Enforcing positiveness on estimated spectral densities

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Estimated spectral densities may often take on negative values in some frequency bands, and hence need be corrected to become positive for all frequencies. The Letter describes a most natural approach to enforcing the positiveness condition on an estimated spectral density, which is shown to lead to a semi-infinite optimisation problem. In the numerical example reported, the latter problem is solved by using the Matlab Optimization Toolbox.

Problem statement: Let $\{\hat{r}_k\}_{k=0}^{n}$ denote the estimated covariances of a discrete time stationary signal. The corresponding estimated spectral density is given by

$$\hat{\phi}(\omega) = \hat{f}(e^{i\omega}) \qquad \hat{f}(z) = \sum_{k=-n}^{n} \hat{r}_{|k|} z^k \tag{1}$$

Hereafter, ω denotes the (angular) frequency variable. As $\hat{\phi}(\omega)$ is an even periodic function of ω, with period 2π, it suffices to consider eqn. 1 for $\omega \in [0, \pi]$. Estimated spectral densities having the above form are ubiquitous in signal processing applications. They are encountered both in nonparametric applications (e.g. based on the Blackman-Tukey approach) and in parametric spectral estimation applications (e.g. by using an MA or ARMA model).

Several commonly-used methods for spectral estimation do not guarantee that $\hat{\phi}(\omega) \geq 0$ for all ω in $[0, \pi]$, and hence they may yield estimates with negative values at some frequencies. As such a situation is not acceptable for most applications, several researchers have proposed methods for enforcing the positiveness condition on an estimated $\hat{\phi}(\omega)$ (see, for example, [1–5]). Some of these methods are simple to apply, but rather ad-hoc [3–5]. Others are optimal, albeit in a limited sense, but they are somewhat intricate from an algorithmic standpoint [1, 2]. This Letter shows that a most general approach to enforcing the positiveness condition on $\hat{\phi}(\omega)$ naturally leads to a semi-infinite optimisation problem, the solution of which can be obtained for instance by using the Matlab Optimization Toolbox.

Proposed solution: Let

$$\varphi(\omega) = [1 \quad \cos\omega \ldots \cos n\omega]^T \tag{2}$$

$$\hat{\rho} = [\hat{r}_0 \quad 2\hat{r}_1 \ldots 2\hat{r}_n]^T \tag{3}$$

By making use of this notation, $\hat{\phi}(\omega)$ can be written as

$$\hat{\phi}(\omega) = \hat{\rho}^T \varphi(\omega) \tag{4}$$

Whenever $\hat{\phi}(\omega)$ is not positive for all $\omega \in [0, \pi]$, a vector ρ which gives a valid spectral density

$$\phi(\omega) = \rho^T \varphi(\omega) \geq 0 \qquad \omega \in [0, \pi] \tag{5}$$

is to be determined such that $\phi(\omega)$ is 'close to' $\hat{\phi}(\omega)$. A natural formulation of this problem is as follows:

$$\min_{\rho} \frac{1}{\pi} \int_0^{\pi} W(\omega)[\hat{\phi}(\omega) - \phi(\omega)]^2 \, d\omega$$
$$\text{subject to} \quad \phi(\omega) \geq 0 \quad \text{for} \quad \omega \in [0, \pi] \tag{6}$$

In eqn. 6, $W(\omega)$ is a weighting function that can be used to emphasise certain frequency bands.

A straightforward calculation shows that eqn. 6 can be re-written in the following more convenient form:

$$\min_{\rho} (\hat{\rho} - \rho)^T Q (\hat{\rho} - \rho)$$
$$\text{subject to} \quad \rho^T \varphi(\omega) \geq 0 \quad \text{for} \quad \omega \in [0, \pi] \tag{7}$$

where

$$Q = \frac{1}{\pi} \int_0^{\pi} W(\omega) \varphi(\omega) \varphi^T(\omega) \, d\omega \tag{8}$$

For simplicity, in the following we consider the common choice $W(\omega) \equiv 1$. For this case Q is readily derived:

$$Q = \begin{bmatrix} 1 & & & 0 \\ & 1/2 & & \\ & & \ddots & \\ 0 & & & 1/2 \end{bmatrix} \tag{9}$$

Matrix Q above differs only slightly from the choice $Q = I$ considered in [1]. Of course, other choices of $W(\omega)$ in eqn. 8 may lead to completely different Q matrices.

Next, we note that eqn. 7 is a semi-infinite optimisation problem which can be solved by a variety of algorithms. Here we use the function 'seminf' in the Optimization Toolbox of Matlab. The initial estimate required to start seminf can be determined as follows. Let ω_1 and ω_2 be such that

$$\hat{\phi}(\omega_1) = \hat{\phi}(\omega_2) = 0 \quad \text{and} \quad \hat{\phi}(\omega) < 0 \quad \omega \in (\omega_1, \omega_2) \tag{10}$$

(ω_1 and ω_2 can be obtained either by rooting $\hat{\phi}(z)$ or by inspection of the plot of $\hat{\phi}(\omega)$). If $\bar{\omega} = (\omega_1 + \omega_2)/2 \neq 0$ or π, then compute

$$z^n \tilde{f}(z) = [z^n \hat{f}(z)/(z-e^{i\omega_1})(z-e^{i\omega_2})(z-e^{-i\omega_1})(z-e^{-i\omega_2})]$$
$$\times (z-e^{i\bar{\omega}})^2 (z-e^{-i\bar{\omega}})^2 \tag{11}$$

If $\bar{\omega} = 0$ or π, then $\tilde{f}(z)$ is redefined as

$$z^n \tilde{f}(z) = [z^n \hat{f}(z)/(z-e^{i\omega_1})(z-e^{i\omega_2})](z-e^{i\bar{\omega}})^2 \tag{12}$$

By construction, $\tilde{\phi}(\omega) \triangleq \tilde{f}(e^{i\omega})$ is non-negative for $\omega \in [\omega_1, \omega_2]$. Continuing in the above simple way, as necessary, we can obtain a corrected spectral estimate that is non-negative for all frequencies. This estimate is used as the starting point for seminf. An improved initial estimate can be obtained by optimally scaling $\tilde{f}(z)$ determined as above. The optimal scaling factor is determined by the minimisation of the criterion in eqn. 7. This amounts to solving a very simple least-squares problem, the details of which are omitted in the interest of brevity.

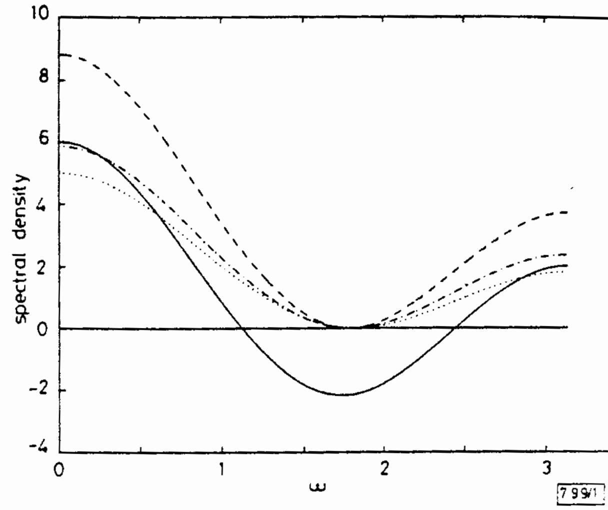

Fig. 1 Initial and corrected spectral densities

——— given $\hat{\phi}(\omega)$
– – – – initial $\tilde{\phi}(\omega)$
· · · · · optimally corrected $\phi(\omega)$
– · – · corrected $\phi(\omega)$ in [1]

Numerical example: For illustration purposes, we consider one of the examples in [1]. Let $\hat{\rho} = [1\ 2\ 3]^T$. The corresponding $\hat{\phi}(\omega)$ does not satisfy eqn. 5, as shown in Fig. 1. The correction method based on eqn. 11 yields $\tilde{\rho} = [3.270\ 2.544\ 3.000]^T$ and $\tilde{\phi}(\omega)$ exhibited in Fig. 1. The optimal correction method, which uses the function seminf (initialised by $\tilde{\rho}$) to solve the semi-infinite optimisation problem eqn. 7, gives $\rho = [1.800\ 1.600\ 1.600]^T$ and $\varphi(\omega)$ shown in Fig. 1. For comparison, Fig. 1 also includes the solution derived in [1], which corresponds to using the slightly different weight $Q = I$ in eqn. 7 (the corresponding ρ is $\rho = [2.154\ 1.743\ 1.960]^T$). As expected, the corrected $\phi(\omega)$ obtained with the method of this Letter is closer to $\hat{\phi}(\omega)$ than is the corrected spectral density derived in [1]. In fact, the values of the criterion in eqn. 7 corresponding to $\tilde{\rho}$, ρ_{opt} and $\rho_{[1]}$ above are 5.300, 1.700 and 1.906, respectively.

In conclusion, it should be noted that the function seminf has been found to work quite properly on all simulations conducted. When initialised as described above, seminf yields the optimally corrected $\phi(\omega)$ almost instantaneously for low dimensional problems. Although using seminf to solve the optimisation problem (eqn. 7) may be expected to be less efficient computationally than using the specialised algorithm of [1], seminf is simple to use and, unlike the method in [1], it does not require much interaction with the user.

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