Communications over Sparse Channels:  
Fundamental limits and practical design

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**Sparse Channels:**

- At large communication bandwidths, channel impulse responses are **sparse**.
- Below left shows channel taps $\mathbf{x} = [x_0, \ldots, x_{L-1}]$, where
  - $x_n = x(nT)$ for bandwidth $T^{-1} = 256$ MHz,
  - $x(t) = h(t) \ast p_{RC}(t)$, and
  - $h(t)$ is generated randomly using 802.15.4a outdoor NLOS specs.

\[ dB \]

![IEEE 802.15.4a outdoor NLOS](image1)

![Measured underwater channel](image2)
**Simplified Channel Model:**

First, let’s *simplify* things to talk concretely about sparse channels. . .

Consider a discrete-time channel that is

- **block-fading** with block size $N$,
- **frequency-selective** with impulse response length $L$ (where $L < N$),
- **sparse** with $S$ non-zero complex-Gaussian taps (where $0 < S \leq L$),

where *both the channel coefficients and support are unknown* to the receiver.

Important questions:

1. What is the *capacity* of this channel?
2. How can we build a *practical* comm system that operates near this capacity?
Noncoherent Capacity of the Sparse Channel:

For the unknown $N$-block-fading, $L$-length, $S$-sparse channel described earlier, we established [1] that

1. In the high-SNR regime, the ergodic capacity obeys

$$C_{\text{sparse}}(\text{SNR}) = \frac{N-S}{N} \log(\text{SNR}) + O(1).$$

2. To achieve the prelog factor $R_{\text{sparse}} = \frac{N-S}{N}$, it suffices to use
   - pilot-aided OFDM (with $N$ subcarriers, of which $S$ are pilots)
   - with joint channel estimation and data decoding.

Key points:

- The effect of unknown channel support manifests only in the $O(1)$ offset.
- Standard non-sparse-channel methods would use $L$ pilots.
- “Compressed channel sensing” would use $S \text{polylog } N$ pilots.

Practical Communication over the unknown Sparse Channel:

We now propose a communication scheme that...

- is practical, with decode complexity $O(N \log_2 N + N|S|)$ per $N$-block,
- delivers outage rates matching the optimal prelog factor $R_{\text{sparse}} = \frac{N-S}{N}$,
- significantly outperforms “compressed channel sensing” (CCS) schemes.

Our scheme uses...

- a conventional transmitter: pilot-aided BICM OFDM,
- a novel receiver: based on belief propagation with the generalized approximate message passing (GAMP) algorithm [3] used in a “turbo” configuration [2].


Factor Graph for pilot-aided BICM-OFDM:

To jointly infer all random variables, we perform loopy-BP via the sum-product algorithm, using AMP approximations in the GAMP sub-graph.
Numerical Results — Perfectly Sparse Channel:

Transmitter:
- LDPC codewords with length $\sim 10000$ bits.
- $2^M$-QAM with $2^M \in \{4, 16, 64, 256\}$ and multi-level Gray mapping.
- OFDM with $N = 1024$ subcarriers.
- $P$ pilot subcarriers and/or $T$ training MSBs.

Channel:
- Length $L = 256 = N/4$.
- Sparsity $S = 64 = N/16$.

Reference Schemes:
- **Pilot-aided LASSO** (i.e., compressed channel sensing) with oracle tuning.
- **Pilot-aided LMMSE, support-aware MMSE**, and **info-bit+support-aware MMSE** channel estimates were also tested.
BER & Outage vs SNR (with $P=L$ pilots & $T=0$ training MSBs):

![Graph showing BER vs SNR for different algorithms: GAMP, LASSO, BSG, and LMMSE. The graph includes BER=0.001 contours for 64-QAM modulation.]

Key points:

- GAMP outperforms both LASSO and the support genie (SG).
- GAMP performs nearly as well as the info-bit+support-aware genie (BSG).
- With $P = L$, all approaches yield prelog factor $R = \frac{N-L}{N} = \frac{3}{4}$, which falls short of the optimal $R_{\text{sparse}} = \frac{N-S}{N} = \frac{15}{16}$. 
**BER & Outage vs SNR (with $P=0$ pilots & $T=SM$ training MSBs):**

- **Key points:**
  - **GAMP** favors $P=0$ pilot subcarriers and $T=SM$ training MSBs.
    - Precisely the necc/suff redundancy of the capacity-maximizing system!
  - **GAMP** achieves the sparse-channel’s capacity-prelog factor, $R_{\text{sparse}} = \frac{N-S}{N}$. 

![Graph showing BER vs SNR with logarithmic scales for BER and SNR. The graph includes a color-coded contour map and a line graph.]
In practice, channel taps are not perfectly sparse, nor i.i.d:

- For example, consider channel taps $x = [x_0, \ldots, x_{L-1}]$, where
  - $x_n = x(nT)$ for bandwidth $T^{-1} = 256$ MHz,
  - $x(t) = h(t) * p_{RC}(t)$, and
  - $h(t)$ is generated randomly using 802.15.4a outdoor NLOS specs.

- The tap distribution varies as the lag increases, becoming more heavy-tailed.
- The big taps are clustered together in lag, as are the small ones.
Proposed channel model:

- Saleh-Valenzuela (e.g., 802.15.4a) models are accurate but difficult to exploit in receiver design.

- We propose a structured-sparse channel model based on a 2-state Gaussian Mixture model with discrete-Markov-chain structure on the state:

\[
p(x_j | d_j) = \begin{cases} 
  \mathcal{CN}(x_j; 0, \mu^0_j) & \text{if } d_j = 0 \quad \text{“small”} \\
  \mathcal{CN}(x_j; 0, \mu^1_j) & \text{if } d_j = 1 \quad \text{“big”}
\end{cases}
\]

\[
\Pr\{d_{j+1} = 1\} = p^1_{j} \Pr\{d_{j} = 0\} + (1 - p^0_{j}) \Pr\{d_{j} = 1\}
\]

- Our model is parameterized by the lag-dependent quantities:
  \[
  \{\mu^1_j\} : \text{big-state power-delay profile} \\
  \{\mu^0_j\} : \text{small-state power-delay profile} \\
  \{p^0_{j1}\} : \text{big-to-small transition probabilities} \\
  \{p^1_{j0}\} : \text{small-to-big transition probabilities}
  \]

- Can learn these statistical params from observed realizations via the EM alg.
Factor graph for pilot-aided BICM-OFDM:

To jointly infer all random variables, we perform loopy-BP via the sum-product algorithm, using AMP approximations in the GAMP sub-graph.
**Numerical results:**

**Transmitter:**
- OFDM with $N = 1024$ subcarriers.
- 16-QAM with multi-level Gray mapping
- LDPC codewords with length $\sim 10000$ yielding spectral efficiency of 2 bpcu.
- $P$ pilot subcarriers and $T$ training MSBs.

**Channel:**
- 802.15.4a outdoor-NLOS (not our Gaussian-mixture model!)
- Length $L = 256 = N/4$.

**Reference Channel Estimation / Equalization Schemes:**
- soft-input soft-output (SISO) versions of LMMSE and LASSO.
- perfect-CSI genie.
BER versus $E_b/N_o$ for $P = 224$ pilots and $T = 0$ training MSBs:

Note 4dB improvement over (turbo) LASSO. Only 0.5dB from perfect-CSI genie!
BER versus $E_b/N_0$ for $P = 0$ pilots and $T = 448$ training MSBs:

Use of training MSBs gives 1dB improvement over use of pilot subcarriers!
Communications over Underwater Channels:

- SPACE-08 Underwater Experiment 2920156F038_C0_S6
- Time-varying channel response estimated using WHOI M-sequence:

- The channel is nearly over-spread: \( f_d T_s L = 20 \times \frac{1}{10000} \times 400 = 0.8 \)!
- Can’t afford to ignore structure of temporal variations!
BICM-OFDM Factor Graph with Temporal Channel Structure:

- Channel taps are modeled as independent Bernoulli-Gaussian processes:
  - each tap’s amplitude follows a temporal Gauss-Markov chain
  - each tap’s on/off state follows a temporal discrete-Markov chain

Performance versus SNR:

Settings:
- experimentally measured underwater channel
- 16-QAM
- 1024 total tones
- 0 pilot tones
- 256 training MSBs
- LDPC length 10k
- LDPC rate 0.5

Exploiting the persistence in channel support and channel amplitudes was critical in this difficult underwater application.
Communications in Impulsive Noise:

- In many wireless and power-line communication systems, the (time-domain) noise is not Gaussian but impulsive.

- The marginal noise statistics are well captured by a 2-state Gaussian mixture (i.e., Middleton class-A) model.

- Noise burstiness is well captured by a discrete Markov chain on the noise state.
Factor Graph for pilot-aided BICM-OFDM:

Numerical Results — Uncoded Case:

Settings:
- 5 channel taps
- GM noise
- 256 total tones
- 15 pilot tones
- 80 null tones
- 4-QAM

Proposed “joint channel/impulsive-noise/symbol” estimation (JCIS) scheme gives \(\sim 15\) dB gain over previous state-of-the-art and performs within 1 dB of MFB!
Numerical Results — Coded Case:

Settings:
- 10 channel taps
- GM noise
- 1024 total tones
- 150 pilot tones
- 0 null tones
- 16-QAM
- LDPC
- Rate 0.5
- Length 60k

Proposed “joint channel/impulsive-noise/symbol/bit” estimation (JCISB) scheme gives \(~15\) dB gain over traditional DFT-based receiver!
Conclusions:

- At wide bandwidths, channel impulse responses are approximately sparse.
  - Sparsity increases the pre-log factor of high-SNR noncoherent ergodic capacity.
  - AMP-based joint channel-estimation/decoding delivers outage rates that empirically match the capacity pre-log factor.
  - Channels impulses are in fact structured-sparse, and exploiting this structure leads to additional performance gains.
  - Sparsity can also be exploited in time-varying channels.

- Impulsive noise is another source of sparsity in communications.
  - AMP-based joint channel-estimation/impulse-estimation/decoding delivers error-rates that approach the matched-filter bound.