. The constraint space $\mathcal{Y}_c$ is defined by the set of all linear estimates $y$ such that $y = (\mathbf{c}+\mathbf{x})^t\mathbf{r}(n)$ with $\mathbf{x}^t\mathbf{c} = 0$.

Note that, unlike the parameter space interpretation offered by Figure 1, the output space interpretation above remains valid in the presence of additive channel noise.

### 3.4 Comparison to MMSE Receiver

From Figure 2, the excess MSE of the unbiased MOE receiver,

$$\mathcal{E}_{\mathrm{umoe}} \stackrel{\triangle}{=} \mathrm{MSE}_{\mathrm{umoe}} - \mathrm{MSE}_{\mathrm{um}},$$

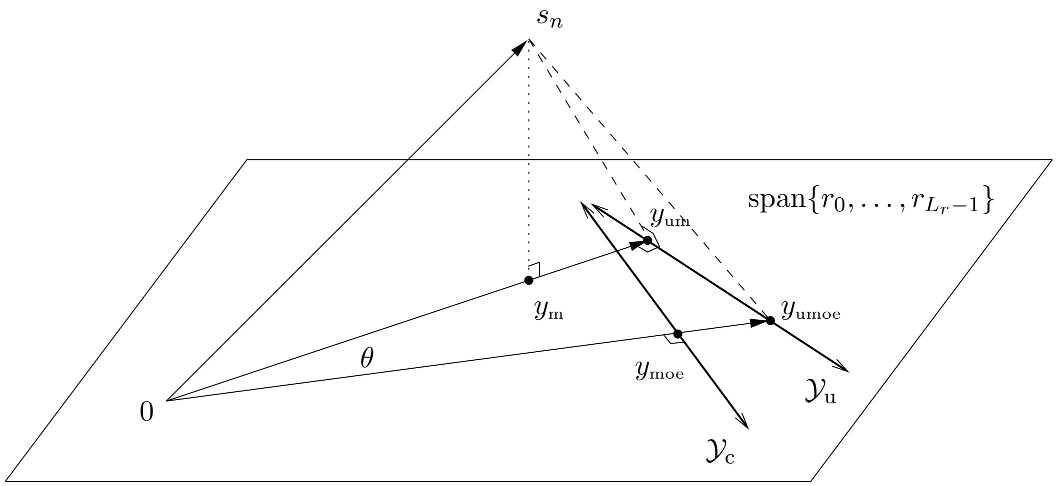


Figure 2: Output space interpretation of MOE and MMSE system responses.

can be calculated as

$$\mathcal{E}_{\mathrm{umoe}} = (\|y_{\mathrm{um}}\| \tan\theta)^2 \tag{9}$$

$$= \frac{\tan^2\theta}{\|\mathcal{H}\mathbf{e}_\delta\|^2_{\mathbf{R}^{-1}}} \tag{10}$$

where

$$\theta \triangleq \cos^{-1}\left(\frac{\mathbf{c}'\mathbf{R}^{-1}\mathcal{H}\mathbf{e}_\delta}{\|\mathbf{c}\|_{\mathbf{R}^{-1}} \cdot \|\mathcal{H}\mathbf{e}_\delta\|_{\mathbf{R}^{-1}}}\right) \tag{11}$$

One interpretation is that steady-state performance of the MOE receiver degrades in proportion to the angle between the "whitened" versions of $\mathbf{c}$ and $\mathcal{H}\mathbf{e}_\delta$, i.e. $\mathbf{R}^{-1/2}\mathbf{c}$ and $\mathbf{R}^{-1/2}\mathcal{H}\mathbf{e}_\delta$, respectively. This follows from

$$\theta = \cos^{-1}\left(\frac{(\mathbf{R}^{-1/2}\mathbf{c})'(\mathbf{R}^{-1/2}\mathcal{H}\mathbf{e}_\delta)}{\|\mathbf{R}^{-1/2}\mathbf{c}\| \cdot \|\mathbf{R}^{-1/2}\mathcal{H}\mathbf{e}_\delta\|}\right). \tag{12}$$

# References


[1] U. Madhow, "Blind adaptive interference suppression for direct-sequence CDMA," *Proceedings of the IEEE*, Oct. 1998.