STATISTICAL ANALYSIS OF TRUE AND EXTRANEOUS MODE ESTIMATES FOR THE TLS-PRONY ALGORITHM *

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ABSTRACT

In this paper we present a statistical analysis of the poles and amplitude coefficients estimated using a TLS-Prony method where signals consist of arbitrary damped exponential terms in noise. Both the true and extraneous modes are considered in the analysis. The derivation for this procedure is based on a first order perturbation analysis; thus the analysis assumes high SNR. We derive the complete covariance matrix for the estimated pole and amplitude coefficient parameters. We also develop the statistics of the mode energies for both the true and extraneous modes. To first order the distributions of the true mode energies are well approximated by Gaussian distributions, and the energies of the extraneous modes are central χ^2 distributed. We verify the theory with Monte-Carlo simulations.

1. INTRODUCTION

The problem of estimating model parameters of noisy exponential signals has been an area of considerable attention in the past few years. This problem has applications in a number of areas, including speech processing, deconvolution, radar and sonar signal processing, array processing, and spectrum estimation. One popular algorithm used in this community is the so-called TLS-Prony algorithm presented in [1, 2]. A number of authors have considered various aspects of this method [3]–[7]. However, none of them has considered the statistical characteristics of the extraneous modes introduced in the algorithm. Nevertheless, it is very important to understand the characteristics of the extraneous modes since they affect the performance of the true mode estimation. Also, based on the statistics of the extraneous modes we can further develop statistical tests for model order selection.

In this paper we present a statistical analysis of the poles and amplitude coefficients estimated using a TLS-Prony method, where signals consist of arbitrary damped exponential terms in noise. Both the true and extraneous modes are considered in the statistical derivation. The statistical derivation for this procedure is based on a first order perturbation analysis; thus the analysis assumes high SNR. We derive the complete covariance matrix for the estimated pole and amplitude coefficient parameters for both the true and extraneous modes in terms of their real and imaginary parts. We verify the statistical theory with Monte-Carlo simulations.

Using these expressions, we also develop the statistics of the mode energies for both the true and extraneous modes. We show that to first order the distributions of the true mode energies are well approximated by Gaussian distributions, and the energies of the extraneous modes are central χ^2 distributed. These energy results can be used to predict the performance of mode energy based criteria for distinguishing between true and extraneous modes.

An outline of this paper is as follows. In Section 2 we present the data model. In Section 3 we present the first order statistics of the model parameters and the mode energies. In Section 4 we present some examples using the statistical expressions. Finally, in Section 5 we conclude the paper.

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2. DATA MODEL AND ESTIMATION PROCEDURE

2.1. Data Model

Assume a data vector y of length m is modeled as a noisy exponential sequence

$$y_q = \sum_{i=1}^{n} x_i p_i^q + e_q \qquad q = 0, 1, \ldots, m-1. \tag{1}$$

There are n distinct exponential modes in the data. Here, it is assumed that $\{e_q\}$ is a zero mean complex white Gaussian noise sequence with variance σ. Equation 1 may be compactly written as

$$y = Ax + e, \tag{2}$$

where $e = [\ e_0 \quad e_1 \ \cdots \ e_{m-1}\ ]^T$, $x = [\ x_1 \quad x_2 \ \cdots \ x_n\ ]^T$, and

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_n \\ \vdots & \vdots & & \vdots \\ p_1^{m-1} & p_2^{m-1} & \cdots & p_n^{m-1} \end{bmatrix}. \tag{3}$$

Note that we can consider the extraneous modes introduced in an Lth order TLS-Prony estimation algorithm (presented below) as true modes by setting the corresponding amplitude coefficients to zero. In this case, A and x in Equation 2 are defined as $x = [\ x_1 \quad x_2 \ \cdots \ x_n \quad 0 \ \cdots \ 0\ ]^T$, and

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_n & p_{n+1} & \cdots & p_L \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_1^{m-1} & p_2^{m-1} & \cdots & p_n^{m-1} & p_{n+1}^{m-1} & \cdots & p_L^{m-1} \end{bmatrix}, \tag{4}$$

where $p_i, i = n+1, n+2, \ldots, L$ are extraneous poles.

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2.2. Parameter Estimation

The backward linear prediction equations are given by:

$$[\; y \;:\; Y \;] \begin{bmatrix} 1 \\ b \end{bmatrix} \approx 0, \tag{5}$$

where

$$b = [\; b_1 \quad b_2 \quad \cdots \quad b_L \;]^T, \tag{6}$$

and where

$$[\; y \;:\; Y \;] = \begin{bmatrix} y_0 & y_1 & \cdots & y_L \\ y_1 & y_2 & \cdots & y_{L+1} \\ \vdots & \vdots & & \vdots \\ y_{m-(L+1)} & y_{m-L} & \cdots & y_{m-1} \end{bmatrix}. \tag{7}$$

Here L is the order of prediction and b is the coefficient vector of the polynomial $B(z)$ given by

$$B(z) = 1 + b_1 z^1 + b_2 z^2 + \cdots + b_L z^L. \tag{8}$$

The choice of L affects the accuracy of the b_i coefficients [5].

The TLS-Prony method considers the effect of noise perturbation of both Y and y, and the TLS solution attempts to minimize the effect of these perturbations on the prediction coefficient vector b (see [2] for details). This is accomplished by obtaining an SVD of the matrix $[\; y \;:\; Y \;]$ and truncating all but the first n singular values to arrive at an estimate $[\; \widehat{y} \;:\; \widehat{Y} \;]$ [2]. This leads to the following modified linear prediction equation

$$\widehat{Y}\widehat{b} \approx -\widehat{y} \tag{9}$$

from which the linear prediction coefficient vector estimate $\widehat{b}$ is found as

$$\widehat{b} = -\widehat{Y}^{+}\widehat{y}, \tag{10}$$

where $^+$ denotes the Moore-Penrose pseudoinverse. Finally, the estimates for the poles are found by

$$\widehat{p}_i = \text{zero}_i\left(\widehat{B}(z)\right), \qquad i = 1, 2, \ldots, L. \tag{11}$$

One method for distinguishing between true and extraneous modes is to use mode energy as a test statistic. The main contribution of this paper is the statistical analysis of the poles and amplitude coefficients for both true and extraneous modes, from which the performance of mode-energy test statistics can be derived.

Once the poles have been estimated, the amplitude coefficients can be found using the pole estimates and Equation 2. This leads to the following least squares equation for the amplitude coefficients,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \widehat{p}_1 & \widehat{p}_2 & \cdots & \widehat{p}_L \\ \vdots & \vdots & & \vdots \\ \widehat{p}_1^{m-1} & \widehat{p}_2^{m-1} & \cdots & \widehat{p}_L^{m-1} \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \vdots \\ \widehat{x}_L \end{bmatrix} \approx y \tag{12}$$

or

$$\widehat{A}\widehat{x} \approx y. \tag{13}$$

The amplitude coefficients can be found from a least squares solution to Equation 13,

$$\widehat{x} = \left(\widehat{A}^{*}\widehat{A}\right)^{-1} \widehat{A}^{*} y = \widehat{A}^{+} y, \tag{14}$$

where * denotes complex conjugate transpose.

3. STATISTICAL ANALYSIS

To analyze parameter statistics we now derive their covariance matrix. This is given in the following theorem. Assume data is given as in Equation 1. Let $\bar{p}_i$ and $\check{p}_i$ be the real and imaginary part, respectively, of each pole p_i, thus $p_i = \bar{p}_i + j\check{p}_i$. Similarly let $\bar{x}$ and $\check{x}$ be the real and imaginary part, respectively, of each amplitude coefficient x_i.

Define following parameter vectors:

$$\begin{aligned} \theta_x &= [\; \bar{x}_1 \quad \bar{x}_2 \quad \cdots \quad \bar{x}_L \quad \check{x}_1 \quad \check{x}_2 \quad \cdots \quad \check{x}_L \;]^T \\ \theta_p &= [\; \bar{p}_1 \quad \bar{p}_2 \quad \cdots \quad \bar{p}_L \quad \check{p}_1 \quad \check{p}_2 \quad \cdots \quad \check{p}_L \;]^T \\ \theta &= [\; \theta_x^T \quad \theta_p^T \;]^T. \end{aligned} \tag{15}$$

The following theorem gives the first order pdf of $\widehat{\theta}$.

Theorem 1: Let $\widehat{\theta}$ denote the TLS-Prony estimate of θ which is given by the estimates found in Equations 11 and 14. Then the first order (as $\sigma \to 0$) pdf of $\widehat{\theta}$ is given by

$$\widehat{\theta} \sim N(\theta, \Sigma_\theta), \tag{16}$$

where

$$\Sigma_\theta = \frac{\sigma}{2} \begin{bmatrix} \bar{U} & -\check{U} & \bar{V}_1 & \check{V}_2 \\ \check{U} & \bar{U} & \check{V}_1 & -\bar{V}_2 \\ \bar{V}_1^T & \check{V}_1^T & \bar{Z}_1 & \check{Z}_2 \\ \check{V}_2^T & -\bar{V}_2^T & \check{Z}_2^T & -\bar{Z}_2 \end{bmatrix}, \tag{17}$$

where the U, V_i, and Z_i matrices are defined in the Appendix and $\bar{\cdot}$ and $\check{\cdot}$ are the real and imaginary operator, respectively.

Proof: See [8]. □

One can easily formulate a similar theorem using the angles and magnitudes of the poles and amplitude coefficients. In fact, it can be achieved via a Jacobian transformation and is derived in [8].

Some conclusions can be drawn from the derivation of Theorem 1. In [5] we derived symmetry properties of the estimates of the true modes for the TLS-Prony algorithm. However, these properties do not carry over to the extraneous modes. For instance, unlike the true poles, the real and imaginary part of an extraneous pole are in general correlated and have unequal variances. Therefore, the concentration ellipse of an extraneous pole may not be a circle.

Based on Theorem 1, we can further analyze the statistics of the mode energies for both true and extraneous modes since one can compute the energy E_i of each of the L modes by

$$E_i = \beta_i^2 \sum_{q=0}^{m-1} \alpha_i^{2q}, \quad i = 1, 2, \ldots, L, \tag{18}$$

where $x_i = \beta_i e^{j\gamma_i}$ and $p_i = \alpha_i e^{j\omega_i}$. Thus, from Theorem 1 we have the following corollaries.

Corollary 1: Let

$$E = [\; E_1 \quad E_2 \quad \cdots \quad E_n \;]^T, \tag{19}$$

denote the parameter vector for the mode energies of the true modes (*i.e.*, the mode energies corresponding to x_i and $p_i, i = 1, 2, \ldots, n$). Let $\widehat{E}$ denote the estimated energies corresponding to the TLS-Prony parameter estimates. Then the first order (as $\sigma \to 0$) pdf of E is given by

$$\widehat{E} \sim N(E, \Sigma_E), \tag{20}$$

where

$$\Sigma_E = 4\left(K_x Q K_x^T + K_x Q_1 K_p^T + K_p Q_1^T K_x^T + K_p Q_2 K_p^T\right), \tag{21}$$

where

$$
\begin{aligned}
Q &= \bar{J}_x \bar{U}^t \bar{J}_x^T - \bar{J}_x \check{U}^t \check{J}_x^T + \check{J}_x \check{U}^t \bar{J}_x^T + \check{J}_x \bar{U}^t \check{J}_x^T \\
Q_1 &= \bar{J}_x \bar{V}_1^t \bar{J}_p^T + \bar{J}_x \check{V}_2^t \check{J}_p^T + \check{J}_x \check{V}_1^t \bar{J}_p^T - \check{J}_x \bar{V}_2^t \check{J}_p^T \\
Q_2 &= \bar{J}_p \bar{Z}_1^t \bar{J}_p^T + \bar{J}_p \check{Z}_2^t \check{J}_p^T + \check{J}_p \check{Z}_2^{tT} \bar{J}_p^T - \check{J}_p \bar{Z}_2^t \check{J}_p^T \\
\bar{J}_x &= \operatorname{diag}\left(\cos(\gamma_1), \cos(\gamma_2), \ldots, \cos(\gamma_n)\right) \\
\check{J}_x &= \operatorname{diag}\left(\sin(\gamma_1), \sin(\gamma_2), \ldots, \sin(\gamma_n)\right) \\
\bar{J}_p &= \operatorname{diag}\left(\cos(\omega_1), \cos(\omega_2), \ldots, \cos(\omega_n)\right) \\
\check{J}_p &= \operatorname{diag}\left(\sin(\omega_1), \sin(\omega_2), \ldots, \sin(\omega_n)\right) \\
K_x &= \operatorname{diag}\left(\frac{E_1}{\beta_1}, \frac{E_2}{\beta_2}, \ldots, \frac{E_n}{\beta_n}\right) \\
K_p &= \operatorname{diag}\left(\frac{\beta_1}{\alpha_1} f_1, \frac{\beta_2}{\alpha_2} f_2, \ldots, \frac{\beta_n}{\alpha_n} f_n\right) \\
f_i &= \frac{\alpha_i^2 - m\alpha_i^{2m} + (m-1)\alpha_i^{2(m+1)}}{(1-\alpha_i^2)^2}.
\end{aligned}
\tag{22}
$$

Here U^t, V_i^t, and Z_i^t are the same as their counterparts in Equation 17 except for replacing A^+ and F by A^{+^t} and F^t which are the first n rows of A^+ and F, respectively (see substitutions in Appendix).

Proof: See [8]. □

Corollary 2: Let $\{\widehat{E_i^e}\}_{i=1}^{L-n}$ denote the estimated energies corresponding to the TLS-Prony parameter estimates of the extraneous mode energies. Then the first order (as $\sigma \to 0$) pdfs of these energies are given by

$$
\widehat{E_i^e} \sim \chi_2^2\left(\sigma_{E_i^e}\right) \qquad i = 1, 2, \ldots, L-n. \tag{23}
$$

where

$$
\begin{aligned}
\sigma_{E_i^e} &= f_i^e \frac{\sigma}{2} \mathrm{Re}\left\{R_i^e B_1 R_i^{e*} - R_i^e B - B^* R_i^{e*} + A_i^{+^e} A_i^{+^{e*}}\right\} \\
f_i^e &= \frac{1 - \alpha_i^{e\,2m}}{1 - \alpha_i^{e\,2}}.
\end{aligned}
\tag{24}
$$

Here $R_i^e = A_i^{+^e} C A T_x^+ T_p F G S^+$ and $A_i^{+^e}$ is the $(n+i)$th row of A^+. Note that $\sigma_{E_i^e}$ are the variances of the Gaussian components of the central χ_2^2 distributions.

Proof: See [8]. □

Using the energy distributions one can compute receiver operation characteristic (ROC) type curves for mode estimation. It is also possible to develop mode detection, and model order selection criteria, based on these energy distributions via a pre-defined energy threshold.

4. EXAMPLE: TEN DAMPED EXPONENTIAL MODE CASE

We consider a model consisting of ten exponential modes. For this case, $m = 40$ data points, $L = 14$, and $\sigma = 0.001$. The amplitude coefficients are chosen so that each true mode energy is unity. Figure 1 presents a comparison between the TLS-Prony estimate theoretical variances and variances obtained using Monte-Carlo simulations. The theoretical variances are shown as two- and three-standard deviation concentration ellipses around each pole. The dots in Figure 1 are pole estimates for both true poles and extraneous poles from each of the 200 Monte-Carlo simulations. From these estimates, we can see that the statistical analysis is in general confirmed. It is clear that the concentration ellipses of the true poles are circles (as proven in [5]). However, the concentration ellipses of the extraneous poles are ellipses for this case.

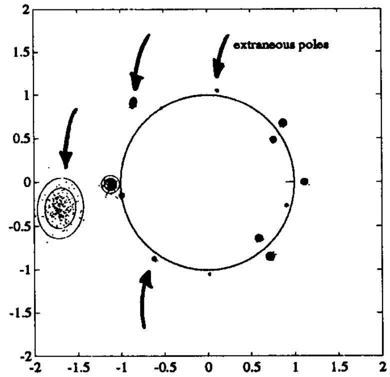

Figure 1. Two- and three-standard deviation concentration ellipses for both true poles and extraneous poles (with arrows) for the case of $m = 40$, $L = 14$, and $\sigma = 0.001$.

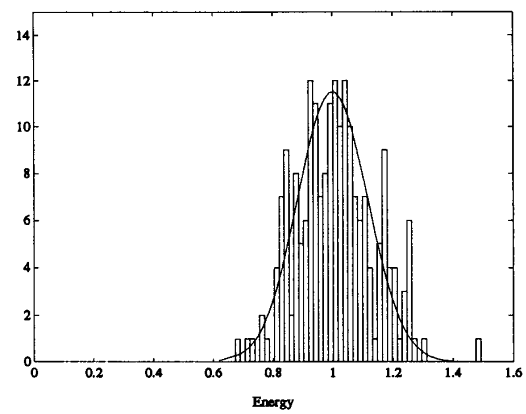

Figure 2. Theoretical energy pdf and histogram for a true mode.

We next verify the theoretical energy distributions given above. Figure 2 shows a comparison between the theoretical pdf and a histogram (obtained from Monte-Carlo simulations) for a true mode in the previous example. It can be seen that the theoretical energy distribution is a good approximation to simulation results in this case. Other true modes have similar results.

Figure 3 shows a comparison between the theoretical pdf and a histogram for an extraneous mode in the previous case. Note that the theoretical predictions agree closely with Monte-Carlo simulations. Other extraneous modes have similar results.

5. CONCLUSIONS

In this paper we have presented a statistical analysis of the well-known TLS-Prony algorithm. We have considered both true and extraneous modes in the method. The parameters include the real and imaginary parts of the poles and the amplitude coefficients. We have also presented a statistical analysis of the energies of the true and extraneous modes. We have shown that the true mode energies are well-approximated by Gaussian distributions, and the

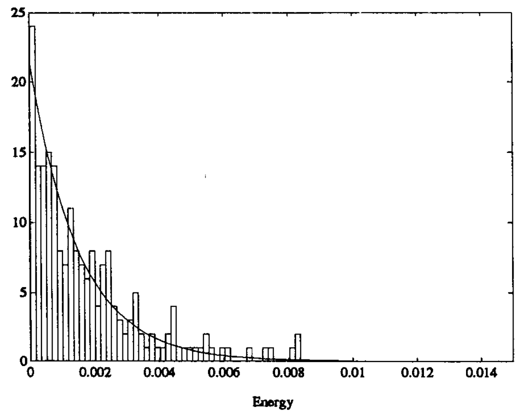

Figure 3. Theoretical energy pdf and histogram for an extraneous mode.

extraneous mode energies are approximately central χ^2 distributed with two degrees of freedom. Simulations have validated the analysis.

APPENDIX

$$\begin{aligned}
U &= R_1 B_1 R_1^* - R_1 B - B^* R_1^* + A^+ A^{+*} \\
V_1 &= (-R_1 B_2 + B_3) R_2^T + (-R_1 B_1 + B^*) R_3^* \\
V_2 &= (-R_1 B_2 + B_3) R_2^T - (-R_1 B_1 + B^*) R_3^* \\
Z_1 &= R_3 B_2 R_2^T + R_2 B_2^T R_3^T + R_3 B_1 R_3^* + R_2 B_4 R_2^* \\
Z_2 &= R_3 B_2 R_2^T + R_2 B_2^T R_3^T - R_3 B_1 R_3^* - R_2 B_4 R_2^* \\
R_1 &= A^+ C A T_x^+ T_p F G S^+ \\
R_2 &= F G P_{S^*}^{\perp} \Psi \\
R_3 &= F G S^+ \\
B_1 &= B B^* \\
B_2 &= B B_3 \\
B_3 &= \begin{bmatrix} I_1^{m\times(m-L)}, I_2^{m\times(m-L)} & \cdots & I_L^{m\times(m-L)} \end{bmatrix} \\
B_4 &= \begin{bmatrix} I_0^{(m-L)\times(m-L)} & \cdots & I_{L-1}^{(m-L)\times(m-L)} \\ \vdots & \ddots & \vdots \\ I_{L-1}^{(m-L)\times(m-L)*} & \cdots & I_0^{(m-L)\times(m-L)} \end{bmatrix} \\
B &= \begin{bmatrix} 1 & b^T & 0 & 0 & \cdots & 0 \\ 0 & 1 & b^T & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & b^T & 0 \\ 0 & \cdots & 0 & 0 & 1 & b^T \end{bmatrix}_{(m-L)\times(m)} \\
\Psi &= \mathrm{diag}\left(\psi^T, \psi^T, \cdots, \psi^T\right)_{L\times(m-L)L} \\
F &= \mathrm{diag}\left(\frac{1}{\eta_1}, \frac{1}{\eta_2}, \ldots, \frac{1}{\eta_L}\right) \\
\eta_i &= \begin{bmatrix} 1 & 2p_i & \cdots & Lp_i^{L-1} \end{bmatrix} b \\
G &= \begin{bmatrix} p_1 & p_1^2 & \cdots & p_1^L \\ p_2 & p_2^2 & \cdots & p_2^L \\ \vdots & \vdots & & \vdots \\ p_L & p_L^2 & \cdots & p_L^L \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
C &= \mathrm{diag}(0, 1, \ldots, m-1) \\
T_x &= \mathrm{diag}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}, 0, \ldots, 0\right)_{L\times L} \\
T_p &= \mathrm{diag}\left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_L}\right). \qquad (25)
\end{aligned}$$

Here $[\ s \ : \ S\ ]$ is the noise free version of $[\ y \ : \ Y\ ]$, $P_{S^*}^{\perp} = I_L - S^* S^{*+}$, I_L is the $L \times L$ identity matrix, $\psi = S^{*+} S^+ s$, and $I_u^{r\times c}$ is an $r \times c$ matrix of zeros except for ones on the uth subdiagonal (the 0th subdiagonal is the main diagonal).

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