

# Source Localization With Isotropic Arrays

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**Abstract**—We consider the effect of unknown signal propagation velocity on direction-of-arrival (DOA) estimation performance using an array of sensors. For general arrays, the DOA estimation variance, as measured by the Cramér–Rao bound (CRB), degrades when the propagation velocity is unknown. In this letter, we show that for both two- and three-dimensional arrays, the CRB on the DOA angle is independent of whether or not the propagation velocity is known if and only if the sensor locations satisfy the isotropy conditions given in earlier work. The results hold for both narrow-band and wideband signals.

**Index Terms**—Array geometry, Cramér–Rao bound (CRB), isotropic array, propagation velocity.

## I. INTRODUCTION

**M**OST direction-of-arrival (DOA) estimation algorithms are based on the relative arrival times of a source signal at sensor elements, and thus DOA estimation performance of a sensor array can be degraded when the speed of propagation  $c$  is unknown. This happens, for example, in seismic signal processing where  $c$  depends on unknown soil or rock conditions between the source and array elements, and in aeroacoustic measurements, where the speed of sound depends on both temperature and wind speed.

The effect of unknown speed of propagation on the DOA estimation performance of a sensor array is considered in [1], where the problem of localizing an unknown source using wideband measurements from an array of sensors is studied. The authors assume that  $c$  is unknown and derive the Cramér–Rao bound (CRB) for the location of the source, which is the source DOA in the far-field case. When  $c$  is unknown, the CRB performance of the array may be degraded, depending on the array geometry used. The authors show that uniform circular array is one array geometry where the source DOA CRB is not affected from  $c$  being unknown.

In this letter, we generalize the result in [1] in three ways. First, we establish the entire class of array geometries for which the source location CRB is independent of  $c$ . In particular, we show that the source location CRB is independent of  $c$  if and only if the array is *isotropic*. Isotropic arrays are arrays for which the far-field source DOA CRB is independent of source arrival angle (when  $c$  is known); their properties are explored in [2]. Second, we give analytical expressions for the loss in DOA accuracy due to unknown  $c$  when the DOA is estimated using an

*anisotropic* array; the loss is parameterized by a simple expression that depends on the eigenvalues of a  $(2 \times 2)$  (for planar arrays) or  $(3 \times 3)$  (for volume arrays) covariance-like matrix that characterizes the array geometry. Finally, we extend the results in [1] from a planar array geometry to the three-dimensional (3-D) case.

## II. SYSTEM MODEL

We assume that: 1) the sensor array has  $N$  identical sensors; 2) each sensor is located at location  $r_i$  for  $i \in [1, N]$ ; and 3)  $r_a = (1/N) \sum_{i=1}^N r_i$  is centroid of these sensors. A single, generally wideband, far-field source  $s(t)$  impinges on the array from direction  $\theta$ . For 3-D geometries,  $\theta = [\phi, \psi]^T$ , and for planar geometries,  $\theta = \phi$ , where the azimuth angle  $\phi$  is measured counterclockwise from the  $x$  axis and the elevation angle  $\psi$  measured from the  $(x, y)$  plane. The signal is assumed to be zero mean and Gaussian. The noise at the sensors is assumed to be independent, zero mean, Gaussian, and independent of the source signal. The observation time  $T$  is partitioned into  $K$  intervals of length  $T_d$ , and a  $J$ -point discrete Fourier transform is applied to each interval. Then [3]

$$x_k(\omega_j) = a_\theta(\omega_j) s_k(\omega_j) + \varepsilon_k(\omega_j), \quad j = 1, \dots, J \\ k = 1, \dots, K \quad (1)$$

where  $x_k(\omega_j)$ ,  $\varepsilon_k(\omega_j)$  are  $(N \times 1)$  vectors, and  $s_k(\omega_j)$  is a scalar. The elements of  $x_k(\omega_j)$ ,  $\varepsilon_k(\omega_j)$ , and  $s_k(\omega_j)$  are the discrete Fourier coefficients of the sensor outputs, the noise, and the signal source at the discrete frequency  $\omega_j$ , respectively. Also

$$a_\theta(\omega_j) = \left[ e^{j\omega_j d_1(\theta)}, e^{j\omega_j d_2(\theta)}, \dots, e^{j\omega_j d_N(\theta)} \right]^T \quad (2)$$

where  $d_k(\theta) = u^T(\theta) \cdot r_k / c$  is the propagation delay associated with the  $k$ th sensor,  $c$  is the speed of propagation and  $u(\theta)$  is the unit vector pointing toward the signal source. For 3-D signals

$$u(\theta) = [\cos(\phi) \cos(\psi), \sin(\phi) \cos(\psi), \sin \psi]^T \quad (3)$$

and for planar signals

$$u(\phi) = [\cos(\phi), \sin(\phi)]^T. \quad (4)$$

Assuming  $T_d$  is long enough, the  $x_k(\omega_j)$  vectors are uncorrelated. A more compact expression for the observations given in (1) is

$$X_k = \begin{bmatrix} x_k(\omega_1) \\ \vdots \\ x_k(\omega_J) \end{bmatrix} \quad (NJ \times 1). \quad (5)$$

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The covariance matrix of the sample vector  $X_k$  is given by

$$R_X = E\{X_k X_k^H\} = \text{diag}(R_1, R_2, \dots, R_J) \quad (6)$$

$$R_j = p_j \cdot a_\theta(\omega_j) a_\theta^H(\omega_j) + n_j \quad (7)$$

and where  $p_j$  and  $n_j$  are the signal and noise power spectral values at frequency  $\omega_j$ , respectively.

### III. CRB FOR THE DOA ANGLE WHEN SPEED OF PROPAGATION IS UNKNOWN

In this section, we give the CRB expression for the DOA angle when the speed of propagation is also unknown, i.e., when the parameter vector to be estimated is  $v = [\theta^T, c]^T$ . We show that the CRB expression is the same whether speed of propagation is known or unknown if and only if the array geometry is isotropic. We consider both planar and 3-D arrays.

#### A. Planar Arrays

Assume that a planar array with elements located at  $r_i = [x_i, y_i]^T, i = 1, \dots, N$  is used to estimate the DOA of a single wideband coplanar signal ( $\theta = \phi$ ).

Since  $\{X_k\}_{k=1}^K$  are independent samples of a zero-mean Gaussian process, the Fisher information matrix (FIM) on the the parameter vector  $v = [\phi, c]^T$  is given by (see also [1], [3], and [4])

$$\text{FIM}(v) = \begin{bmatrix} J_{\phi\phi} & J_{\phi c} \\ J_{\phi c}^T & J_{cc} \end{bmatrix} \quad (8)$$

$$J_{mn} = K \cdot \text{tr} \left\{ R_X^{-1} \frac{\partial R_X}{\partial v_m} R_X^{-1} \frac{\partial R_X}{\partial v_n} \right\} \quad (9)$$

where  $R_X$  is given in (6). The entries of the FIM are given by

$$\begin{aligned} J_{\phi\phi} &= u_\phi^T B u_\phi \cdot P \\ J_{\phi c} &= -\frac{1}{c} u_\phi^T B u \cdot P \\ J_{cc} &= \frac{1}{c^2} u^T B u \cdot P \end{aligned}$$

where  $u_\phi = (du(\phi)/d\phi)$  and

$$P = \frac{2KN}{c^2} \sum_{j=1}^J \frac{\omega_j^2}{n_j} p_j \left( 1 - \frac{n_j}{p_j N + n_j} \right) \quad (10)$$

$$B = \frac{1}{N} \sum_{i=1}^N (r_i - r_a)(r_i - r_a)^T. \quad (11)$$

The narrowband case is the special case that  $J = 1$  above. In either the narrowband or wideband case,  $P$  is a nonnegative scalar whose form has no bearing on the results that follow; rather, the results rely on the structure of  $J$  as characterized by the form of the matrix  $B$ .

When the speed of propagation is known the CRB on the DOA angle is given by (e.g., see [2])

$$\text{CRB}_0(\phi) = J_{\phi\phi}^{-1}. \quad (12)$$

When the speed of propagation is unknown, the CRB on the DOA is given by the upper left entry of  $\text{FIM}^{-1}(v)$  given in (8) (see also [1])

$$\text{CRB}(\phi) = P^{-1} [u_\phi^T B u_\phi - F_c]^{-1} \quad (13)$$

where  $F_c$  is the penalty term due to the unknown speed of propagation and given by

$$F_c = \frac{[u_\phi^T B u]^2}{u^T B u}. \quad (14)$$

We use the notation  $\text{CRB}_0(\phi)$  for the DOA CRB when  $c$  is known, and  $\text{CRB}(\phi)$  for the DOA CRB when  $c$  is unknown.

In the above CRB derivation, we assumed for simplicity that the signal and noise spectral values (i.e.,  $p_j$  and  $n_j$ ) are known. However, the results that follow do not change if these spectra are unknown, because the FIM is block-diagonal in these unknown spectral values whether or not  $c$  is known.

In [2], an isotropic planar array is defined to be one whose single-source CRB is independent of the source arrival angle  $\phi$ . It is shown that a planar array is isotropic if and only if its array covariance matrix [given in (11)] is in the form  $B = kI_2$  where  $k$  is a positive constant and  $I_2$  is the  $(2 \times 2)$  identity matrix. The matrix  $B$  is similar to a covariance or moment-of-inertia matrix if the array elements are considered as unit masses at their element locations. With the following theorem, we will show that unknown speed of propagation does not affect the CRB performance of the array if and only if the array is isotropic.

**Theorem 1:** For a planar array, the penalty term due to the unknown speed of propagation  $F_c$  is zero for all arrival angles  $\phi \in [0, 2\pi)$  if and only if the array is isotropic.

*Proof:* Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues ( $\lambda_1 \geq \lambda_2 \geq 0$ ), and let  $e_1$  and  $e_2$  be the eigenvectors of the array covariance matrix  $B$  in (11)

$$B = [e_1 \ e_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix}. \quad (15)$$

Since  $[e_1 \ e_2]$  is unitary,  $e_1$  and  $e_2$  can be written as

$$e_1 = [-\sin \gamma \ \cos \gamma]^T \quad (16)$$

$$e_2 = [\cos \gamma \ \sin \gamma]^T \quad (17)$$

for some angle  $\gamma$  (see Fig. 1 for a geometric interpretation). Inserting (4), (11), (15), (16), and (17) into (14) yields

$$F_c = \frac{(\lambda_1 - \lambda_2)^2 \sin^2(\beta) \cos^2(\beta)}{\lambda_1 \sin^2(\beta) + \lambda_2 \cos^2(\beta)} \quad (18)$$

where  $\beta = \gamma - \phi$ . From (18), we see that  $F_c$  is zero for all  $\beta$  if and only if  $\lambda_1 = \lambda_2$ , which is equivalent to  $B = kI_2$  with  $k = \lambda_1 = \lambda_2$ . ■

In [1], it is shown that  $F_c$  is zero for uniform circular arrays. With Theorem 1, we extend this result to all isotropic arrays, and we also show that isotropy is a necessary and sufficient condition.

When an array is not isotropic, the penalty term  $F_c$  depends on the source DOA. From (18),  $F_c = 0$  (i.e., there is no loss of CRB accuracy when  $c$  is unknown) when  $\beta = (l\pi/2)$  for integer  $l$ ; this happens when the signal DOA coincides with one

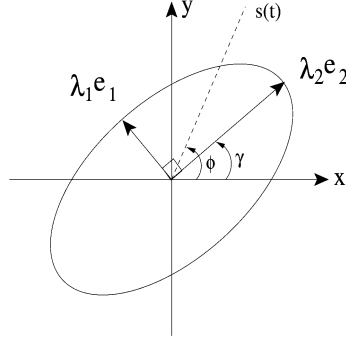


Fig. 1. Ellipse representation of the array covariance matrix  $[B$  in (11)] corresponding to a 2-D array.

of the eigenvectors of  $B$  (see Fig. 1). We define  $\alpha = \lambda_2/\lambda_1$ ;  $\alpha$  can be viewed as a measure of anisotropy for a planar array, since  $0 \leq \alpha \leq 1$  and  $\alpha = 1$  corresponds to an isotropic array. We can write the ratio of the CRBs for unknown and known  $c$  in terms of  $\alpha$

$$\frac{\text{CRB}(\phi)}{\text{CRB}_0(\phi)} = 1 + \frac{(1 - \alpha)^2 \sin^2(2\beta)}{4\alpha}. \quad (19)$$

This ratio shows the relative loss in the CRB performance of the array when speed of propagation is unknown as a function of the anisotropy of the array. Notice that for a given  $\alpha$ , the ratio in (19) is maximum for  $\beta = (2l + 1)\pi/4$  and integer  $l$ . Fig. 2 shows the relation between the worst case  $\text{CRB}(\phi)/\text{CRB}_0(\phi)$  and  $\alpha$  for  $\beta = \pi/4$ .

As can be seen from Fig. 2, the DOA accuracy loss is large when  $\alpha \approx 0$ , which is the case in nearly linear arrays (for linear arrays, there is an ambiguity between DOA and  $c$ , and they cannot be jointly estimated without additional prior knowledge; thus,  $\text{CRB}(\phi)$  is infinite in this case). We can conclude that nearly linear arrays may have very poor DOA estimation performance in media where speed of propagation is subject to change or cannot be determined exactly. On the other hand, moderately anisotropic arrays exhibit a small loss in DOA performance. For example, the CRB ratio in (19) is  $\leq 2$  for  $\alpha \geq 0.172$ , and this is at a worst case DOA.

We remark that [1] and [5] also consider the scenario where the impinging signal is a near-field source; they give the CRB expression for the  $x$  and  $y$  locations of the source when the speed of propagation is unknown. The result in Theorem 1 applies to far-field sources only; there does not appear to be a class of geometries for which unknown  $c$  imposes no loss of performance for near-field source localization.

### B. Three-Dimensional Arrays

In this section, we will generalize the results in the previous section for 3-D arrays. Assume a 3-D array ( $r_i = [x_i, y_i, z_i]^T$ ) is used to estimate the DOA of a 3-D signal arriving at angle ( $\theta = [\phi, \psi]^T$ ) where  $\phi$  is the azimuth and  $\psi$  is the elevation angles.

The FIM on the parameter vector  $[\theta, c]^T$  is given by

$$\text{FIM}(v) = \begin{bmatrix} J_{\theta\theta} & J_{\theta c} \\ J_{\theta c}^T & J_{cc} \end{bmatrix}$$

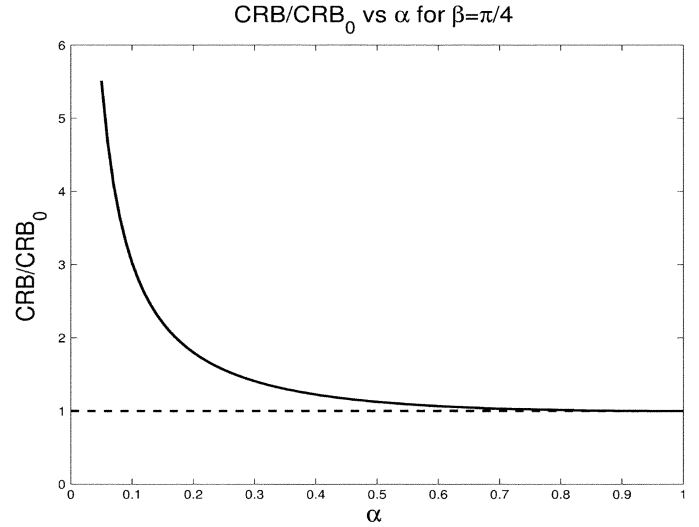


Fig. 2. Relation between isotropy of the array and relative loss in the CRB due to the unknown speed of propagation.

where

$$J_{\theta\theta} = \begin{bmatrix} u_\phi^T \\ u_\psi^T \end{bmatrix} B [u_\phi \ u_\psi] \cdot P \quad (20)$$

$$J_{\theta c} = -\frac{1}{c} \begin{bmatrix} u_\phi^T B u \\ u_\psi^T B u \end{bmatrix} \cdot P \quad (21)$$

$$J_{cc} = \frac{1}{c^2} u^T B u \cdot P. \quad (22)$$

Here  $[u_\phi \ u_\psi] = [\partial u(\theta)/\partial \phi \ \partial u(\theta)/\partial \psi]$ , and  $P$  and  $B$  are the same as in (10) and (11).

When the speed of propagation is known, the CRB on the DOA angle  $\theta$  is

$$\text{CRB}_0(\theta) = J_{\theta\theta}^{-1}. \quad (23)$$

When the speed of propagation is unknown, using the matrix inversion lemma it can be shown that the CRB on the DOA angle  $\theta$  is given by

$$\text{CRB}(\theta) = P^{-1} \left( \begin{bmatrix} u_\phi^T \\ u_\psi^T \end{bmatrix} B [u_\phi \ u_\psi] - F_c \right)^{-1} \quad (24)$$

where the  $(2 \times 2)$  positive semidefinite matrix  $F_c$  is the penalty term due to the unknown speed of propagation and is given by

$$F_c = \frac{1}{u^T B u} J_{\theta c} J_{\theta c}^T. \quad (25)$$

It is useful to introduce a scalar measure of DOA estimation accuracy. The mean square angular error (MSAE) is such a measure, and it is defined as the mean-squared angle between the true DOA unit vector  $u$  and its estimate  $\hat{u} = [\hat{\phi}, \hat{\psi}]^T$  (see [6] for derivation and details on MSAE). It can be shown [6] that the MSAE is bounded by

$$\text{MSAE}(\theta) > \text{MSAE}_B(\theta) = \cos^2(\psi) \text{CRB}_0(1, 1) + \text{CRB}_0(2, 2) \quad (26)$$

where  $\text{CRB}_0(i, j)$  is the  $(i, j)$ th element of  $\text{CRB}_0$  given in (23). In [2], an isotropic 3-D array is defined to be one whose bound on the MSAE is constant for all  $\theta$ . It is shown in [2] that  $B =$

$kI_3$  is a necessary and sufficient condition for isotropic performance. As in the planar case, there is no penalty in the CRB due to the unknown speed of propagation for 3-D arrays when the array geometry is isotropic. The following theorem establishes this result.

**Theorem 2:** For a 3-D array, the penalty term due to the unknown speed of propagation  $F_c$  is zero for all arrival angles  $\theta$  if and only if the array is isotropic.

*Proof:* Let  $\lambda_1, \lambda_2$ , and  $\lambda_3$  be the eigenvalues ( $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ ) and  $e_1, e_2$ , and  $e_3$  be the eigenvectors of the array covariance matrix  $B$  of a 3-D array. We can align the  $x, y$ , and  $z$  axes with the eigenvectors of  $B$  by a coordinate-system rotation. Thus, without loss of generality, we can choose the coordinate system such that  $e_1 = [1, 0, 0]^T$ ,  $e_2 = [0, 1, 0]^T$ , and  $e_3 = [0, 0, 1]^T$ , i.e., the  $(x, y, z)$  axes coincide with the eigenvectors. Then

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

From (27), it can be seen that  $F_c$  is zero if and only if  $J_{\theta c} = [0, 0]^T$ , since  $u^T B u$  is finite. With the chosen coordinate system, the vector  $J_{\theta c}$  in (21) becomes

$$J_{\theta c} = \begin{bmatrix} -(\lambda_1 - \lambda_2) \sin \phi \cos \phi \cos^2 \psi \\ [\lambda_3 - (\lambda_1 \cos^2 \phi + \lambda_2 \sin^2 \phi)] \sin \psi \cos \psi \end{bmatrix}. \quad (27)$$

From (27), it can be seen that  $J_{\theta c} = [0 \ 0]^T$  for all  $\phi \in [0, 2\pi)$ ,  $\psi \in (-\pi/2, \pi/2)$  if and only if  $\lambda_1 = \lambda_2 = \lambda_3$ , which is equivalent to the condition  $B = kI_3$  with  $k = \lambda_1$ . ■

#### IV. CONCLUSION

For both 2-D and 3-D arrays, we showed that when the array geometry is isotropic, direction-of-arrival performance in terms

of CRB is independent of whether or not the speed of propagation is known. Isotropic arrays not only provide uniform CRB performance for all possible DOA angles but also are robust to an unknown speed of propagation. We derived the performance loss in DOA accuracy due to unknown  $c$ , and we showed that while linear arrays have the worst performance loss, moderately anisotropic arrays have only a slight performance loss in DOA accuracy.

The results presented assumed that the propagation velocity is not a function of frequency. For propagation in dispersive media, the propagation speed changes with frequency and must be estimated for each frequency bin (or group of bins). The results above apply to this case as well: the Fisher information matrix, and thus the CRB, is block-diagonal in the DOAs and propagation velocities if and only if the array is isotropic.

The results presented apply to far-field sources. There does not appear to be a class of array geometries for which unknown  $c$  imposes no loss of location estimation performance for near-field sources.

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