The Effect of Timing Offset on Fractionally-Spaced CMA

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1 Introduction

In this paper, we analyze the impact that choice sampling phase has on the transient and steady-state behaviors of FSE-CMA. Our analysis is enabled by the introduction of an interpolation operator into the standard FSE system model.

2 System Model

Consider the $T/2$-spaced equalization scenario illustrated by Figure 2. A $T$-spaced source process $\{s_n\}$ is transmitted through a baseband-equivalent channel with impulse response $h(t)$ and bandlimited\(^1\) additive noise process $w(t)$. The received signal $x(t)$ is sampled (uniformly) at $T/2$-spaced intervals, and the resulting sequence $r_k = x((k - \rho)T/2)$ is applied to an FIR equalizer with length-$N_f$ impulse response vector $f$. The delay parameter $\rho \in (-1,1)$ allows for an arbitrary sampling phase offset. Finally, the equalizer output is decimated in forming the soft decisions $\{y_n\}$.

![Figure 1: T/2-spaced equalization model](image)

Figure 1: $T/2$-spaced equalization model, showing channel $h(t)$, additive noise $w(t)$, sampling phase offset $\rho$, and fractionally-spaced equalizer $f$.

The discrete multirate equivalent of Figure 1 is shown in Figure 2, where the continuous-time channel response, noise, and channel output have been replaced by their $T/2$-sampled equivalents: $h_k = h(kT/2)$, $w_k = w(kT/2)$, and $x_k = x(kT/2)$. Assuming an FIR channel, the nonzero channel samples can be collected into a length-$N_h$ impulse response vector $c$. In the discrete model, the receiver sampling phase offset is implemented by the interpolation operator $T_\rho$, effecting the equivalent of a $\rho T/2$ second delay from $\{x_k\}$ to $\{r_k\}$. The construction of $T_\rho$ will be discussed in Section 3.

The band-spaced system response taking $\{s_n\} \rightarrow \{y_n\}$ is characterized by the length-$N_q$ vector $q$, so that in the absence of noise $y_n = s^T(n)q$, where $s(n) = (s_n,s_{n-1},\ldots,s_{n-N_q+1})^T$ is a

\(^1\)For simplicity, we assume that $w(t)$ is bandlimited to $|\omega| < 2\pi/T$. 

1
Figure 2: Discrete system model, with interpolation operator \( T_\rho \) implementing \( \rho T/2 \)-second sampling phase offset.

The channel matrix

\[
H = \begin{pmatrix}
    h_1 & h_0 \\
    h_3 & h_2 & h_1 & h_0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    h_{N_h-1} & h_{N_h-2} & \cdots & h_3 & h_2 & \cdots & h_{N_h-1} & h_{N_h-2}
\end{pmatrix}
\]

is constructed so that \( q = Hf \) (in the case of zero sampling phase offset, i.e., \( \rho = 0 \)) \[1\]. Collecting the previous \( N_f \) channel output and noise samples into \( x(n) \) and \( w(n) \), respectively, yields \( x(n) = Hs(n) + w(n) \). The timing phase offset between \( x(n) \) and \( r(n) \) may be approximated by an interpolation operator \( T_\rho \in \mathbb{R}^{N_f \times N_f} \):

\[
r(n) = T_\rho x(n).
\]

In this case, the system response incorporating \( \rho T/2 \) seconds of sampling offset takes the form

\[
q = HT_\rho f.
\]

Section 4 discusses how this \( \mathbb{R}^{N_f \times N_f} \) construction of \( T_\rho \) is a particular case of a more general (and arbitrarily accurate) construction.

3 The Interpolation Operator

The following sections detail the construction of interpolation operators acting on finite-length sampled data records which approximate the action of the continuous-time delay operator

\[
T_\rho^{(c)} : x(t) \mapsto x(t - \rho T/2) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \rho T/2 - \tau) d\tau,
\]

where \( \delta(t) \) denotes the Dirac delta.

3.1 Construction of \( T_\rho \)

Fourier theory specifies that the frequency domain equivalent of (3) is

\[
T_\rho^{(c)} : X(e^{j\omega}) \mapsto e^{-j\omega \rho T/2} X(e^{j\omega}),
\]

where \( X(e^{j\omega}) \) is the Fourier transform of \( x(t) \).
The frequency domain definition (4) motivates our treatment of discrete-time signals. First, however, we establish some common notation. Let \( \mathbf{x} \in \mathbb{C}^N \) denote a vector of time-domain samples and \( \hat{\mathbf{x}} \in \mathbb{C}^N \) denote its DFT, such that

\[
\hat{x}_m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-j2\pi mk/N} \quad \text{for} \quad 0 \leq m \leq N - 1.
\]

The DFT has an equivalent matrix representation \( \mathbf{W} \in \mathbb{C}^{N \times N} \) where

\[
[\mathbf{W}]_{m,n} = \frac{1}{\sqrt{N}} e^{-j2\pi mn/N},
\]

so that \( \hat{\mathbf{x}} = \mathbf{W}\mathbf{x} \). It is important to note that \( \mathbf{W} \) is both unitary (\( \mathbf{W}\mathbf{W}^H = \mathbf{W}^H\mathbf{W} = \mathbf{I} \)) and symmetric (\( \mathbf{W} = \mathbf{W}^t \)). For notational simplicity, we henceforth assume that \( N \) is even.

Consider the discrete equivalent of the frequency domain operator in (4):

\[
\hat{T}_\rho : \hat{\mathbf{x}} \mapsto \Phi \hat{\mathbf{x}},
\]

where \( \Phi \) is a diagonal matrix with entries

\[
[\Phi]_{m,m} := \begin{cases}
  e^{-j2\pi pm/N} & 0 \leq m < N/2, \\
  1 & m = N/2, \\
  e^{-j2\pi (m-N)/N} & N/2 < m \leq N - 1.
\end{cases}
\]

The conjugate symmetry along the diagonal of \( \Phi \) ensures that the corresponding time-domain operation

\[
\mathbf{T}_\rho : \mathbf{x} \mapsto \mathbf{W}^H \Phi \mathbf{W} \mathbf{x}
\]

is real-valued. The properties listed below follow from inspection:

**Claim 1.** The operator \( \mathbf{T}_\rho \in \mathbb{R}^{N \times N} \) defined in (6) has the following properties:

1. \( \mathbf{T}_\rho \) is unitary,
2. \( \mathbf{T}_\rho = \mathbf{T}_\rho^t \),
3. \( \mathbf{T}_\rho \) is circulant.

Figure 3 demonstrates the interpolation capabilities of \( \mathbf{T}_\rho \) using SPIB\(^2\) channel \#1 and various values of \( \rho \).

4 Application of \( \mathbf{T}_\rho \)

The frequency-domain construction (6) implies that \( \mathbf{T}_\rho \) implements a “circulant” resampling; i.e., \( \{x_k\} \) are treated as samples of a cyclic waveform \( x(t) \) on \( (-\infty, \infty) \) with the property that \( x(t) = x(t - NT/2) \) \( \forall t \) where \( x_k = x(kT/2) \).

\(^2\)The Rice University Signal Processing Information Base (SPIB) microwave channel database resides at [http://spib.rice.edu/spib/microwave.html](http://spib.rice.edu/spib/microwave.html).
Figure 3: SPIB impulse response interpolated using four values of $\rho$. Solid stems indicate the original $T/2$-spaced samples, while dotted stems indicate the delayed versions obtained using $T_\rho$ with $\rho = \{0, 0.25, 0.5, 0.75\}$.

Though I have not proved it yet, I believe that $T_\rho$ in (6) is the best $L_2$ approximation of (3) over $\mathbb{R}^{N \times N}$. Note, however, that a frequency-weighted $L_2$ design may better suit the signals typically encountered in the $T/2$-spaced equalization scenario. Does such a $T_\rho$ still satisfy the properties in Claim 1? What about the following $H_\infty$-like optimization?

$$T_\rho := \arg \min_{T \in \mathbb{R}^{N \times N}} \max_{x \in \mathbb{R}^N} \frac{\| \sum_n [Tx]_n e^{-j\omega n} - D_\rho(\omega) \sum_n [x]_n e^{-j\omega n} \|_{L_2}}{\|x\|_2},$$

where

$$D_\rho(\omega) := e^{-j\omega \rho} \text{ for } \omega \in (-\pi, \pi).$$

5 Applications of the Delay Operator

As suggested earlier, the delay operator in (6) may prove useful in FSE analysis. Consider the SVD of the (non-delayed) real-valued channel convolution matrix: $H = USV^t$. Incorporating the delay operator into the convolution operation yields the matrix

$$\tilde{H} := HT_\rho^t = USV^t T_\rho^t = US(T_\rho V)^t = US\tilde{V}^t,$$

where the unitary property of $T_\rho$ implies that $\tilde{V} := T_\rho V$ is also unitary. In other words, the $\rho$-delay leads to a convolution-matrix-like quantity $\tilde{H}$ with SVD $\tilde{H} = US\tilde{V}^t$ and preserves the singular values and left singular vectors of $H$. As it is known that $V$ determines the orientation of CMA regions of convergence in equalizer space [Chung;Draft;ROC], the unitary transformation $T_\rho$ describes the rotation of ROC boundaries as a function of sampling phase.

Now consider noise entering the system of Figure 2 just after the channel. The combined operation of delay plus equalization can be represented by the modified equalizer impulse response vector

$$\tilde{r} := T_\rho^t r.$$
The CMA cost function, for source and noise processes both white and real-valued, then takes the form

\[ J_{CM} | \mathbb{R} = (\kappa_s - 3) \sum_{i=0}^{P-1} h_i^4 + 3\|q\|^2 + 3\sigma_w^4 \|\tilde{f}\|^2 + 6\sigma_w^2 \|q\| \|\tilde{f}\|^2 - 2\kappa_s (\|q\|^2 + \sigma_w^2 \|\tilde{f}\|^2) + \kappa_s^2 \]

where \(\sigma_w^2\) is the noise variance and \(\kappa_s\) is the (normalized) source kurtosis. But since the unitary operation \(T_n\) preserves both the \(\ell_2\) norm and the reachable system subspace (as confirmed by SVD above), the CMA cost function remains essentially invariant to changes in sampling time offset. This contradicts the (unconvincing) claims of [2]. Simulations should be performed as a confirmation.

6 Other Questions

- Relationships to Zak Transform [3]?

References

