Signal Processing 21 (1990) 349-350
Elsevier

SHORT COMMUNICATION

ON BIASED ESTIMATORS AND THE UNBIASED CRAMÉR–RAO LOWER BOUND

Petre STOICA

Department of Automatic Control, Polytechnic Institute of Bucharest, Splaiul Independentei 313, Bucharest, Romania

Randolph L. MOSES

Department of Electrical Engineering, The Ohio State University, 2015 Neil Avenue, Columbus, OH 43210, U.S.A.

Received 23 October 1989
Revised 15 March 1990

Abstract. The goal of this note is to provide a simple example of a biased estimator whose mean square error (MSE) is less than the unbiased Cramér–Rao lower bound (UCRLB).

Zusammenfassung. Der Zweck dieser Mitteilung ist es, ein einfaches Beispiel dafür anzugeben, daß der mittlere quadratische Fehler eines Schätzers mit Bias kleiner sein kann als die untere Cramér-Rao-Schranke im biasfreien Fall.

Résumé. L'objectif de cette note est de fournir un exemple simple d'un estimateur biaisé dont l'erreur quadratique moyenne (MSE) est inférieure à la borne inférieure de Cramér–Rao pour le cas non biaisé (UCRLB).

Keywords. Biased estimators, unbiased Cramér–Rao lower bound, mean square error, sample variance.

1. Introduction

Much of the research literature in the areas of signal processing, system identification and parameter estimation contains comparisons of the mean square error (MSE) of a specific estimator to the Cramér–Rao lower bound for unbiased estimators (abbreviated as UCRLB). The existence of the bias in the studied estimator is sometimes considered to be the reason for its inflated MSE in comparison with the UCRLB. However, the MSE of a biased estimator is not necessarily greater than the UCRLB. The purpose of this work is to make this fact widely known to the signal processing community by means of a simple example.

There are several examples in the literature which indicate that use of a biased estimator may result in a reduction in the MSE. For example, ridge regression and minimum MSE linear estimators of parameters in regression models are known to be biased and to have MSEs less than the UCRLB [1]. However these estimators depend on the true (unknown) parameters and, therefore, are not realizable. In the following section we give a simple example of a realizable biased estimator which is statistically more efficient than any unbiased estimator in the sense that its MSE is less than the UCRLB.

2. An example

Let $\{y_t\}_{t=1}^N$ be a sequence of independent and identically distributed Gaussian random variables with zero mean and with (finite) variance denoted by σ^2. Consider the following estimate of σ^2:

$$\hat{\sigma}^2 = \frac{\alpha}{N} \sum_{t=1}^{N} y_t^2, \tag{1}$$

where $\alpha > 0$. Typically α is chosen equal to 1 to make $\hat{\sigma}^2$ an unbiased estimator. As we will show, when $\alpha = 1$ the estimator (1) is statistically efficient, that is, its MSE is equal to the UCRLB. In the following, however, we will choose α so as to minimize the MSE of $\hat{\sigma}^2$, and show that this MSE is lower than the UCRLB.

Straightforward calculations give

$$\mathrm{E}\{\sigma^2\} = \alpha\sigma^2 \tag{2}$$

and

$$\begin{aligned}\mathrm{MSE}(\hat{\sigma}^2) &\triangleq \mathrm{E}\{(\hat{\sigma}^2 - \sigma^2)^2\} \\ &= \mathrm{E}\{\hat{\sigma}^4\} + \sigma^4(1-2\alpha) \\ &= \frac{\alpha^2}{N^2}\sum_{t=1}^{N}\sum_{s=1}^{N}\mathrm{E}\{y_t^2 y_s^2\} + \sigma^4(1-2\alpha) \\ &= \frac{\alpha^2}{N^2}(N^2\sigma^4 + 2N\sigma^4) + \sigma^4(1-2\alpha) \\ &= \sigma^4\left[\alpha^2\left(1+\frac{2}{N}\right) + (1-2\alpha)\right].\end{aligned} \tag{3}$$

It is straightforward to show that the minimum value of $\mathrm{MSE}(\hat{\sigma}^2)$ is obtained for

$$\alpha_{\min} = \frac{N}{N+2}, \tag{4}$$

and this minimum MSE is given by

$$\min \mathrm{MSE}(\hat{\sigma}^2) = \frac{2\sigma^4}{N+2}. \tag{5}$$

Next consider the UCRLB of any unbiased estimate of σ^2. The log-likelihood function of the sequence $\{y_1, \ldots, y_N\}$ is the following:

$$L = -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{N} y_t^2. \tag{6}$$

Thus

$$\frac{\partial L}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^{N} y_t^2 \tag{7}$$

and

$$\frac{\partial^2 L}{\partial(\sigma^2)^2} = \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{t=1}^{N} y_t^2. \tag{8}$$

It follows from (8) that [1, 2]

$$\begin{aligned}\mathrm{UCRLB} &\triangleq -\left[\mathrm{E}\left\{\frac{\partial^2 L}{\partial(\sigma^2)^2}\right\}\right]^{-1} \\ &= \left(-\frac{N}{2\sigma^4} + \frac{N\sigma^2}{\sigma^6}\right)^{-1} = \frac{2\sigma^4}{N}.\end{aligned} \tag{9}$$

Clearly the UCRLB in (9) is greater than the minimum MSE in (5). This provides a simple example of a biased realizable estimator whose MSE is less than the UCRLB. In addition, the sample variance formula (1), (4) obtained as a byproduct of this example may be of interest in its own right.

Finally, note from (7) and (3) that the maximum-likelihood estimator of $\hat{\sigma}^2$ is given by (1) with $\alpha = 1$, and its MSE is equal to the UCRLB.

References

[1] J.P. Norton, *An Introduction to Identification*, Academic Press, New York, 1986.

[2] T. Söderström and P. Stoica, *System Identification*, Prentice-Hall, London, 1989.