Determining the Closest Stable Polynomial to an Unstable One

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***Abstract*—This paper considers the following problem: given a polynomial whose zeros do not all lie on or inside the unit circle, find the closest polynomial whose zeros are all on or inside the unit circle. The measure of closeness used is the weighted Euclidean distance in coefficient space. The algorithm can be extended to other measures of closeness, as well. Because the direct minimization on the coefficient space is difficult, we approach the problem in Schur coefficient space. In this way, the stability condition is easily guaranteed. We develop a very efficient algorithm for obtaining the optimum solution.**

I. Introduction

In time series modeling and system identification problems, one often obtains an estimate of an autoregressive (denominator) polynomial. Depending on the particular estimator used, this polynomial may or may not be "stable"; that is, it may or may not have all its zeros inside the unit circle [1], [2]. Examples of autoregressive (AR) estimators which do not guarantee stability include the covariance and prewindow methods [1], and most singular value decomposition-based methods [3]. In addition, nearly all noniterative methods of ARMA modeling first estimate the AR coefficients by using some form of the extended Yule–Walker equations; these methods almost never guarantee that the estimate AR polynomial is stable [1], [3], [4]. Few system identification algorithms ensure stability of the estimate either [2], [5].

Many applications require that the estimated denominator polynomial be stable. This is especially true in system identification applications, and in time series analysis applications which use the model as a synthesis filter (speech synthesis is one example [6]). Because most estimation algorithms do not guarantee stability, the following problem is of interest: given a polynomial whose zeros are not all inside the unit circle, find a "close" polynomial whose zeros are all inside the unit circle. We call this the stabilization problem.

There are several ways to stabilize an unstable polynomial. One method is to find the zeros of the unstable polynomial, and if any zero has magnitude greater than one, change it to have magnitude equal to (or slightly less than) one. In this case, the stable polynomial is "close" to the original one in the sense of minimizing a distance measure based on the zero locations of the polynomials. In some applications this zero is reflected inside the unit circle by using the reciprocal of its magnitude; this choice has the property that $|B(z)| = |A(z)|$ on the unit circle (although the phases of these polynomials will differ). Another method based on the Schur parameters (or reflection coefficients) associated with a polynomial could be used: find the reflection coefficient sequence of the given polynomial (using the Levinson-Durbin recursions), and change any Schur parameter with magnitude greater than one to one which is (slightly less than) one in magnitude [7].

This paper considers solutions to the stabilization problem that minimize the error between the polynomial coefficients of the original and stabilized polynomial. The reason for working in coefficient space is that most algorithms which estimate these polynomials directly estimate the coefficients of the polynomials (rather than the Schur parameters or the zeros corresponding to that polynomial). Since the polynomial coefficients are being estimated, it is natural to stabilize the polynomial by perturbing these estimated coefficients as little as possible.

More specifically, we use the weighted l_2 distance measure in coefficient space as the measure of closeness. The reason for this choice is that most of the polynomial coefficient estimation methods in time series analysis and systems identification give coefficient estimates which are asymptotically Gaussian distributed as the number of data points used to estimate the coefficients becomes large [1], [8]; this Gaussian distribution is obtained even when the data themselves are non-Gaussian. Since the polynomial coefficients are approximately Gaussian, the l_2 error is the most natural distance metric to use in perturbing these coefficients. Moreover, asymptotic variance expressions for these coefficient estimates have been obtained for several algorithms [1], [9]; in this case, the inverse of the covariance matrix can be used as a weighing matrix in a weighted l_2 coefficient norm to form the distance measure. A stable polynomial whose (weighted) distance from the given polynomial is minimum has the interpretation of a minimum variance solution to the stabilization problem.

Although we minimize an error in coefficient space for the stabilization problem, we find that working in the Schur parameter space is easier because the stability condition is readily guaranteed. A related problem involving optimization of a covariance sequence was studied in [10], where the Schur parameter space was used to guarantee that a covariance sequence is nonnegative definite. We then employ the alternate minimization method [11] to derive a computationally efficient algorithm for solving the stabilization problem.

While the algorithm we present uses the l_2 norm, it readily generalizes to other l_p norms as well; the only difference is that the error function is no longer quadratic in the parameters for $p \neq 2$, so the alternate minimization of the error function becomes more complex.

An outline of this paper is as follows. In Section II, we present a formal statement of the problem. In Section III, some properties of the stability sets both in polynomial coefficients

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and Schur parameters are discussed. These properties form the foundations of our results. In Section IV, an efficient algorithm for solving this minimization problem is given. In Section V, some examples are given to illustrate the algorithm.

II. Problem Statement

Assume we are given the real vector $b = [b_1, \cdots, b_n]^T$, and that its associated polynomial

$$B(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_n \tag{1}$$

has at least one zero z_0 satisfying $|z_0| > 1$. We are interested in finding another vector $a = [a_1, \cdots, a_n]^T$ which is close to b, and such that its associated polynomial

$$A(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \tag{2}$$

has all its zeros on or inside the unit circle. The measure of error we use is the weighted Euclidean (l_2) distance

$$J = \|a - b\|_W^2 \triangleq (a - b)^T W(a - b) \tag{3}$$

for some given positive definite weighting matrix W.

Consider the set of coefficients corresponding to stable polynomials:

$$S_a = \{a | A(z) = 0 \Rightarrow |z| \leq 1\}. \tag{4}$$

The stabilization problem can then be stated as follows:

Problem SP: Given a vector $b \notin S_a$, find a vector $a^\circ \in S_a$ such that $J = (a^\circ - b)^T W(a^\circ - b)$ is minimized over all $a \in S_a$.

III. Characterization of the Stability Set

In order to solve the above stabilization problem, it is useful to establish some basic properties of the stability set S_a.

Let us first introduce the concepts of Schur parameters (also known as reflection coefficients in the signal processing literature). For any polynomial of degree k, we define [12]

$$\Phi_k^*(z) = z^k \Phi_k(z^{-1}). \tag{5}$$

Then for an arbitrarily given real vector $[r_1, \cdots, r_n]$ (the vector of Schur parameters), we can determine a set of polynomials $\Phi_i(z)$, $i = 0, 1, \cdots, n$ through the following recursive formulas:

$$\Phi_i(z) = z\Phi_{i-1}(z) + r_i \Phi_{i-1}^*(z), \qquad \Phi_0(z) = 1. \tag{6}$$

The following two properties of $\Phi_n(z)$ are useful in the stability study of the discrete time system analysis [13]:

1) $\Phi_n(z)$ has all its zeros inside the unit circle $|z| = 1$ if and only if $|r_i| < 1$ for $i = 1, 2, \cdots, n$.

2) Assume $|r_i| < 1$ for $i = 1, 2, \cdots, (n - 1)$. Then $\Phi_n(z)$ has all its zeros on the unit circle $|z| = 1$ if and only if $|r_n| = 1$.

From these two properties, the following property readily follows:

3) If $|r_i| \leq 1$ for $i = 1, \cdots, n$, then $\Phi_n(z)$ has all its zeros on the unit disk $|z| \leq 1$.

With the help of property 3, we can count the number of zeros inside the unit circle and the number of zeros on the unit circle of $\Phi_n(z)$. Let us express $\Phi_k(z)$, $k = 1, \cdots, n$ in terms of a coefficient vector $[a_{k1}, \cdots, a_{kk}]$ where

$$\Phi_k(z) = z^k + a_{k1}z^{k-1} + \cdots + a_{kk}. \tag{7}$$

It is easy to show that the vector $[a_{k1}, \cdots, a_{kk}]$ and the vector $[r_1, \cdots, r_k]$ are related by

$$\begin{cases} a_{ki} = a_{(k-1)i} + r_k a_{(k-1)(k-i)} & \text{for } i = 1, \cdots, (k-1) \\ a_{kk} = r_k. \end{cases} \tag{8}$$

We know that the stability region of polynomial $A(z) = \Phi_n(z)$ in terms of Schur parameters $[r_1, \cdots, r_n]$ is completely specified by the unit hypercube:

$$S_r = \{|r_i| \leq 1, i = 1, \cdots, n\}. \tag{9}$$

This set is a closed and bounded subset of R^n. Since the a_i parameters are continuous functions of the Schur parameters, we have the following:

Theorem: a) S_a is a bounded and closed subset of R^n. b) Let B_s denote the boundary of S_a. Then if $a \in B_s$, there is at least one zero z_0 of (2) satisfying $|z_0| = 1$.

It is obvious that the set S_r is convex, but the set S_a is not convex in general. As an example, $a = [3, 3, 1]^T$ and $b = [-3, 3, -1]^T$ are in S_a, but $\alpha a + (1 - \alpha)b = [0, 3, 0]^T$ with $\alpha = 0.5$ is not in S_a. Since S_a is not a convex set, there may not be a unique solution to the stabilization problem. However, since S_a is closed, any solution to the stabilization problem will lie on the boundary of S_a.

IV. Solution to the Stabilization Problem

From the previous discussion we know that the a_i coefficients can be expressed in terms of the Schur parameters $[r_1, \cdots, r_n]$ and the stability set S_a can be transformed to the stability set S_r. We will find the optimal solution a° by working on the set S_r.

For fixed $[r_1, \cdots, r_{k-1}, r_{k+1}, \cdots, r_n]$, each $a_i (i = 1, \cdots, n)$ is a linear function of r_k. Thus J will be (at most) a quadratic function of r_k, and its minimum is easily found. Since we need only three points to specify a quadratic function, we calculate the following three points: $Q_{-1} = J$ at $r_k = -1$, $Q_0 = J$ at $r_k = 0$, and $Q_1 = J$ at $r_k = 1$. Then the quadratic function can be found as

$$J_k = J(r_k) = \alpha r_k^2 + \beta r_k + \gamma \tag{10}$$

where

$$\alpha = \tfrac{1}{2}[Q_{-1} + Q_1 - 2Q_0], \; \beta = \tfrac{1}{2}[Q_1 - Q_{-1}], \; \gamma = Q_0.$$

Note that $\alpha \geq 0$. The minimum value of J_k on $-1 \leq r_k \leq 1$ with respect to r_k can be found very easily as

$$\begin{cases} \gamma - 0.25\beta^2/\alpha & \text{if } -1 \leq -0.5\beta/\alpha \leq 1 \\ \gamma + \beta & \text{if } \alpha = 0 \quad \text{and } \beta \leq 0 \\ \gamma - \beta & \text{if } \alpha = 0 \quad \text{and } \beta \geq 0 \\ \gamma & \text{if } \alpha = 0 \quad \text{and } \beta = 0. \end{cases} \tag{11}$$

To use this quadratic function property, the alternative minimization procedure is used (see [11] for a description of the alternative maximization procedure). The algorithm for finding the optimal Schur parameters r° is as follows:

1) Initialize $r^0 \in S_r$ and accuracy ϵ. Set $i = 1$.

2) Calculate r_k^i by using (10) and (11) successively for $k = 1, \cdots, n$.

3) If $\max_{1 \le k \le n} |r_k^{i-1} - r_k^i| \le \epsilon$, then go to 4); otherwise increase i and go to 2).

4) Calculate a° from r^i by (8).

Note that the above algorithm is based on a (weighted) l_2 error norm, which results in a quadratic error function in (10). If an l_p norm is used for $1 \le p < \infty$, the above algorithm can still be used; the only difference is that J_k is no longer quadratic in r_k, so its minimum on $-1 \le r_k \le 1$ must be found by some other means. For $1 < p < \infty$, J_k is a continuously differentiable function of r_k, and can be minimized using standard techniques. For $p = 1$ or $p = \infty$, J_k is not continuously differentiable, and more care must be taken to find its minimum.

In the case $\alpha = 0$ and $\beta = 0$, J_k is independent of r_k. Theoretically, we can choose any number on $[-1, 1]$ for r_k. However, we must be careful about the selection of r_k values, otherwise the algorithm can get "stuck." As an example, for $n = 2$ we know

$$a_1 = r_1(1 + r_2), \qquad a_2 = r_2.$$

Assume $b = [2, -2]^T$ and that the initial value is $r^0 = [0, -1]^T$. Then if we do the minimization procedure with respect to r_1 with $r_2 = -1$ fixed, we have $\alpha = 0$ and $\beta = 0$. If we select $r_1 = 0$, then the minimization with respect to r_2 gives $r_2 = -1$. Thus, $r^1 = [0, -1]^T$ and the algorithm fails to converge to the true solution of $r^\circ = [1, 0.5]^T$. However, if we select any value for r_1 on $[-1, 1]$ except 0, we can continue the minimization procedure. In fact, if we chose any number $r_1 \in [-1, 1]$ and any number $r_2 \in (-1, 1]$ as the initial values, the algorithm converges to the true minimum. In general, if the algorithm "sticks" at an iteration, we can perturb the coefficients slightly and proceed.

Since S_a is not convex, there is the possibility of convergence to a local minimum. One way to overcome this situation is to try several initial values. Because we can use the quadratic function property, we have a very efficient minimization algorithm for each initial condition. The choice of a "good" initial guess can often eliminate problems of convergence to local minima. In most applications, the given polynomial is an estimate of a stable polynomial, and is therefore expected to be not too far away from a stable polynomial. In this case a good initial guess can be found using the following procedure [7].

- From $B(z)$ compute the Schur parameters $r^b = [r_1^b, \cdots, r_n^b]^T$.
- For each i, if $|r_i^b| > 1$ replace it by $\operatorname{sign}(r_i^b) \cdot (1 - \alpha)$ for some small positive α (we use $\alpha = 0.001$). This modified Schur sequence is used as the initial sequence r^0.

Note that $r^0 \in S_r$. Also, r^0 is close to r^b, and their corresponding polynomials are close to each other [7].

Another initial guess can be found using the following procedure.

- Find all the zeros $\{z_i\}$ of $B(z) = 0$.
- Form another sequence $\{z_i'\}$, where $z_i' = z_i$ if $|z_i| < 1$ and z_i' and $= (1 - \alpha)z_i/|z_i|$, for some small positive α, if $|z_i| \ge 1$.
- Use $\{z_i'\}$ as the zeros to form a polynomial $B'(z)$.
- From $B'(z)$ compute the Schur parameters, and use them as the initial r^0.

This initial guess requires more computation, as the zeros of $B(z)$ must be found. However, we found that this method gives a better initial guess than does the previous method.

We close this section by extending the minimization to the case where the leading polynomial coefficient is not unity. If the unstable polynomial is given by

$$B(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n \tag{12}$$

and the corresponding stable polynomial is

$$C(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_n. \tag{13}$$

We can write $C(z)$ as

$$C(z) = c_0[z^n + a_1 z^{n-1} + \cdots + a_n] = c_0 A(z) \tag{14}$$

with $a_i = c_i/c_0$, $i = 0, 1, \cdots, n$. Thus we can determine a_i in terms of Schur parameter vector r, and c_0 can be determined by minimizing

$$\begin{aligned} J_k &= (c - b)^T W (c - b) \\ &= (a^T W a) c_0^2 - 2(a^T W b) c_0 + b^T W b \end{aligned} \tag{15}$$

where $a = [1, a_1, \cdots, a_n]^T$, $b = [b_0, b_1, \cdots, b_n]^T$, and $c = [c_0, c_1, \cdots, c_n]^T$.

Taking the derivative with respect to c_0 of (15) and setting it to zero, we have

$$(a^T W a) c_0 - a^T W b = 0$$

which gives

$$c_0 = \frac{a^T W b}{a^T W a}. \tag{16}$$

The corresponding J_k function is given by

$$J_k = b^T W b - f(r_k) \tag{17}$$

where

$$f(r_k) = \frac{(a^T W b)^2}{a^T W a} = \frac{d_0 r_k^2 + d_1 r_k + d_2}{e_0 r_k^2 + e_1 r_k + e_2}. \tag{18}$$

It is obvious that $d_0 \ge 0$, $e_0 \ge 0$, $d_2/e_2 \ge 0$, and

$$d_0 r_k^2 + d_1 r_k + d_2 \ge 0, \qquad e_0 r_k^2 + e_1 r_k + e_2 > 0.$$

Minimizing J_k is equivalent to maximizing $f(r_k)$ with respect to r_k over $[-1, 1]$. Except when $f(r_k) = $ constant, we can prove that $f(r_k)$ has a unique maximum on $[-1, 1]$, and this maximum can be found analytically in a similar manner as before. In this way, we can still make use of the alternating projection algorithm in the more general case.

V. Examples

Below we present some examples which illustrate the theory discussed above. To study the computation time, we consider searching the minimum in all directions once (J_k, $k = 1, \cdots, n$) as one step. In all cases the accuracy is $\epsilon = 10^{-5}$.

Example 1: $n = 2$ and $W = I$.

All of the examples for $n = 2$ are quite simple. Moreover, the minimum point to the stabilization problem can be found immediately by inspection of Fig. 1. For the all $n = 2$ cases we obtain the optimal solution within two iterations of the algorithm.

Example 2: Consider the stable polynomial from [14]

$$A(z) = z^4 - 2.7607 z^3 + 3.8106 z^2 - 2.6535 z + 0.9238$$

so $a = [-2.7607, 3.8106, -2.6535, 0.9238]^T$.

We add Gaussian noise $N(0, 0.0001)$ to the vector a. If the perturbed coefficients give an unstable polynomial, we stabilize it by using the algorithm discussed; if the perturbed coefficients

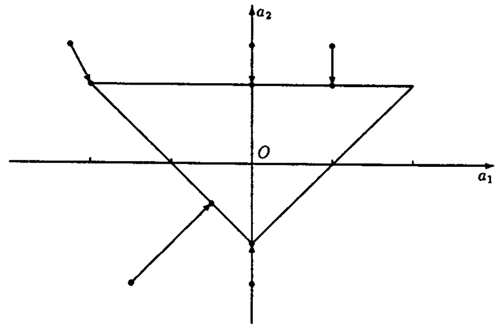

Fig. 1. Optimal solutions for $n = 2$.

TABLE I
SOLUTIONS TO EXAMPLE 2 USING OPTIMAL METHOD

		Before Stabilization		After Stabilization	
a	True Value	Mean	Stdev	Mean	Stdev
a_1	−2.7607	−2.7621	0.0100	−2.7625	0.0083
a_2	3.8106	3.8086	0.0091	3.8079	0.0091
a_3	−2.6535	−2.6555	0.0102	−2.6560	0.0083
a_4	0.9238	0.9237	0.0105	0.9233	0.0077
Avg Distance		3.9556×10^{-4}		2.7901×10^{-4}	

give a stable polynomial, we will not do anything. The weighing matrix W is chosen as the identity matrix, and 50 Monte Carlo simulations are performed.

Table I shows the mean and standard deviation of the polynomial coefficients before and after stabilization. In this case, 30 of the 50 polynomials required stabilization. It can be seen that the standard deviation is lower for the stabilized polynomial coefficients. Table I also shows the average distance square (J) between the true polynomial coefficients and the perturbed polynomial coefficients (before and after stabilization, respectively). For the stabilized case, a smaller distance measure is observed.

Figs. 2 and 3 show the zero distribution of 50 simulations before and after stabilization, respectively.

In this example, the minimization procedure converged to the optimal solutions for 26 out of 30 stabilizations using the modified Schur parameters as the initial conditions. For the case of finding the initial conditions from the zeros of $B(z)$, the minimization procedure converged to the optimal solution for all 30 stabilizations.

To compare this stabilization method with others, stabilized the polynomial using two other methods. In the first method, we computed the zeros of the noisy polynomial, and any zeros outside the unit circle were reflected inside; that is, the magnitude of the zero was inverted, and the angle of the zero was left unchanged. We note that this method is the asymptotic result of the planar least squares inverse (PLSI) techniques applied twice to the polynomial [15, pp. 173–174], [16, pp. 234–236]. In the PLSI technique, a Levinson recursion is used to find a stable inverse polynomial; applying this method twice gives a stable approximant to the original unstable polynomial. The PLSI stabilization method is computationally efficient because we need only compute two Levinson recursions. As the order of the first inverse polynomial approaches infinity, the PLSI method approaches the zero reflecting method discussed above.

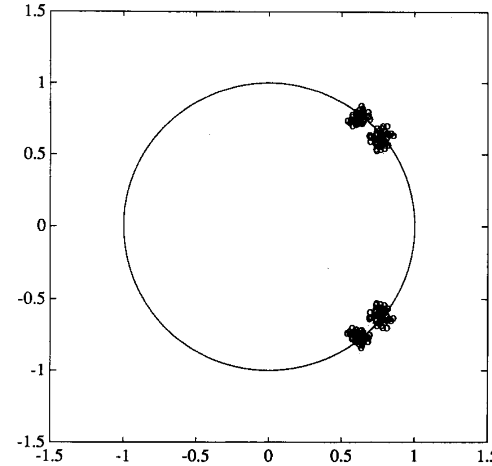

Fig. 2. Zero distribution before stabilization of example 2.

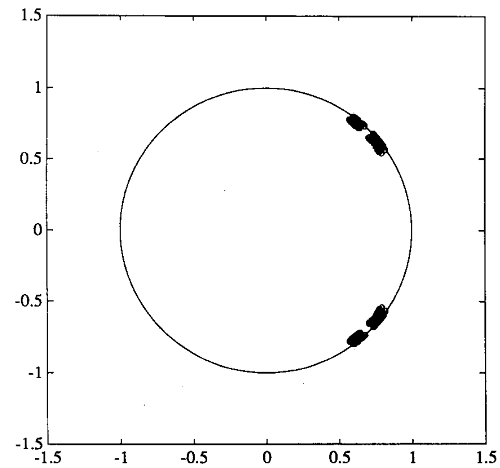

Fig. 3. Zero distribution after stabilization of example 2, using optimal method.

The second method of stabilization is similar to the first method except that the magnitude of the zero is set to one instead of inverted. This method provides the minimum distance stable polynomial, where distance is measured in "zero space."

The results of the above two stabilization methods are shown in Table II. It can be seen that both of these stabilization methods give much higher distances to the true polynomials than the optimal method does; in this case, the squared error J is about two orders of magnitude higher. Figs. 4 and 5 show the zero plots for the two other stabilization methods. Comparing these with the zero plots for the original and optimally stabilized polynomials in Figs. 2 and 3, we see similar results, with the optimal method giving slightly tighter zero clusters than the other two methods.

Example 3: This example considers stabilization of an estimated polynomial $B(z)$ obtained by AR modeling of a stochas-

TABLE II
SOLUTIONS TO EXAMPLE 2 USING ZERO MOVING METHODS

a	True Value	Zero Reflecting		Zero to Unit Circle	
		Mean	Stdev	Mean	Stdev
a_1	−2.7606	−2.7167	0.0730	−2.7389	0.0381
a_2	3.8106	3.6874	0.1905	3.7462	0.0991
a_3	−2.6535	−2.5348	0.1876	−2.5931	0.0979
a_4	0.9238	0.8689	0.0833	0.8953	0.0432
Avg Distance		8.3744×10^{-2}		2.2723×10^{-2}	

TABLE III
SOLUTIONS TO EXAMPLE 3 USING OPTIMAL METHOD

a	True Value	Before Stabilization		After Stabilization	
		Mean	Stdev	Mean	Stdev
a_1	0.1000	0.0457	0.1896	0.0425	0.1943
a_2	1.6600	1.5601	0.2105	1.5305	0.2161
a_3	0.0930	0.0446	0.2076	0.0425	0.1957
a_4	0.8469	0.7599	0.2434	0.7327	0.2369
Avg Distance		0.1826		0.1789	

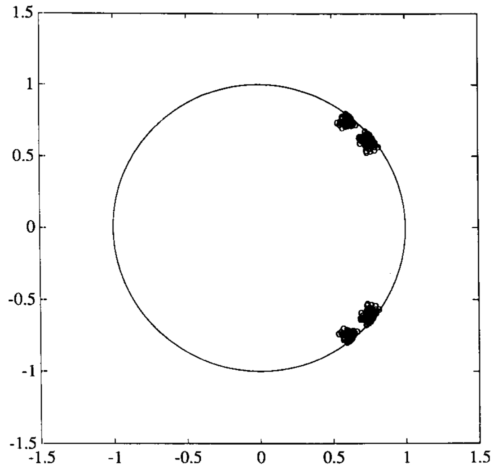

Fig. 4. Zero distribution after stabilization of example 2, using zero reflection method.

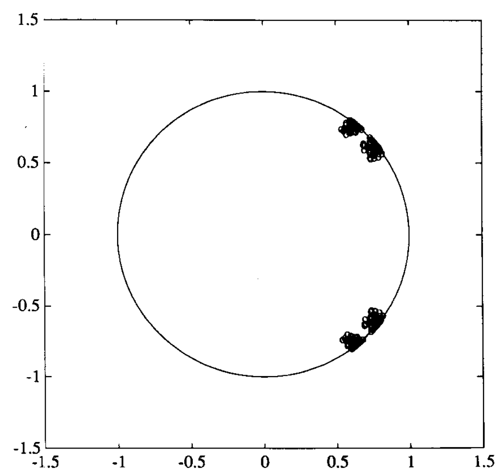

Fig. 5. Zero distribution after stabilization of example 2, moving unstable zeros to the unit circle.

tic time series. The data is generated by an AR(4) model with

$$A(z) = z^4 + 0.1z^3 + 1.66z^2 + 0.093z + 0.8649.$$

This example was taken from [14]. Thus, the data $x(n)$ is generated by the recursion

$$x(k) = -\sum_{i=1}^{4} a_i x(k-i) + u(k)$$

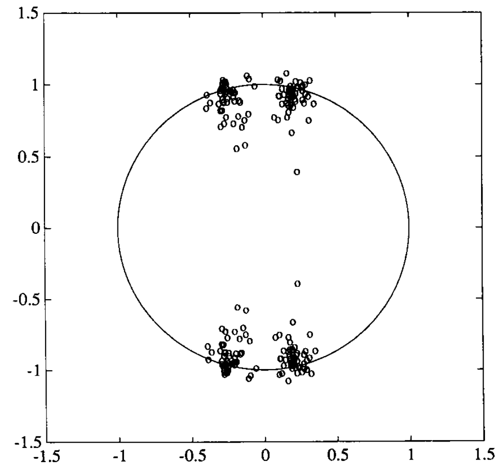

Fig. 6. Zero distribution before stabilization of example 3.

where $\{u(k)\}$ is a $N(0, 0.5)$ white noise sequence. We generate 50 sets of data points $\{x(k)\}_{k=0}^{19}$ and from each set we obtain an estimate of the AR parameter vector a using the covariance method (see [1]). If the estimated polynomial is not stable, we stabilize it using the minimization algorithm.

It is known that such an AR parameter vector estimate is asymptotically a Gaussian distributed random vector. The covariance matrix of the estimate can be found in, e.g., [8, p. 212]. For this vector, the natural distance metric is the weighted l_2 norm with the weighting matrix W chosen to be the inverse of the asymptotic covariance matrix of AR parameter vector; in this way, the stabilization procedure corresponds to a minimum variance update. Therefore, we have used the inverse of the covariance matrix as the weighting matrix in this example.

Table III shows the means and standard deviations of the estimated AR coefficients before and after stabilization (20 of the 50 polynomials were unstable). Table III also shows the average distance square (J) from the true polynomial coefficients to the realized polynomial coefficients before and after stabilization, respectively. For the stabilized case, a smaller distance measure is also observed. The distance measure is only slightly smaller because the variance of the estimates is the dominant factor in this distance.

Figs. 6 and 7 show the zero distribution of 50 simulations before and after stabilization, respectively. While the stabilized zeros are all within the unit circle, there was not much movement needed to change the unstable polynomials to stable ones. In this example, the minimization procedure did not converge to the optimal solution only for one of the 20 stabilizations using

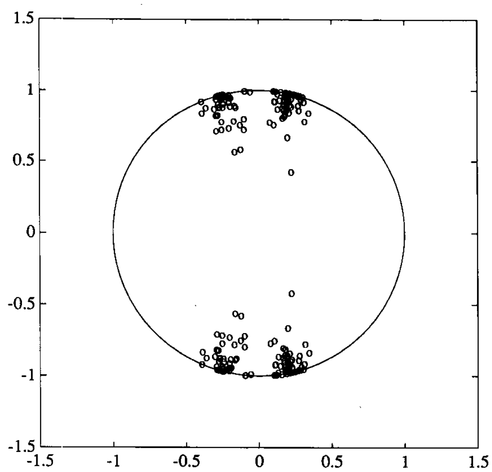

Fig. 7. Zero distribution after stabilization of example 3, using the optimal method.

the modified Schur parameters as the initial conditions. It converged to the optimal solutions for all 20 stabilizations for the initial conditions from the zeros of $B(z)$.

The spectral peaks corresponding to the stabilized polynomials will be very sharp because the stabilized AR estimates have poles on the unit circle. However, if it is known *a priori* that the spectral peaks must have a certain minimum bandwidth, then the stabilization procedure can be modified to give poles which lie in the disk $\{|z| < 1 - \epsilon\}$ for some appropriate value of ϵ. Such a modification will be dependent on the particular application of the modeling procedure.

VI. Conclusions

We have considered the problem of finding the closest stable polynomial to a given unstable one. The measure of error between these two polynomials is the weighted Euclidean distance in coefficient space. This problem has no closed form solution in general. We developed an efficient minimization procedure using the alternating projection approach in Schur parameter space. In each step in the iteration we minimize a scalar quadratic function, which is very efficient to implement. As a result, each iteration of the algorithm requires only $4.5n^2 + 1.5n + 2$ multiplications and one division. For the case $W = I$, identity matrix, there are only $1.5n^2 + 1.5n + 2$ multiplications and one division per iteration. Simulation examples illustrate the effectiveness of the algorithm for both a polynomial stabilization application and an autoregressive (AR) modeling example.

As a final note, we used an l_2 distance as a measure of closeness in polynomial coefficient space. However, other distance measures could easily be employed in this procedure. If other distance measures are used, the alternating projection approach can still be used, with only a small change in the error minimization procedure for the error function $J(r_k)$.

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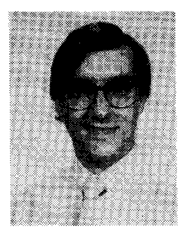

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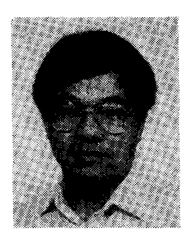

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