A Bayesian Approach to Array Geometry Design

Ülkü Oktel and Randolph L. Moses, *Senior Member, IEEE*

***Abstract*— In this paper we consider the design of planar arrays that optimize direction-of-arrival (DOA) estimation performance. We assume that the single source DOA is a random variable with a known prior probability distribution and the sensors of the array are constrained to lie in a region with an arbitrary boundary. The Cramér-Rao Bound (CRB) and the Fisher Information Matrix (FIM) for single source DOA constitute the basis of the optimality criteria. We relate the design criteria to a Bayesian CRB criterion and to array beamwidth; we also derive closed-form expressions for the design criteria when the DOA prior is uniform on a sector of angles. We show that optimal arrays have elements on the constraint boundary, thus providing a reduced dimension iterative solution procedure. Finally, we present example designs.**

***Index Terms*— array design, planar arrays, direction of arrival estimation, Cramér-Rao bound**

I. Introduction

Direction-of-arrival (DOA) estimation from the outputs of an array of sensors is an important and well-studied problem with many applications in radar, sonar, and wireless communications. A large number of DOA estimation algorithms and analytical performance bounds have been developed (see, *e.g.*, [1]). The DOA estimation performance of an array strongly depends on the number and locations of the array elements. In this paper we consider planar array geometry design for "good" DOA estimation performance.

A number of researchers have considered the design of arrays to achieve or optimize desired performance goals. Much of the array design literature is devoted to linear arrays ([2]-[6]). For planar arrays, performance comparisons of some common array geometries are given in [7]-[9]. In [10], a measure of similarity between array response vectors is introduced and a tight lower bound for this similarity measure is derived. This bound is suggested as a performance criterion in the sense that the array with highest bound has best ambiguity resolution. In [11], differential geometry is used to characterize the array manifold and an array design framework based on these parameters is proposed. In [12], a sensor polynomial is constructed using prespecified performance levels, such as detection resolution thresholds and CRBs on error variance, and roots of the polynomial are the sensor locations of the desired linear or planar array. The Dolph-Chebyshev criterion is proposed for optimal element positioning in [13]. The method proposed in [13] minimizes the mainlobe area while satisfying the prespecified sidelobe levels.

Ü. Oktel is with Aselsan, Inc., Ankara 06370, Turkey (e-mail: uoktelaselsan.com.tr).

R. L. Moses is with the Department of Electrical and Computer Engineering, The Ohio State University, Columbus, OH 43210 USA (e-mail: randy@ece.osu.edu).

Most of the above papers consider designs for a single desired DOA or they implicitly assume that the DOA is equally likely in all directions. In many applications, including radar, sonar, and wireless base station design, the DOA of interest may be constrained to lie in a sector, or may be more likely in some directions than others. In this paper, we consider design of optimal planar arrays for such scenarios by modeling the DOA of the single source as a random variable whose prior probability distribution function (pdf) characterizes any prior constraints or arrival angle likelihood. To keep the paper concise, we present results for planar arrays that estimate DOA in azimuth angle only. However, the design method and main results also apply to volume arrays, and to arrays which estimate the DOA in both azimuth and elevation. In addition, the results apply to both narrowband and wideband arrays.

We adopt a Bayesian approach and employ the average CRB and average FIM as design criteria. We relate both the average FIM and average CRB to the Bayesian CRB (also called the global CRB). The CRB gives a lower bound on the variance of any unbiased estimate of a non-random parameter. The Bayesian CRB is a lower bound on the mean-squared error of the estimate of a random parameter and is independent of any particular estimator [14].

Because the array locations are nonlinear functions of the resulting cost criteria, closed-form solutions are not available except in a few special cases; thus, we adopt nonlinear function minimization techniques. We show that the optimal element locations lie on the boundary of the element constraint region, so the dimension of the minimization problem can be reduced from $2m$ to m, where m is the number of array elements. In the case of FIM criterion, the function to be minimized is a quadratic function of the array elements, so efficient quadratic optimization procedures can be used.

Both the CRB and FIM are closely related to the mainlobe width of the array [15], [13], [16]. We show that average FIM and average CRB can be interpreted as the average mainlobe width of the array, averaged over the steering angle. Arrays that have small mainlobe width perform well in moderate or high SNRs and they have high resolution.

An outline of this paper is as follows: In Section II we describe the system model, state our assumptions and give the expression for the CRB on the single source DOA. In Section III we introduce the performance measures and define the optimization problems. We discuss that both CRB and FIM based cost functions can be related to the Bayesian CRB. We also give the closed form integrals for FIM and CRB based cost functions when the probability distribution of the DOA is uniform. In Section IV, we prove that sensors of the both CRB-optimal and FIM-optimal arrays should lie on the boundary of the constraint region. We give example optimal array designs in Section V. Section VI concludes the paper.

II. System Model and Single Source CRB

We consider an array of m identical sensors on the (x, y) plane. Each sensor is located at $r_i = [x_i, y_i]^T$ for $i \in [1, m]$. We define $r_a = [x_a, y_a]^T = \frac{1}{m}\sum_{i=1}^{m} r_i$ as the centroid of these sensors. The array is represented by the $2 \times m$ array location matrix

$$\underline{\mathbf{r}} = [r_1, r_2, \ldots, r_m] = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \\ y_1 & y_2 & \ldots & y_m \end{bmatrix} \tag{1}$$

A single, narrowband far field source $s(t)$ centered at frequency $\omega_c = \frac{2\pi}{\lambda}$ and coplanar with the array impinges on the array from direction θ. A set of N snapshots are sampled by the array, giving the $m \times 1$ measurement vectors

$$x(t) = A_\theta(t)s(t) + n(t) \quad t = 1, 2, \ldots, N \tag{2}$$

where $n(t)$ is the $m \times 1$ noise vector, $s(t)$ is a scalar, and

$$A_\theta = \left[e^{j\frac{2\pi}{\lambda}d_1(\theta)}, e^{j\frac{2\pi}{\lambda}d_2(\theta)}, \ldots, e^{j\frac{2\pi}{\lambda}d_m(\theta)}\right]^T \tag{3}$$

where $d_k(\theta) = \frac{u^T(\theta)\cdot r_k}{c}$ is the propagation delay associated with the k^{th} sensor, c is the speed of propagation, and $u(\theta) = [\cos(\theta), \ \sin(\theta)]^T$ is the unit vector pointing towards the signal source. The noise at the sensors is assumed to be white Gaussian, and independent of the source signal.

Under the system model described above, the Fisher Information Matrix (FIM) for the DOA estimate from measurements $\{x(t)\}_{t=1}^{N}$ is given by [17], [18] and some simple algebra:

$$\begin{aligned} FIM(\underline{\mathbf{r}}, \theta) &= G(\underline{\mathbf{r}}, \theta) \cdot P & (4) \\ G(\underline{\mathbf{r}}, \theta) &= \frac{du(\theta)}{d\theta}^T B \frac{du(\theta)}{d\theta} & (5) \\ B &= \frac{1}{m}(\underline{\mathbf{r}} - \underline{\mathbf{r}}_A)(\underline{\mathbf{r}} - \underline{\mathbf{r}}_A)^T & (6) \end{aligned}$$

where P is an SNR term that is independent of the source DOA θ and of the array geometry, and where $\underline{\mathbf{r}}_A = [r_a, r_a, \ldots, r_a]$ is the $2 \times m$ array centroid matrix (see also [19]). The CRB on the DOA estimate is the inverse of the FIM given in (4).

For the purpose of array design, the narrowband signal assumption is not needed. If $s(t)$ is wideband, the expression for the FIM is still given by (4)–(6); only the expression for P changes (see [18], [16]).

III. Problem Statement and Cost Functions

We are interested in array geometry designs, *i.e.*, the selection of $\underline{\mathbf{r}}$, that yield good DOA estimation performance. We assume that the single source DOA is a random variable characterized by a known prior pdf $f(\theta)$. We further assume that the sensor elements are constrained to lie in a closed, connected region $D_\Gamma \subset \mathbb{R}^2$ which is bounded by a closed curve Γ. Let $D = D_\Gamma \times \cdot \times D_\Gamma \subset \mathbb{R}^{2\times m}$ denote the constraint region for the array element location matrix; thus, an admissible array geometry satisfies $\underline{\mathbf{r}} \in D$.

In determining optimal array designs we adopt a Bayesian approach and propose two different but related cost functions. We define a CRB-optimal array $\underline{\mathbf{r}}_{\mathbf{C}}$ as one whose element locations satisfy:

$$\underline{\mathbf{r}}_{\mathbf{C}} = \arg\min_{\underline{\mathbf{r}}\in D} J_C(\underline{\mathbf{r}}) \tag{7}$$

where the CRB cost function $J_C(\underline{\mathbf{r}})$ is given by:

$$\begin{aligned} J_C(\underline{\mathbf{r}}) &= E_\theta\{CRB(\underline{\mathbf{r}}, \theta)\} = \int_{-\pi}^{\pi} CRB(\underline{\mathbf{r}}, \theta) f(\theta) d\theta \\ &= \int_{-\pi}^{\pi} \frac{1}{FIM(\underline{\mathbf{r}}, \theta)} f(\theta) d\theta \end{aligned} \tag{8}$$

Similarly, we define the FIM-optimal array $\underline{\mathbf{r}}_{\mathbf{F}}$ by:

$$\underline{\mathbf{r}}_{\mathbf{F}} = \arg\max_{\underline{\mathbf{r}}\in D} J_F(\underline{\mathbf{r}}) \tag{9}$$

$$J_F(\underline{\mathbf{r}}) = E_\theta\{FIM(\underline{\mathbf{r}}, \theta)\} = \int_{-\pi}^{\pi} FIM(\underline{\mathbf{r}}, \theta) f(\theta) d\theta \tag{10}$$

In general, $\underline{\mathbf{r}}_{\mathbf{C}} \neq \underline{\mathbf{r}}_{\mathbf{F}}$ because of the integrations in (8) and (10).

The FIM cost criterion is a quadratic function of the array locations. From equations (4) and (10), we have

$$\begin{aligned} J_F(\underline{\mathbf{r}}) &= P \cdot \mathrm{tr}\left\{\frac{1}{m}(\underline{\mathbf{r}} - \underline{\mathbf{r}}_A)^T K (\underline{\mathbf{r}} - \underline{\mathbf{r}}_A)\right\} & (11) \\ K &= \int_{-\pi}^{\pi} \frac{du(\theta)}{d\theta} \frac{du(\theta)}{d\theta}^T f(\theta)\, d\theta & (12) \end{aligned}$$

which is quadratic in $\underline{\mathbf{r}}$. Quadratic optimization functions are useful because they lead to closed-form solutions for certain array boundaries, and they permit the use of efficient quadratic programming techniques for iteratively solving (9) when closed-form solutions are not available.

A. Relationship to Bayesian CRB

The cost functions $J_C(\underline{\mathbf{r}})$ and $J_F(\underline{\mathbf{r}})$ can be related to the Bayesian CRB as we show below. The Bayesian CRB has been proposed for random parameter estimation and is a lower bound on the mean-squared error of the estimated parameter (see [14]). The Bayesian CRB is a global bound that includes the *a priori* DOA information encoded in the prior probability distribution function.

For our problem the Bayesian CRB on the DOA angle θ, denoted $BCRB(\underline{\mathbf{r}}, \theta)$, is given by

$$BCRB(\underline{\mathbf{r}}, \theta) = [J_F(\underline{\mathbf{r}}) + I_\Theta]^{-1} \tag{13}$$

where $J_F(\underline{\mathbf{r}})$ is given in equation (10) and

$$I_\Theta = E_\theta\left\{\frac{\partial^2 \ln f(\theta)}{\partial \theta^2}\right\} \tag{14}$$

is the Fisher information of the prior. Since I_Θ is independent of array geometry, the FIM-optimal array $\underline{\mathbf{r}}_F$ also minimizes the Bayesian CRB on the DOA angle. The Bayesian CRB can also be related to $J_C(\underline{\mathbf{r}})$. Since $(\cdot)^{-1}$ is a convex function for positive arguments, by Jensen's inequality it follows that

$$\frac{1}{E_\theta\{FIM(\underline{\mathbf{r}}, \theta)\}} \leq E_\theta\left\{\frac{1}{FIM(\underline{\mathbf{r}}, \theta)}\right\} \tag{15}$$

Combining (13) and (15) gives the following relation:

$$BCRB(\underline{\mathbf{r}},\theta) = \frac{1}{J_F(\underline{\mathbf{r}}) + I_\Theta} \leq \frac{1}{J_F(\underline{\mathbf{r}})} \leq J_C(\underline{\mathbf{r}}) \tag{16}$$

The Bayesian CRB is thus bounded above by the CRB cost function and the CRB-optimal array $\underline{\mathbf{r}}_C$ minimizes that bound.

In [20] it is shown that the Bayesian CRB is unrealistically low for uniform distributions on θ since the term I_Θ tends to infinity for uniform distributions. The FIM-optimal array design above minimizes the Bayesian CRB, but removes the term in the Bayesian CRB that tends to infinity and that is anyway independent of the array element locations; similar comments apply to $J_C(\underline{\mathbf{r}})$. Thus, the functions $J_C(\underline{\mathbf{r}})$ and $J_F(\underline{\mathbf{r}})$ appear to be better suited than the Bayesian CRB for array design in scenarios where the DOA angle has uniform distribution.

The FIM and CRB are derived using a small perturbation analysis. The resulting bounds are tight bounds for maximum likelihood estimates of DOA for high SNR, but they are typically not tight bounds at low SNR, mainly because they do not take into account the effects of high sidelobes or ambiguity directions in the array beampattern. Other possible bounds for random parameter estimation are the Ziv-Zakai lower bound (ZZLB) and the Weiss-Weinstein lower bound (WWLB) [20], [15], [21]. A design framework based on WWLB is presented for linear arrays in [22]. Although the ZZLB and WWLB provide tighter and more realistic bounds (especially at low SNRs) they are computationally intensive to determine. When optimizing for a single DOA, the computational expense may be acceptable, but when the optimization criterion contains a range of DOAs as in (8) or (10), the computation of the ZZLB becomes significant. For most cases, minimizing (8) (or maximizing (10)) involves an iterative search for $\underline{\mathbf{r}}$. With the FIM criterion, the integral in (12) is evaluated once, whereas with the CRB criterion, the integral in (8) must be evaluated at each iteration on $\underline{\mathbf{r}}$. Both integrals are computationally simple. In contrast, the ZZLB involves computing an integral for *every* θ in the support of $f(\theta)$ and at each iteration on $\underline{\mathbf{r}}$; this is a significant increase the required computational load. An approximate closed-form expression for the ZZLB is derived in [15], but the approximation assumes that the array geometry is such that sidelobes of the beampattern are not significant, which is precisely the assumption we would attempt to avoid in replacing $FIM(\underline{\mathbf{r}},\theta)$ with a different bound. Thus, to keep computation tractable while maintaining a criterion that is based on a bound that is tight above a threshold SNR, we adopt the FIM and CRB criteria.

B. Relationship to Beamwidth

The beamwidth of the mainlobe for a delay-and-sum beamformer is proportional with the square root of the CRB (with the asymptotic standard deviation). Using the second order Taylor series approximation of the array gain around the steering angle θ_0, one can approximate the half-power beamwidth of the array as $\frac{\lambda_0}{2\pi}\sqrt{\frac{1}{G(B,\theta_0)}}$ [19], [15], [16]. The CRB cost function can thus be thought of as the average beamwidth of the array (averaged over steering angle) and the CRB-optimal array gives the minimum average beamwidth. For high SNRs, the mainlobe width is a good indicator of the DOA estimation performance and resolution of the array.

C. Cost Functions for Uniform Prior Distributions

It is possible to obtain closed form expressions for the integrals in the cost functions $J_C(\underline{\mathbf{r}})$ and $J_F(\underline{\mathbf{r}})$ when the DOA to be estimated has uniform probability distribution over a subset of $[-\pi,\pi]$. Such a special case is useful in many practical scenarios. For example, when no prior knowledge is available about the DOA, one typically assumes $f(\theta)$ is uniformly distributed on $[-\pi,\pi]$. In addition, for arrays that monitor a certain sector of angles, the prior DOA pdf may be chosen as uniformly distributed in that sector.

Assume $f(\theta)$ is uniformly distributed in the interval $[\theta_1,\theta_2] \subset [-\pi,\pi]$ with $\theta_1 < \theta_2$. By suitably rotating the element constraint region, we can without loss of generality take $\theta_1 = -\frac{\theta_0}{2}$ and $\theta_2 = \frac{\theta_0}{2}$.

The array geometry dependent term $G(\underline{\mathbf{r}},\theta)$ in the CRB can be written as a function of the eigenvalues λ_i and entries B_{ij} of the array covariance matrix B given in equation (6). Let $\lambda_1 \geq \lambda_2$ be the eigenvalues of B. A straightforward calculation gives:

$$G(\underline{\mathbf{r}},\theta) = a - b\cos(2\theta - \alpha) \tag{17}$$

$$a = \frac{B_{11}+B_{22}}{2} = \tfrac{1}{2}(\lambda_1+\lambda_2) \tag{18}$$

$$b = \sqrt{\left(\frac{B_{11}-B_{22}}{2}\right)^2 + B_{12}^2} = \tfrac{1}{2}(\lambda_1-\lambda_2) \tag{19}$$

$$\alpha = \tan^{-1}\left(\frac{B_{11}-B_{22}}{2B_{12}}\right) \tag{20}$$

When $f(\theta)$ is uniform over $[-\frac{\theta_0}{2},\frac{\theta_0}{2}]$, the FIM and CRB cost functions are given by:

$$J_F(\underline{\mathbf{r}}) = a - b\cos\alpha\frac{\sin\theta_0}{\theta_0} \tag{21}$$

$$J_C(\underline{\mathbf{r}}) = \frac{1}{\theta_0}\frac{1}{\sqrt{\lambda_1\lambda_2}}\left[\tan^{-1}\left(\sqrt{\frac{\lambda_1}{\lambda_2}}\tan\left(\tfrac{\theta_0-\alpha}{2}\right)\right) + \tan^{-1}\left(\sqrt{\frac{\lambda_1}{\lambda_2}}\tan\left(\tfrac{\theta_0+\alpha}{2}\right)\right)\right] \tag{22}$$

Note that for $\theta_0 = \pi$ (that is, $f(\theta)$ is uniformly distributed on $[-\pi,\pi)$) then

$$J_F(\underline{\mathbf{r}}) = \frac{1}{2}(\lambda_1+\lambda_2) = a \tag{23}$$

$$J_C(\underline{\mathbf{r}}) = \frac{1}{\sqrt{\lambda_1\lambda_2}} = \frac{1}{\sqrt{a^2-b^2}} \tag{24}$$

The term $a = \frac{1}{2}\operatorname{tr}(B)$ can be interpreted as an average aperture size, and increases as the sensors are moved away from the origin. The term b can be interpreted as an isotropy term — the array has constant CRB performance for all angles if and only if $b = 0$ (see [16]), and larger values of b correspond to larger changes in CRB performance as a function of DOA θ. Thus, we see that the FIM criterion attempts to maximize the average aperture, whereas the CRB criterion also tends to favor isotropic arrays. These properties are seen in the examples in Section V.

IV. Boundary Result

For general boundaries or prior pdfs, it is not possible to find an analytic solution for either of the optimization problems in (8) or (10), so iterative optimization procedures are employed. The optimal solution is found as a $2m$-dimensional search for element locations $\{x_i, y_i\}_{i=1}^m$. In this section we show that optimal solutions have all array elements on the boundary of the constraint region. If the constraint region D_Γ is convex, this boundary result is a direct consequence from optimization theory since $G(\underline{\mathbf{r}}, \theta)$ is a convex function in $\mathbb{R}^{2m}$. In this section we show that for the optimal array all elements are on the boundary even for nonconvex constraint regions.

The boundary result not only reduces the search dimension from $2m$ to m, but provides a convenient ordering of elements along the boundary to eliminate the nonuniqueness of solution corresponding to interchanging element locations of two or more elements. In particular, by parameterizing the boundary Γ as $\Gamma(t)$ for $t \in [0,1]$ and correspondingly parameterizing each array element location r_i as a point by $r(t_i)$ on the boundary, we reduce the search space to the compact subset $[t_1, \ldots, t_m]^T \in \mathbb{R}^m$ with $0 < t_1 < \cdots < t_m \leq 1$.

To establish the boundary result, we first show that moving a sensor away from the array centroid increases the term $G(\underline{\mathbf{r}}, \theta)$ that appears in the CRB. We note that $G(\underline{\mathbf{r}}, \theta)$ is invariant to translation of the entire array (see (5) and (6)), we will without loss of generality assume $\underline{\mathbf{r}}_A = 0$.

Lemma 1: Assume that $\underline{\mathbf{r}}$ is the array location matrix of an m element array centered at the origin. Consider another array $\underline{\tilde{\mathbf{r}}}$ formed by moving the j^{th} sensor, away from the origin:

$$\underline{\tilde{\mathbf{r}}} = [r_1, r_2, \ldots, (1+\beta)r_j, r_{j+1}, \ldots, r_m] \tag{25}$$

where $\beta > 0$. Then

$$\begin{array}{rcll} G(\underline{\tilde{\mathbf{r}}}, \theta) & > & G(\underline{\mathbf{r}}, \theta) & \text{for } \theta \in [-\pi, \pi) - \{\gamma, \gamma+\pi\} \\ G(\underline{\tilde{\mathbf{r}}}, \theta) & = & G(\underline{\mathbf{r}}, \theta) & \text{for } \theta \in \{\gamma, \gamma+\pi\} \end{array}$$

where $\gamma = \tan^{-1}\left(\frac{y_j}{x_j}\right)$.

Proof: For $\underline{\tilde{\mathbf{r}}}$ in (25), the array centroid is $\tilde{r}_a = \frac{\beta}{m} r_j$ and the corresponding centroid matrix is $\underline{\tilde{\mathbf{r}}}_A = \frac{\beta}{m}[r_j, \ldots, r_j]$. Then

$$\begin{aligned} G(\tilde{\underline{\mathbf{r}}}, \theta) &= \frac{\partial u^T}{\partial \theta}(\underline{\tilde{\mathbf{r}}} - \underline{\tilde{\mathbf{r}}}_A)(\underline{\mathbf{r}} - \underline{\tilde{\mathbf{r}}}_A)^T \frac{\partial u}{\partial \theta} \\ &= \frac{\partial u^T}{\partial \theta}\left(\underline{\mathbf{r}}\underline{\mathbf{r}}^T + \left(2\beta + \frac{m-1}{m}\beta^2\right) r_j r_j^T \right. \\ &\qquad \left. - \beta r_j \frac{1}{N}\sum_{k=1}^m r_k\right)\frac{\partial u}{\partial \theta} \\ &= G(\underline{\mathbf{r}}, \theta) + \underbrace{\left(2\beta + \frac{m-1}{m}\beta^2\right)\frac{\partial u^T}{\partial \theta} r_j r_j^T \frac{\partial u}{\partial \theta}}_{\zeta(\theta)} \end{aligned}$$

Since $\beta > 0$, $\zeta(\theta) \geq 0$ and is equal to zero only when $\frac{\partial u}{\partial \theta} \perp r_j$, or, equivalently, when $\theta = \gamma$ or $\theta = \gamma + \pi$. ■

When a sensor is moved away from the array centroid, the term $G(\underline{\mathbf{r}}, \theta)$ strictly increases except at two DOA angles. As long as $f(\theta)$ has support region greater than just these two angles, the optimization criteria $J_C(\underline{\mathbf{r}})$ and $J_F(\underline{\mathbf{r}})$ will strictly increase. Thus, optimal arrays have all elements on the boundary of the constraint region. The following theorem establishes this result.

Theorem 1: Assume the prior pdf $f(\theta)$ has support on a set of nonzero measure. Then elements of the FIM-optimal and CRB-optimal arrays $\underline{\mathbf{r}}_{\mathbf{F}}$ and $\underline{\mathbf{r}}_{\mathbf{C}}$ lie on the design constraint boundary Γ.

Proof: We will prove Theorem 1 using $J_C(\underline{\mathbf{r}})$; the proof for $J_F(\underline{\mathbf{r}})$ is nearly identical. Let $\underline{\mathbf{r}} = [r_1, \ldots, r_m]$ be the array location matrix of an optimal array, so $\underline{\mathbf{r}}$ is a solution to (7). Assume without loss of generality that the centroid of this array is at the origin.

Assume that sensor r_j (for some $1 \leq j \leq m$) is not on the boundary Γ. Then there is a neighborhood around r_j that lies in D_Γ. Consider another array with element locations given by $\underline{\tilde{\mathbf{r}}}$ in (25) where $\beta > 0$ is chosen such that $\tilde{r_j} \in D_\Gamma$. By Lemma 1,

$$J_C(\underline{\tilde{\mathbf{r}}}) = \int_{-\pi}^{\pi} P^{-1} G(\underline{\tilde{\mathbf{r}}}, \theta)^{-1} f(\theta) d\theta < J_C(\underline{\mathbf{r}})$$

which contradicts the statement that $\underline{\mathbf{r}}$ minimizes $J_C(\underline{\mathbf{r}})$ in (7). Thus, every optimal array must have all elements on the boundary Γ. ■

From the discussion above Theorem 1, the assumption that the pdf $f(\theta)$ has support of nonzero measure can be replaced by the (weaker) assumption that $f(\theta)$ has nonzero measure on a set of greater than two points.

Theorem 1 provides a qualitative explanation for array geometries designed according to the criteria proposed in [13] and [6]. The aim in [13] is to find the array that minimize the mainlobe area while satisfying a sidelobe constraint. In [13] it is noted that optimal designs have most elements either on or near the constraint boundary. Without the sidelobe level constraint, minimizing the mainlobe width corresponds to minimizing the CRB criterion, because the single source CRB is directly related to mainlobe width (see, *e.g.*, [19], [15], [16]); by Theorem 1, all array elements would be on the boundary in this case. Apparently, the sidelobe constraint does not significantly alter the array placement. In [6], an ML estimator (which asymptotically achieves the CRB) is used to estimate the DOAs and the optimal nonuniform *linear* array is designed by minimizing the variance of the DOA estimates. Since the CRB describes the asymptotic ML performance, we expect elements to lie on the boundary in this case as well.

V. Examples

We present two examples of arrays designed using the cost functions $J_C(\underline{\mathbf{r}})$ and $J_F(\underline{\mathbf{r}})$. First, consider an example in which the sensors are constrained to lie in a disk of radius R_0. By Theorem 1, all elements of the optimal array satisfy $|r_i| = R_0$. For the cases in which either $f(\theta)$ is a uniform distribution over $[0, \pi]$ or $[-\pi, \pi)$, it can be shown that every solution for $\underline{\mathbf{r}}_{\mathbf{C}}$ and $\underline{\mathbf{r}}_{\mathbf{F}}$ is an isotropic array. Isotropic arrays

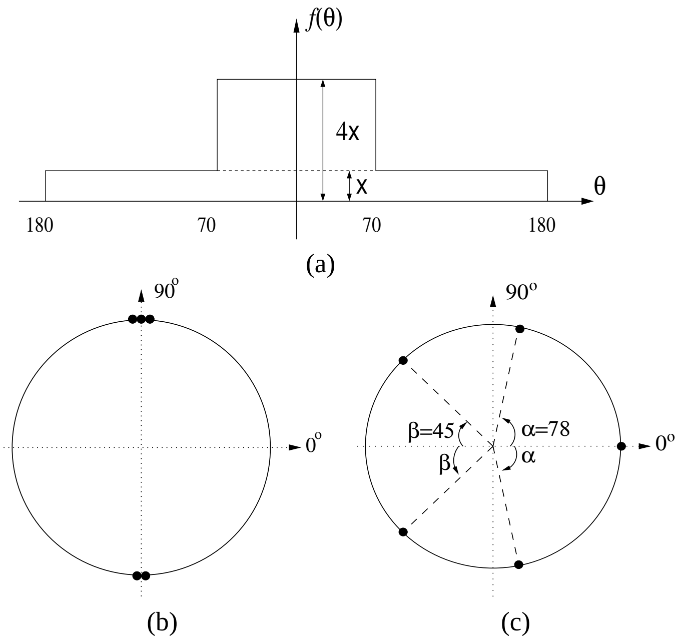

Fig. 1. An example array geometry design when the boundary of the constraint region is a circle. (a) The pdf, $f(\theta)$, of the single source DOA, (b) 5-element FIM-optimal array, (c) 5-element CRB-optimal array, $\alpha = 78°$, $\beta = 45°$.

are studied in [16]; a planar isotropic array is one whose single source CRB is independent of the arrival angle θ.

If the prior DOA pdf is not uniform as above, then the optimal array is no longer isotropic. As an example, consider the circular constraint region as above along with a DOA pdf shown in Figure 1(a). The pdf represents a scenario where the signal can be impinging on the array from any direction but it is expected primarily from a particular sector. Figure 1(b) shows the resulting 5-element FIM-optimal array (which is also the Bayesian CRB-optimal array); note that all elements cluster at the top and bottom, to give the widest aperture for signals arriving around $0°$. This is similar to the optimum linear array design, in which half of the array elements are clustered at each end of the constraint line segment [23]. The CRB-optimal array geometry, in Figure 1(c) has elements spread around the boundary. While the CRB-optimal array is not isotropic (an isotropic array has equally-spaced elements), it is nearly so, but has lower beamwidths for arrival angles close to $0°$ and $180°$. These observations agree with the maximum aperture character of FIM-optimal arrays and the aperture-and-isotropic character of CRB-optimal arrays as discussed at the end of Section III.C. In addition, the CRB-optimal array designs generally have lower sidelobe levels than do FIM-optimal designs. For these reasons, the CRB-optimal designs are preferable in most applications.

VI. Conclusions

We have considered optimal planar array designs using the average CRB and average FIM as performance criteria. Prior information on the source DOA is encoded as a prior probability density function; this allowed us to address applications in which not all directions are equally likely. Closed form expressions for the optimization criteria $J_C(\underline{\mathbf{r}})$ and $J_F(\underline{\mathbf{r}})$ were derived when DOA prior is uniform on a sector of angles.

We showed that sensors of both CRB-optimal and FIM-optimal arrays lie on the boundary of the array constraint region, even when that region is not convex. As a result, the dimension of the design optimization is reduced from $2m$ to m.

We also related the two proposed optimization criteria to the Bayesian CRB and to average array beamwidth. We showed that the FIM-optimal array also minimizes the Bayesian CRB, and the CRB-optimal criterion bounds the Bayesian CRB. The boundary result, in conjunction with expressions relating the CRB to the array beamwidth, provided geometric interpretation of the optimality criteria and the resulting array designs.

Because the CRB is a realistic bound for moderate to high SNRs and can be optimistic for low values of SNR, the design criteria we consider apply to moderate or high SNR applications. In [20] it is shown that the WWLB and ZZLB provides tighter bounds than the BCRB and they converge to the BCRB above a threshold SNR. However, the use of the WWLB or ZZLB increases the computational cost of the approach substantially. In practice our geometry designs should perform well for SNRs that are above this threshold SNR. For lower SNR applications, it may be interest to design arrays with low sidelobe levels. The methods presented in this paper are based on the FIM and CRB, whose properties relate to mainlobe width but not sidelobe level. The CRB-based designs tend to have lower sidelobe levels than the FIM-based designs, and are therefore preferable in most applications. If low sidelobe level is an additional design requirement, the FIM and CRB criteria may be combined or augmented with other criteria or constraints, such as those in [10], [13], to obtain desired sidelobe performance; alternately, a ZZLB- or WWLB-based criterion can be used.

Although we have focused on design of planar arrays that estimate azimuth DOA, the methods apply to other scenarios as well. The design method applies to volume arrays, to DOA angles in azimuth and elevation, and to both narrowband and wideband arrays. The design procedure follows because in all of these cases the expression for the Fisher information matrix can be partitioned as in equations (4)–(6), where P is a scalar and $G(\underline{\mathbf{r}}, \theta)$ contains the array geometry terms.

References

[1] H. L. Van Trees, *Optimum Array Processing*. New York: Wiley, 2002.

[2] V. Murino, "Simulated annealing approach for the design of unequally spaced arrays," in *ICASSP International Conference on Acoustics, Speech, and Signal Processing*, vol. 5, (Detroit, USA), pp. 3627–3630, 1995.

[3] A. B. Gershman and J. F. Böhme, "A note on most favorable array geometries for DOA estimation and array interpolation," *IEEE Signal Processing Letters*, vol. 4, pp. 232–235, August 1997.

[4] D. Pearson, S. U. Pillai, and Y. Lee, "An algorithm for near-optimal placement of sensor elements," *IEEE Transactions on Information Theory*, vol. 36, pp. 1280–1284, November 1990.

[5] H. Alnajjar and D. W. Wilkes, "Adapting the geometry of a sensor subarray," in *ICASSP International Conference on Acoustics, Speech, and Signal Processing*, vol. 4, (Minneapolis, USA), pp. 113–116, 1993.

[6] X. Huang, J. P. Reilly, and M. Wong, "Optimal design of linear array of sensors," in *ICASSP International Conference on Acoustics, Speech, and Signal Processing*, vol. 2, (Toronto, Canada), pp. 1405–1408, 1991.

[7] J.-W. Liang and A. J. Paulraj, "On optimizing base station antenna array topology for coverage extension in cellular radio networks," in *IEEE 45th Vehicular Technology Conference*, vol. 2, (Stanford, CA), pp. 866–870, 1995.

[8] Y. Hua, T. K. Sarkar, and D. D. Weiner, "An L-shaped array for estimating 2-D directions of wave arrival," *IEEE Transactions on Antennas and Propagation*, vol. 39, pp. 143–146, February 1991.

[9] A. Manikas, A. Alexiou, and H. R. Karimi, "Comparison of the ultimate direction-finding capabilities of a number of planar array geometries," in *IEE Proceedings on Radar, Sonar and Navigation*, pp. 321–329, 1997.

[10] M. Gavish and A. J. Weiss, "Array geometry for ambiguity resolution in direction finding," *IEEE Transactions on Antennas and Propagation*, vol. 44, pp. 889–895, June 1996.

[11] A. Manikas, A. Sleiman, and I. Dacos, "Manifold studies of nonlinear antenna array geometries," *IEEE Transactions on Signal Processing*, vol. 49, pp. 497–506, March 2001.

[12] N. Dowlut and A. Manikas, "A polynomial rooting approach to super-resolution array design," *IEEE Transactions on Signal Processing*, vol. 48, pp. 1559–1569, June 2000.

[13] M. Viberg and C. Engdahl, "Element position considerations for robust direction finding using sparse arrays," in *Conference Record of the Thirty Third Asilomar Conference on Signals, Systems and Computers*, vol. 2, (Pacific Grove, CA), pp. 835–839, 1999.

[14] H. L. Van Trees, *Detection, estimation, and modulation theory*, vol. 1. New York, Wiley, 1968.

[15] K. L. Bell, Y. Ephraim, and H. L. Van Trees, "Explicit Ziv-Zakai lower bound for bearing estimation," *IEEE Transactions on Signal Processing*, vol. 44, pp. 2810–2824, November 1996.

[16] Ü. Baysal and R. L. Moses, "On the geometry of isotropic arrays," *IEEE Transactions on Signal Processing*, vol. 51, pp. 1469–1478, June 2003.

[17] A. J. Weiss and B. Friedlander, "On the Cramér-Rao bound for direction finding of correlated signals," *IEEE Transactions on Signal Processing*, vol. 41, pp. 495–499, 1993.

[18] M. A. Doron and E. Doron, "Wavefield modeling and array processing. iii. resolution capacity," *IEEE Transactions on Signal Processing*, vol. 42, pp. 2571–2580, October 1994.

[19] H. Messer, "Source localization performance and the array beampattern," *Signal Processing*, vol. 28, pp. 163–181, August 1992.

[20] H. Nguyen and H. L. Van Trees, "Comparison of performance bounds for DOA estimation," in *IEEE Seventh SP Workshop on Statistical Signal and Array Processing*, (Quebec City, Canada), pp. 313–316, 1994.

[21] K. L. Bell, Y. Steinberg, Y. Ephraim, and H. L. Van Trees

[22] F. Athley, "Optimization of element positions for direction finding with sparse arrays," in *Proceedings of the 11th IEEE Signal Statistical Signal Processing Workshop on Statistical Signal Processing*, (Singapore), pp. 516–519, 2001.

[23] V. H. MacDonald and P. M. Schultheiss, "Optimum passive bearing estimation in a spatially incoherent noise environment," *Journal of the Acoustical Society of America*, vol. 46, pp. 37–43, 1969.