Signal Processing 26 (1992) 95–118
Elsevier

On the unit circle problem: The Schur–Cohn procedure revisited

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Received 11 September 1990
Revised 11 July 1991

Abstract. This paper presents several results concerned with finding the zero distribution of a polynomial with respect to the unit circle using variants of the Schur–Cohn procedure. First, new and simple proofs of the Schur–Cohn procedure in the regular and singular cases are presented. A new method for handling one type of singular case is also developed. Next, we consider several aspects of the 'inverse problem', in which one wishes to alter a given polynomial to have a prescribed zero distribution. Three methods for forcing all zeros of a polynomial to lie inside the unit circle are derived. Also, two algorithms for solving singular inverse problems are presented; specifically, one corrects a symmetric polynomial to ensure that all its zeros lie on the unit circle, and the other corrects a symmetric polynomial to ensure that none of its zeros lie on the unit circle. The inverse problems have applications in spectral estimation, signal processing and system identification.

Zusammenfassung. Verschiedene Ergebnisse werden vorgestellt zur Frage des Auffindens der Lage von Polynomnullstellen in Bezug auf den Einheitskreis; debei werden Varianten der Schur–Cohn-Prozedur verwendet. Erstens werden neue und einfache Beweise für die Schur–Cohn-Prozedur im regulären wie im singulären Fall vorgestellt. Auch wird eine neue Methode zur Behandlung eines bestimmten singulären Falles entwickelt. Als nächstes betrachten wir verschiedene Aspekte des 'inversen Problems', in welchem man ein gegebenes Polynom so ändern möchte, daß es eine vorgeschriebene Nullstellenverteilung aufweist. Drei Verfahren werden hergeleitet, die eine Lage aller Nullstellen eines Polynoms innerhalb des Einheitskreises erzwingen. Auch werden zwei Algorithmen zur Lösung inverser singulärer Probleme vorgestellt; insbesondere korrigiert eines davon ein symmetrisches Polynom so, daß alle seine Nullstellen sicher auf dem Einheitskreis liegen, und das andere stellt sicher, daß nach Korrektur eines symmetrischen Polynoms keine seiner Nullstellen auf dem Einheitskreis liegen. Die inversen Probleme lassen sich bei der Spektralschätzung, der Signalverarbeitung und der Systemidentifkation anwenden.

Résumé. Cet article presente plusieurs résultats concernant la détermination de la position des zéros d'un polynôme par rapport au cercle unité en utilisant des variantes de la procédure de Schur–Cohn. Premièrement, des démonstrations simples et nouvelles de cette dernière procédure sont présentées dans les cas régulier et singulier. Une nouvelle méthode pour utiliser l'un des deux cas singuliers est également développée. Enfin, nous considérons plusieurs aspects du 'problème inverse', dans lequel on souhaite modifier le polynôme donné pour qu'il ait une distribution de zéros imposée. Deux méthodes permettant de forcer tous les zéros d'un polynôme à appartenir à l'intérieur du cercle unité sont établies. Aussi, deux algorithmes pour résoudre les problèmes inverses dans le cas singulier sont présentés, spécifiquement, l'un corrige un polynôme symétrique pour qu'il ait tous ses zéros sur le cercle unité, et l'autre corrige un polynôme symétrique pour qu'il n'ait aucun zéro sur le cercle unité. Les problèmes inverses ont des applications en estimation spectrale, en traîtement du signal et dans l'identification des systèmes.

Keywords. Schur–Cohn procedure, polynomial zero distribution, discrete-time system stability, wide-sense stability, stabilization, frequency estimation, spectral estimation.

1. Introduction

The unit circle problem consists of determining the distribution of the zeros of a given polynomial with respect to the unit circle. This is an important problem for many applications in control, signal processing and system identification areas and there are various approaches available for solving it (see, e.g., [1, 3–5, 8–10, 12, 14–20, 25, 26, 28, 29, 32–35] and the references therein).

The Schur–Cohn (S–C) procedure is perhaps the most frequently used approach to solve the unit circle problem. It may be used in one of its several equivalent forms (see, e.g., [3–5, 14, 26, 28]). In this paper our discussion will concentrate on the so-called 'table form' which is very convenient for calculations by hand and may be easily programmed on a computer. This 'table form' requires $O(n^2)$ arithmetic operations, where n is the degree of the polynomial under test. More efficient implementations of the S–C type procedures are available, for example those introduced in [3, 5] and [17, 19] are about two times faster than the 'table form' implementation if n is very large (if $n \leqslant 10$ then the 'table form' implementation is faster). We refer to the above references for more details on implementation, as our main concern in this paper is not the computational issue. In fact for the applications we have in mind that n is usually small and the 'table form' implementation of the S–C procedure is quite convenient from a computational standpoint.

The main purpose of this paper may be summarized as follows. We introduce two basic results on the zero distribution for some polynomials of a special form. These results turn out to be sufficient for analyzing the S–C procedures in both the regular and the singular cases. We present simple new proofs of the S–C test, which provide additional insight into the properties of the test. Our analysis of the S–C procedure is partially tutorial but we felt motivated to include it in this paper since proofs of the S–C table form tests are not readily available in the literature (see [3] for a similar comment). Furthermore our analysis suggests a simple way for handling one of the two types of singular cases which may appear within the S–C procedure. There are several treatments in the literature of the singular cases [1, 3–5, 7, 10, 14, 19, 26, 35]. We will comment on some of them later. Here we make the remark that most of the earlier studies are imcomplete or somewhat informal. An important exception is the detailed formal study [3], which deals with the problem of establishing whether the zeros of a given polynomial lie inside and on the unit circle (the so-called wide-sense stability problem). However, this problem is less general than that considered herein and in fact the main results of [3] can be obtained as special cases of our results. We may also remark in the present context that the relation between S–C stability test and the Levinson procedure for solving stationary linear prediction problems is very well understood in the regular case but not completely understood in the singular case. Better understanding this relationship in the singular case appears to be an interesting research topic (see [2, 10, 11, 13] for some relevant studies of this aspect).

This paper also presents a number of new results on what we call the 'inverse problem'. This is the problem of altering a given polynomial to obtain one with a given zero distribution with respect to the unit circle. Even though this problem appears to have a number of important applications in system identification, time series analysis and spectral estimation, a formal treatment of it does not seem to be available in literature.

An outline of this paper is as follows. In Section 2 we establish two key lemmas concerned with the zero distribution of a special class of polynomials. In Section 3 we apply these two lemmas to prove the validity of the Schur–Cohn procedure in the regular case. We also consider the 'inverse problem' in the regular case. We develop three procedures for correcting a polynomial to ensure that all its zeros lie inside the unit circle. In Section 4 we consider the singular cases of the Schur–Cohn procedure. We develop a simple new procedure for handling one type of singular case, and give a simple proof of Cohn's result for the second singular case. Finally, in Section 5 we consider two important singular inverse problems: correcting a symmetric (also called self reciprocal) polynomial to ensure all of its zeros lie on the unit circle, and correcting it to ensure that none of its zeros lie on the unit circle.

2. Notation and basic mathematical results

We consider throughout this paper polynomials with real coefficients. Let

$$A(z)=a_0+a_1z+\cdots+a_nz^n \quad (a_0\cdot a_n\neq 0).$$

Let $A^*(z)$ denote the reciprocal polynomial

$$A^*(z)=z^nA(z^{-1})$$
$$=a_n+a_{n-1}z+\cdots+a_0z^n.$$

We will let n_A denote the degree of $A(z)$, and we will let n_A^+, n_A^- and n_A^0 denote the number of zeros of $A(z)$ which are located outside, inside and on the unit circle $C_1\triangleq\{|z|=1\}$. Note that $|A(z)|=|A^*(z)|$ on C_1, so $A(z)$ and $A^*(z)$ have the same zeros on C_1, and in particular, $n_A^0=n_{A^*}^0$. Note also that $n_A^+=n_{A^*}^-$ and $n_A^-=n_{A^*}^+$. Finally, let $A'(z)=(\mathrm{d}/\mathrm{d}z)A(z)$.

In this section we deal with polynomials of the following special form:

$$P(z)=K(z)B(z)+L(z)B^*(z), \tag{2.1}$$

where K, L and B are polynomials, and where $K(z)$ and $L(z)$ have no zeros on C_1. For polynomials of the form (2.1) it is relatively easy to make statements on zero distribution, using the Rouché's theorem. For ease of reference we include a statement of this theorem [27, p. 218].

ROUCHÉ'S THEOREM. If two functions $f(z)$ and $g(z)$ are analytic on and inside the closed path C, $f(z)\neq 0$ on C, and $|f(z)|>|g(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

Our first basic result is the following.

LEMMA A. Consider the polynomial $P(z)$ defined by (2.1).

(i) If $|K(z)|>|L(z)|$ on C_1, then

$$n_P^-=n_B^-+n_K^-, \tag{2.2}$$
$$n_P^+=n_P-n_B^0-n_B^--n_K^-, \tag{2.3}$$
$$n_P^0=n_B^0, \tag{2.4}$$

(ii) If $|K(z)|<|L(z)|$ on C_1, then

$$n_P^-=n_B^++n_L^-, \tag{2.5}$$
$$n_P^+=n_P-n_B^0-n_B^+-n_L^-, \tag{2.6}$$
$$n_P^0=n_B^0. \tag{2.7}$$

PROOF. See Appendix A.

If $|K(z)|=|L(z)|$ on C_1, the study of the zero distribution of $P(z)$ is a bit more complicated. In such a case exact assertions on the distribution of the zeros of $P(z)$ cannot be made unless additional assumptions are introduced. Our second basic result deals with this case.

LEMMA B. Consider (2.1) and assume that $|K(z)|=|L(z)|$ on C_1. Assume also that $P(z)$ has full degree: $n_P=\max(n_K,n_L)+n_B$.

(i) The following inequalities hold:

$$n_P^-\leqslant\min(n_B^-+n_K^-,\ n_B^++n_L^-), \tag{2.8a}$$
$$n_P^+\leqslant\min(n_B^+-n_K^-,\ n_B^--n_L^-)+\max(n_K,n_L), \tag{2.8b}$$
$$n_P^0\geqslant n_B^0+|n_B^++n_L^--n_B^--n_K^-|. \tag{2.8c}$$

(ii) Let $z_1,\ldots,z_J$ denote the zeros of $P(z)$ on C_1 which are not zeros of $B(z)$. Assume that $P'(z_i)\neq 0$ for $1\leqslant i\leqslant J$.

(iia) If for each $i\in 1,\ldots,J$, the following inequality is satisfied:

$$\mathrm{Re}\left\{\frac{K(z_i)B(z_i)}{z_iP'(z_i)}\right\}<0$$
$$\Leftrightarrow\quad \mathrm{Re}\left\{\frac{L(z_i)B^*(z_i)}{z_iP'(z_i)}\right\}>0, \tag{2.9}$$

then

$$n_P^- = n_B^- + n_K^-, \quad (2.10a)$$

$$n_P^+ = \max(n_K, n_L) + n_B^- - n_L^-. \quad (2.10b)$$

(*iib*) *If for each* $i \in 1, \ldots, J$, *the following inequality is satisfied*:

$$\mathrm{Re}\left\{\frac{K(z_i)B(z_i)}{z_i P'(z_i)}\right\} > 0$$

$$\Leftrightarrow \quad \mathrm{Re}\left\{\frac{L(z_i)B^*(z_i)}{z_i P'(z_i)}\right\} < 0, \quad (2.11)$$

then

$$n_P^- = n_B^+ + n_L^-, \quad (2.12a)$$

$$n_P^+ = \max(n_K, n_L) + n_B^+ - n_K^-. \quad (2.12b)$$

PROOF. See Appendix B.

3. The Schur–Cohn test in the regular case

3.1. The direct problem

Let $A_n(z)$ denote a given polynomial of degree n

$$A_n(z) = a_{0,n} + a_{1,n}z + \cdots + a_{n,n}z^n. \quad (3.1)$$

The 'direct problem' consists of finding the distribution of the zeros of $A_n(z)$ with respect to C_1. To solve this problem, consider the sequence of decreasing order polynomials found by using the following recursion[1]:

$$zA_{k-1}(z) = A_k(z) - \phi_k A_k^*(z),$$
$$k = n, n-1, \ldots, 1, \quad (3.2)$$
$$\phi_k \triangleq a_{0,k}/a_{k,k}.$$

[1] Note that by using (3.2), one reduces the zeroth order term of $A_k(z)$. An alternative way would be to reduce the highest-order term of $A_k(z)$ by using the recursion

$$A_{k-1}(z) = A_k(z) - (a_{k,k}/a_{0,k})A_k^*(z).$$

Using this recursion leads to similar results with those corresponding to (3.2); however, (3.2) seems to be the more commonly used recursion (see, e.g., [4, 26]).

It is assumed that $A_n(z)$ is such that

$$a_{k,k} \neq 0, \quad k = 0, \ldots, n. \quad (3.3)$$

Note from (3.2) that

$$a_{k-1,k-1} = a_{k,k}[1 - (a_{0,k}/a_{k,k})^2]. \quad (3.4)$$

Thus, the condition (3.3) is equivalent to

$$a_{n,n} \neq 0, \quad |a_{0,k}| \neq |a_{k,k}|,$$
$$\Leftrightarrow \quad |\phi_k| \neq 1, \quad k = 1, \ldots, n. \quad (3.5)$$

The condition above designates the 'regular case'. The 'singular case', in which (3.5) fails to hold, is discussed in the next section.

Under the condition (3.5) it is rather easy to solve the direct problem. The following result is attributed originally to Schur and Cohn even if in its present form it looks quite different from the original results (Schur and Cohn results evolved towards the form of results presented below through the work of many researchers including Jury, Åström and Raible).

THEOREM 1. Assume that the condition (3.5) is satisfied. Let s^- *denote the number of* $a_{k,k}$ *coefficients,* $k = n-1, n-2, \ldots, 0$, *which have the same sign as* $a_{n,n}$. *Then*

$$n_{A_n}^- = s^-, \quad n_{A_n}^+ = n - s^-, \quad n_{A_n}^0 = 0. \quad (3.6)$$

PROOF. Theorem 1 follows immediately from Theorem 2 below (set $p = 0$, $n_{A_0}^- = n_{A_0}^+ = n_{A_0}^0 = 0$ there). □

Next we present a more general result that will be quite useful when analyzing the singular cases of the S–C procedure in Section 4. This result is also useful for the present analysis. Specifically, it can be used to save computation whenever in the recursive calculation of (3.2) one arrives at a polynomial with known zero distribution.

THEOREM 2. Let $A_p(z)$ *be computed from* $A_n(z)$ *using (3.2) for* $k = n, n-1, \ldots, p+1$, *and assume* $a_{p,p} \neq 0$. *Let* $s_{\mu,\nu}^-$ ($s_{\mu,\nu}^+$) *denote the number of elements in the sequence* $\{a_{k,k}\}_{k=\mu}^{\nu-1}$ ($\nu > \mu$) *which have the* *same sign as* $a_{v,v}$ $(-a_{v,v})$, *and let* s^- (s^+) *be a short notation for* $s_{p,n}^-$ $(s_{p,n}^+)$. *Then*

(*i*) *If* $a_{p,p} \cdot a_{n,n} > 0$,

$$n_{A_n}^- = n_{A_p}^- + s^-, \tag{3.7a}$$

$$n_{A_n}^+ = n_{A_p}^+ + s^+, \tag{3.7b}$$

$$n_{A_n}^0 = n_{A_p}^0. \tag{3.7c}$$

(*ii*) *If* $a_{p,p} \cdot a_{n,n} < 0$,

$$n_{A_n}^- = n_{A_p}^+ + s^-, \tag{3.8a}$$

$$n_{A_n}^+ = n_{A_p}^- + s^+, \tag{3.8b}$$

$$n_{A_n}^0 = n_{A_p}^0. \tag{3.8c}$$

PROOF. See Appendix C.

3.2. The inverse problem

Let us assume that we are given a polynomial $A_n(z)$ which satisfies the regularity condition (3.5). Then its zero distribution may be established using Theorem 1. Now, let us suppose that we want an approximant of $A_n(z)$, with a given zero distribution (i.e., with a specified number of zeros inside and outside C_1). Determination of such an approximant is what we call the 'inverse problem'. Most commonly, such a problem occurs in parameter estimation and system identification applications where we may need a stable approximant (i.e., one with all zeros lying inside the unit circle) of a given possibly unstable polynomial [22, 31]. A simple solution to this class of inverse problems may be obtained as described below.

Let us observe that (3.2) can be rewritten as

$$A_{k-1}^*(z) = -\phi_k A_k(z) + A_k^*(z). \tag{3.9}$$

Combining (3.2) and (3.9), we get

$$\begin{bmatrix} zA_{k-1}(z) \\ A_{k-1}^*(z) \end{bmatrix} = \begin{bmatrix} 1 & -\phi_k \\ -\phi_k & 1 \end{bmatrix} \begin{bmatrix} A_k(z) \\ A_k^*(z) \end{bmatrix}. \tag{3.10}$$

Since $|\phi_k| \neq 1$, the matrix in (3.10) is nonsingular. Thus, (3.10) can be rewritten as

$$\begin{bmatrix} A_k(z) \\ A_k^*(z) \end{bmatrix} = \frac{1}{1-\phi_k^2} \begin{bmatrix} 1 & \phi_k \\ \phi_k & 1 \end{bmatrix} \begin{bmatrix} zA_{k-1}(z) \\ A_{k-1}^*(z) \end{bmatrix}, \tag{3.11}$$

which in turn gives

$$A_k(z) = [zA_{k-1}(z) + \phi_k A_{k-1}^*(z)]/(1-\phi_k^2). \tag{3.12}$$

Equations (3.2) and (3.12) are commonly called the 'backward recursion' and 'forward recursion' equations, respectively. Any polynomial $A_n(z)$ (satisfying condition (3.5)) can be converted to the set $\{a_{0,0}, \phi_1, \ldots, \phi_n\}$, and conversely, using these two equations.

Recall that our inverse problem is to obtain a stable approximant of a given polynomial $A_n(z)$. To this end we can proceed by using one of the following two methods which correct the sequence $\{\phi_k\}$ associated with $A_n(z)$. Note that if this sequence is such that $|\phi_k| < 1$ for $1 \leqslant k \leqslant n$, then all of the elements in the sequence $\{a_{k,k}\}_{k=0}^n$ have the same sign, which implies that the corresponding polynomial $A_n(z)$ is stable (c.f. Theorem 1).

METHOD A

Step 0. Test whether $A_n(z)$ is stable. If it is, then set $\bar{A}_n(z) = A_n(z)$. Otherwise, determine a stable approximant $\bar{A}_n(z)$ in the following steps.

Step 1. For each $k = n, n-1, \ldots, 1$,

(a) Compute ϕ_k.

(b) If $|\phi_k| \leqslant \alpha$, then leave it unchanged ($\alpha \in (0, 1)$).

(c) If $|\phi_k| > \alpha$, then replace it by $\bar{\phi}_k = \alpha$ sign(ϕ_k). Also, replace $a_{k,k}$ by $\bar{a}_{k,k} = a_{0,k}/\bar{\phi}_k$ (alternatively, replace $a_{0,k}$ by $\bar{a}_{0,k} = a_{k,k}\bar{\phi}_k$). Use the 'bar terms' in all subsequent computations of the backward recursion (3.2).

Step 2. Using (3.12), compute the stable approximant $\bar{A}(z)$ from $\{\bar{\phi}_k\}_{k=1}^n$ and $\bar{a}_{0,0}$ (note that $\bar{a}_{0,0}$ has only a scaling effect which is eliminated by the next operation). Normalize $\bar{A}_n(z)$ such that $\bar{a}_{k,n} = a_{k,n}$ for some k (usually for $k = n$).

The threshold α in the above method should not be chosen too close to one, contrary to what might be expected. For example, one may choose $\alpha = 0.75$ or 0.8. This recommendation can be motivated as follows. If $|\phi_k|$ is only slightly less than 1 (e.g., $|\phi_k| = 0.99$) then $|\phi_{k-1}|$ will in general be quite large since $a_{k-1,k-1}$ will be small (see (3.4)). Thus, ϕ_{k-1} will have to be significantly truncated, and this may affect the approximation error adversely.

Note that Step 0 in Method A is required to avoid correcting a possibly stable polynomial $A_n(z)$ in Steps 1 and 2. Note also that the truncation of ϕ_k to the interval $[-\alpha, \alpha]$ (e.g., $\alpha = 0.75$) is neither necessary nor sufficient for ensuring that $|\phi_{k-1}|$ is 'small'. Method B presented in the following is designed to eliminate the above drawbacks of Method A.

METHOD B

Step 1. For each $k = n, n-1, \ldots, 1$

(a) Compute ϕ_k.

(b) If $|\phi_k| < 1$, then leave it unchanged.

(c) If $1 \leqslant |\phi_k| \leqslant 1.5$, then replace ϕ_k by $\hat{\phi}_k = 0.95\,\mathrm{sign}(\phi_k)$. also, replace $a_{k,k}$ by $\hat{a}_{k,k} = a_{0,k}/\hat{\phi}_k$ (alternatively, replace $a_{0,k}$ by $\hat{a}_{0,k} = a_{k,k}\hat{\phi}_k$). Use the 'hat terms' in all subsequent computations of the backward recursion (3.2).

(d) If $|\phi_n| > 1.5$, then replace it by $\hat{\phi}_n = 0.95\,\mathrm{sign}(\phi_n)$.
If $|\phi_k| > 1.5$ for $k < n$, then decrease $|\phi_{k+1}|$ in steps of 0.05, correct $a_{k+1,k+1}$ (or $a_{0,k+1}$) accordingly, and recompute ϕ_k until either $|\phi_k| \leqslant 1.5$ (thus one of cases (b) or (c) occurs) or $|\phi_{k+1}| < 0.5$. In the latter case, replace ϕ_k by $\hat{\phi}_k = 0.95\,\mathrm{sign}(\phi_k)$ and go to $k-1$.

Step 2. As in Method A.

The various threshold values in Method B (1.5, 0.95, 0.05, 0.5) were arrived at by experiment, and other values may produce improved approximations in specific applications. However, in our experiments, the approximation error has not changed significantly as these parameters have been varied.

It should be noted that Methods A and B are not optimal in the sense of minimizing some norm of the difference between the given polynomial and its stable approximant. Furthermore, these methods are based on the transformation $\{a_{i,n}\} \to \{\phi_i\}$ which may be quite sensitive to small perturbations in the coefficient sequence $\{a_{i,n}\}$ as is shown in Examples 3.1 and 3.2 below. Nevertheless, they are computationally simple means of determining a 'suboptimal' stable approximant of a given polynomial; their results may be refined by more sophisticated methods if so desired (see e.g., [22]).

EXAMPLE 3.1. This example illustrates the performance of Methods A and B for stabilizing noise-perturbed versions of the polynomial

$$A_6(z) = z^6 - 0.284z^5 - 0.2226z^4 + 0.0527z^3 + 0.3254z^2 - 0.2135z - 0.6161.$$

This polynomial has zeros at $0.98\,\mathrm{e}^{\pm j0.3\pi}$, $0.9\,\mathrm{e}^{\pm j0.7\pi}$, 0.99 and -0.8. White noise with zero mean and standard deviation $\sigma = 0.1$ was added to each coefficient except the z^6 coefficient, resulting in a perturbed monic polynomial. Methods A and B were applied to stabilize the resulting polynomial. This experiment was repeated 50 times and the errors between the original and stabilized polynomial coefficients were computed by

$$D_k = \sqrt{\frac{1}{50}\sum_{i=1}^{50} (\hat{a}_{k,i} - a_k)^2}, \quad k = 0, 1, \ldots, 5. \tag{3.13}$$

The resulting errors from Methods A and B are shown in Table 1. Also shown are the errors for the perturbed polynomials before stabilization; these errors are approximately equal to σ as they should be. However, for the stabilized polynomials it can be seen that some of the coefficient errors for Methods A and B are much larger than σ, indicating that the corrected polynomials are 'far' from the original polynomial. This large error is also

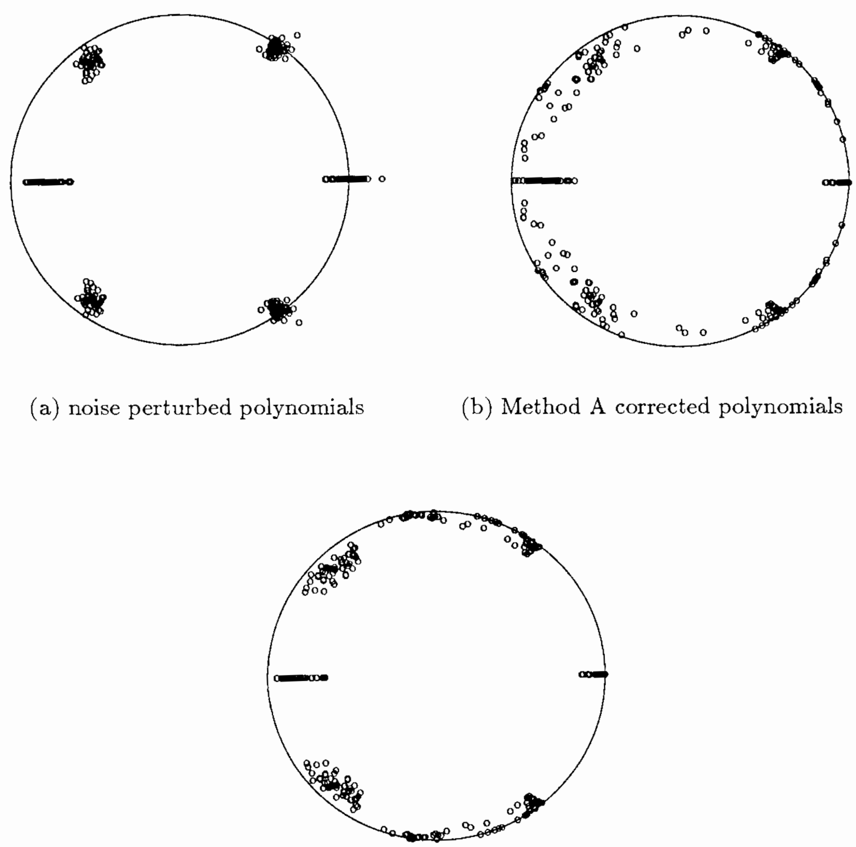
Table 1

Errors of uncorrected and corrected polynomial coefficients corresponding to Examples 3.1 and 3.2

	D_1	D_2	D_3	D_4	D_5	D_6
Uncorrected	0.1166	0.1145	0.1022	0.0986	0.0874	0.0992
Method A	0.1062	0.6146	1.1020	0.2714	1.0731	0.9701
Method B	0.1037	0.7581	0.9775	0.1633	0.7920	1.0955
Method C	0.1750	0.0900	0.1067	0.0859	0.0724	0.0831

evident in the zero locations of the stabilized polynomials. Figure 1 shows the zeros of the uncorrected and corrected polynomials, and it is evident that the corrected polynomials zeros are not always close to the original polynomial zeros.

The primary reason that the corrected polynomials are far from the original one is that the correction procedures in Methods A and B rely on correcting the $\{\phi_k\}$ sequence, and that small errors in the polynomial coefficients can give rise to large



(a) noise perturbed polynomials

(b) Method A corrected polynomials

(c) Method B corrected polynomials

Fig. 1. Zero locations of polynomials in Example 3.1.

Table 2
Errors in the reflection coefficient sequence corresponding to Example 3.1

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
True sequence	−0.6948	0.1331	0.0394	−0.1910	−0.6262	−0.6161
Five sample sequences	0.4479	1.6876	−2.6804	−0.7421	−0.7276	−0.5699
	−0.7648	1.8952	0.9607	7.3625	−1.0385	−0.7077
	−4.5244	−0.9857	−11.0594	0.9972	−20.6429	−0.9755
	−0.8711	14.3291	0.8840	−0.5183	−0.8550	−0.5872
	1.2651	−4.8264	−0.8516	−0.1443	−0.5582	−0.6336
Mean of 50	−0.6365	0.0991	−0.0752	3.2267	−1.0999	−0.6211
St. dev of 50	1.7957	10.1654	2.0788	18.3636	2.8107	0.1165

deviations in the $\{\phi_k\}$ sequence. This is shown in Table 2, where the 'reflection coefficient' sequences $\{\phi_k\}$ of five of the perturbed polynomials in the Monte-Carlo experiment are shown. Even though the coefficients of these polynomials are perturbed by no more than $2\sigma = 0.2$, the reflection coefficients are sometimes very different from the unperturbed reflection coefficients. The ill-conditioning of the transformation from the $\{a_k\}$ to the $\{\phi_k\}$ sequence is caused by the division by $(1-\phi_k^2)$ in (3.12); when the zeros of $A_n(z)$ are near the unit circle, this term approaches zero. There does not seem to be a direct way of overcoming this inherent ill-conditioning problem for Methods A and B.

One way to circumvent the ill-conditioning problem is to stabilize a polynomial by directly adjusting the polynomial coefficients, and use the reflection coefficient sequence only as a stability test. This leads to Method C.

METHOD C

Step 0. Test whether $A_n(z)$ is stable; if so, then $\bar{A}_n(z) = A_n(z)$. If not, determine a stable approximant $\bar{A}_n(z)$ in the following steps.

Step 1. Let $\hat{A}_n(z) = a_{n,n}z^n + a_{n-1,n}\alpha_0 z^{n-1} + \cdots + a_{1,n}\alpha_0^{n-1}z + a_{0,n}\alpha_0^n$ for some $\alpha_0 \in (0, 1]$. Test whether $\hat{A}_n(z)$ is stable.

- If $\hat{A}_n(z)$ is stable, use a bisecting method I times on $\alpha \in [\alpha_0, 1]$ to determine the largest α for which $\hat{A}_n(z)$ is stable.
- If $\hat{A}_n(z)$ is not stable, reduce α_0 by some given amount β and repeat Step 1.

Note that Method C does not change the angles of the zeros of $A_n(z)$; this method scales the magnitudes of the zeros until the maximum magnitude is less than one. As a result, if an unstable polynomial has zeros not too far outside the unit circle, the Method C stabilized polynomial will have zeros (and polynomial coefficients) which are close to those of the unstable polynomial. In addition, Method C is computationally efficient; for most choices of α_0 and I, this method is faster than Method B.

EXAMPLE 3.2. Figure 2 shows the zeros of the 50 stabilized polynomials using Method C; these results use the same polynomials as in Fig. 1. For this example, $\alpha_0 = 0.8$, $\beta = 0.2$ and $I = 5$. The polynomial coefficient errors for this method are shown in Table 1. It can be seen that the stabilized polynomial zeros are much closer to the original polynomial zeros using Method C. The polynomial

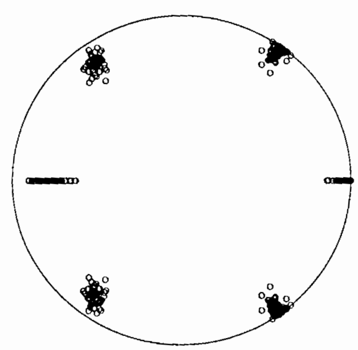


Fig. 2. Zero locations of Method C corrected polynomials, corresponding to Fig. 1.

coefficient errors are also much smaller with Method C than with either Methods A or B.

In general, method C is expected to work well when n is not too large, and when the errors on the polynomial coefficients are not too large. The small order restriction results from the exponential perturbation on the polynomial coefficients; α_0^n is close to zero when n is large. The small perturbation restriction arises from the fact that the set of all polynomial coefficients which give stable polynomials is not convex; thus, for large perturbations, the projection of an unstable polynomial vector onto the stability set may be 'far' from the original polynomial. This latter issue is not particular to Method C, but true of all stabilizing methods, even optimal methods [22]. On the other hand, applications which use polynomial stabilization often satisfy the constraints that neither n nor the perturbations are too large.

4. The direct problem in the singular case

During the recursive computation of (3.2) one may encounter a polynomial $A_p(z)$ of the following form

$$\begin{aligned} A_p(z) &= (a_0 + a_1 z + \cdots + a_m z^m) \\ &\quad + (a_{m+1} z^{m+1} + \cdots + a_{p-m-1} z^{p-m-1}) \\ &\quad + \beta z^{p-m}(a_m + a_{m-1} z + \cdots + a_0 z^m) \quad (a_0 \neq 0) \\ &\triangleq F(z) + R(z) + \beta z^{p-m} F^*(z), \end{aligned} \tag{4.1}$$

where $\beta = \pm 1$, and where $0 \leqslant m \leqslant \lfloor p/2 \rfloor$ ($\lfloor x \rfloor$ denotes the integer part of x). Assume that

$$a_{m+1} \neq \beta a_{p-m-1}. \tag{4.2}$$

The condition above ensures that m is the largest order for which $A_p(z)$ can be written in the form of (4.1). Note that if $m = \lfloor p/2 \rfloor$ then $R(z) = 0$ in (4.1).

It may be argued that in applications it is unlikely to arrive at polynomials $A_p(z)$ of the form (4.1) due to round-off errors. However, we may obtain polynomials $A_p(z)$ having the symmetry of (4.1) to within a pre-imposed numerical accuracy; moreover, the polynomial $A_n(z)$ itself may be of the form (4.1).

For $A_p(z)$ given by (4.1) we have $\phi_p = \beta = \pm 1$, so from (3.2)

$$\begin{aligned} A_{p-1}(z) &= \frac{1}{z} R(z) - \beta z^{p-1} R(z^{-1}) \\ &= a_{m,p-1} z^m + \cdots + a_{p-m-2,p-1} z^{p-m-2}. \end{aligned} \tag{4.3}$$

Thus the first m and the last $m+1$ coefficients of $A_{p-1}(z)$ are equal to zero. This situation is usually referred to as the *singular case* of the Schur–Cohn test. When it occurs, ϕ_{p-1} cannot be computed, and the recursion (3.2) must be stopped. In the following we will discuss how to handle the singular case. It will be convenient to consider separately the following two types of singular cases:
Singular case I: $A_{p-1}(z) \not\equiv 0$,
Singular case II: $A_{p-1}(z) \equiv 0$.

4.1. Singular case I

Theorem 2 in Section 3 suggests the following method to handle Singular case I: When an $A_p(z)$ of the form (4.1) is obtained during the recursive computation of (3.2), replace it by a polynomial $\bar{A}_p(z)$ with the same zero distribution as $A_p(z)$ but with $|\bar{\phi}_p| \neq 1$, and continue the recursion using $\bar{A}_p(z)$. This is the way to proceed that is commonly used in literature (e.g., [26, 35], etc.) even though a proof of its validity is not provided; also, the importance of the sign of $a_{p,p} a_{n,n}$ (c.f. Theorem 2) is not mentioned.

The problem which remains is to find a polynomial $\bar{A}_p(z)$ having the same zero distribution as $A_p(z)$. In [26], following a result by Cohn, it is suggested to form $\bar{A}_p(z)$ as

$$\bar{A}_p(z) = (z^{m+1} + g) A_p(z) \tag{4.4}$$

for certain g with $|g| > 1$. Note that $\bar{A}_p(z)$ has degree $m+p+1$ and has the same zeros inside and on C_1 as $A_p(z)$. Alternatively, in [35] it is suggested to set

$$\bar{A}_p(z)=(z+\sigma)A_p(z) \tag{4.5}$$

for some $\sigma \neq \pm 1$. This solution might seem simpler than that of [26] since the degree of (4.5) is smaller than that of (4.4). However, this is not true. To see this, define $\tilde{F}(z)$ such that $F(z)=a_0+z\tilde{F}(z)$. Then we have from (4.1), (4.5)

$$\begin{aligned}\bar{A}_p(z)&=zA_p(z)+\sigma[a_0+z\tilde{F}(z)\\&\quad+R(z)+\beta z^{p-m}F^*(z)]\\&=z[A_p(z)+\sigma\tilde{F}(z)+z^{-1}\sigma R(z)\\&\quad+\sigma\beta z^{p-m-1}F^*(z)]+a_0\sigma,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\bar{A}_p^*(z)&=(1+\sigma z)A_p^*(z)\\&=z\sigma A_p^*(z)+z^{p-m}F^*(z)\\&\quad+z^{m+1}R^*(z)+\beta[a_0+z\tilde{F}(z)]\\&=z[\sigma A_p^*(z)+z^{p-m-1}F^*(z)\\&\quad+z^mR^*(z)+\beta\tilde{F}(z)]+\beta a_0.\end{aligned} \tag{4.7}$$

Since $\bar{\phi}_p=\sigma/\beta=\sigma\beta$, we get from (4.6) and (4.7)

$$\begin{aligned}\bar{A}_{p-1}(z)&=\frac{1}{z}[\bar{A}_p(z)-\sigma\beta\bar{A}_p^*(z)]\\&=A_p(z)-\sigma^2\beta A_p^*(z)+\sigma z^{-1}R(z)\\&\quad-\sigma\beta z^mR^*(z),\end{aligned}$$

which is still in the Singular case I form, but now the $F(z)$ polynomial corresponding to $\bar{A}_{p-1}(z)$ has degree $m-1$. Thus, we need to multiply $\bar{A}_{p-1}(z)$ by $(z+\sigma)$ again, and this procedure must be repeated m times.

The discussion above shows that the procedure based on (4.4) is more efficient computationally than that which uses (4.5). However, (4.4) is not a very attractive choice for $\bar{A}_p(z)$ either, because $\bar{A}_p(z)$ has a (much) higher degree than $A_p(z)$ and this leads to an increase in the computational burden. Furthermore, Singular case I may occur more than once for a given polynomial $A_n(z)$. Since in every such case we have to replace $A_p(z)$ by a higher-order polynomial, it is not a priori clear in how many steps such a procedure will eventually terminate. It would be very convenient from both a computational and a theoretical standpoint to replace $A_p(z)$ with a polynomial $\bar{A}_p(z)$ of degree p. Such a polynomial exists as shown in the following.

THEOREM 3. Consider the polynomial $A_p(z)$ given by (4.1). Then the polynomial

$$\bar{A}_p(z)=A_p(z)+\rho z^{-(m+1)}[A_p(z)-\beta A_p^*(z)],$$
$$|\rho|<\tfrac{1}{2} \tag{4.8}$$

has the same zero distribution (with respect to C_1) as $A_p(z)$ does. Furthermore $|\bar{a}_{p,0}/\bar{a}_{p,p}|\neq 1$ for every $|\rho|\in(0,1/2)$ except possibly for $\rho=-2a_0/(a_{m+1}-\beta a_{p-m-1})$.

PROOF. See Appendix D.

The polynomial $\bar{A}_p(z)$ is easy to construct. Note that $z^{-1}[A_p(z)-\beta A_p^*(z)]$ in (4.8) is equal to $A_{p-1}(z)$ and is obtained during the iteration with (3.2). We illustrate the use of $\bar{A}_p(z)$ in (4.8) by means of a numerical example.

EXAMPLE 4.1. Let

$$\begin{aligned}A_9(z)&=4z^9-6z^8-4z^7+2z^6+5z^5\\&\quad-14z^4-8z^3+4z^2-6z-4\\&=4(z-2)(z^4+2)(z^3+\tfrac{1}{2})(z+\tfrac{1}{2}).\end{aligned}$$

Note that

$$n_{A_9}^-=4,\quad n_{A_9}^+=5,\quad n_{A_9}^0=0.$$

Use of Theorems 2 and 3 to resolve Singular case I which occurs during the application of the Schur–Cohn procedure leads to the results of Table 3. From Table 3 we obtain the correct zero distribution from the signs of the $a_{k,k}$ and $\bar{a}_{k,k}$ coefficients (shown as bold face numbers in the table).

Note that other ways of handling Singular case I may be found in [1, 14]. Stated briefly, the basic idea of both the ε-approach of [14] and the method of [1] (which is based on infinitesimally contracting and/or expanding C_1) is to perturb the coefficients

Table 3

Schur–Cohn table to Example 4.1, illustrating Singular case I

A_9	**4**	−6	−4	2	5	−14	−8	4	−6	−4	$\phi_9=-1$
$-\phi_9 A_9^*$	−4	−6	4	−8	−14	5	2	−4	−6	4	
zA_8	0	−12	0	−6	−9	−9	−6	0	−12	0	Singular case I
$\frac{1}{3}A_8$	0	0	−4	0	−2	−3	−3	−2	0	−4	
$\bar{A}_9$	4	−6	−8	2	3	−17	−11	2	−6	−8	$\bar{\phi}_9=-2$
$-\bar{\phi}_9\bar{A}_9^*$	−16	−12	4	−22	−34	6	4	−16	−12	8	
$\bar{A}_8$	**−12**	−18	−4	−20	−31	−11	−7	−14	−18		$\bar{\phi}_8=1.5$
$-\bar{\phi}_8\bar{A}_8^*$	27	21	10.5	16.5	46.5	30	6	27	18		
$\bar{A}_7$	**15**	3	6.5	−3.5	15.5	19	−1	13			$\bar{\phi}_7=0.867$
$-\bar{\phi}_7\bar{A}_7^*$	−11.267	0.867	−16.467	−13.433	3.033	−5.633	−2.6	−13			
$\bar{A}_6$	**3.733**	3.867	−9.667	−16.933	18.533	13.367	−3.6				$\bar{\phi}_6=-0.964$
$-\bar{\phi}_6\bar{A}_6^*$	−3.471	12.889	17.871	−16.329	−9.611	3.729	3.6				
$\bar{A}_5$	**0.262**	16.756	7.905	−33.262	8.923	17.095					$\bar{\phi}_5=65.273$
$-\bar{\phi}_5\bar{A}_5^*$	−1115.9	−582.4	2171.1	−516.0	−1093.7	−17.095					
$\bar{A}_4$	**−1115.6**	−565.6	2179.0	−549.2	−1084.8						$\bar{\phi}_4=0.9724$
$-\bar{\phi}_4\bar{A}_4^*$	1054.8	534.1	−2118.8	550	1084.8						
$\bar{A}_3$	**−60.76**	−31.59	60.17	0.8							$\bar{\phi}_3=-0.0132$
$-\bar{\phi}_3\bar{A}_3^*$	0.011	0.792	−0.416	−0.8							
$\bar{A}_2$	**−60.75**	−30.79	59.76								$\bar{\phi}_2=-0.9836$
$-\bar{\phi}_2\bar{A}_2^*$	58.8	−30.3	−59.76								
$\bar{A}_1$	**−1.975**	−61.1									$\bar{\phi}_1=30.93$
$-\bar{\phi}_1\bar{A}_1^*$	1889.2	61.1									
$\bar{A}_0$	**1887.2**										

of $A_n(z)$ in such a way that Singular case I is eliminated (within the ε-approach this is done implicitly). To ensure that no other singular cases are introduced by this perturbation, both these methods require algebraic manipulations; as such they do not seem attractive for analysis of high-order polynomials or for numerical implementation.

4.2. Singular case II

Consider now the case where $A_{p-1}(z)\equiv 0$. This may happen if and only if $A_p(z)$ is given by (4.1) with no $R(z)$ term. In other words,

$$A_p(z)=F(z)+\beta z^{p-m}F^*(z), \qquad (4.9)$$

where $m=\lfloor p/2\rfloor$. In this case

$$A_p^*(z)=\begin{cases}A_p(z) & \text{if } \beta=+1,\\ -A_p(z) & \text{if } \beta=-1.\end{cases}$$

We say that $A_p(z)$ is 'symmetric' when $\beta=+1$ and 'skew-symmetric' when $\beta=-1$.

Since $A_{p-1}(z)\equiv 0$, the procedure of the previous subsection is of no help here. However, this singular case can be handled by using an interesting result due to Cohn (see, e.g., [3, 26]). We will present a simple proof of that result. First, however, we need the following preliminary result.

LEMMA 1. Let $A(z)=a_0+a_1z+\cdots+a_nz^n$. Then

$$nA^*(z)=z[A^*(z)]'+[A'(z)]^*. \qquad (4.10)$$

PROOF. Equation (4.10) follows by a simple calculation (see [3]). □

It follows from Lemma 1 above that $A_p(z)$ given by (4.9) satisfies

$$pA_p(z)=zA_p'(z)+\beta[A_p'(z)]^*. \qquad (4.11)$$

We can now state and prove Cohn's result.

LEMMA 2. Let the polynomial $A_p(z)$ be symmetric or skew-symmetric. Then $A_p(z)$ and $A_p'(z)$ have the same number of zeros outside C_1.

PROOF. See Appendix E.

Next we describe the use of Lemma 2 and Theorem 2 for handling the case of a polynomial $A_p(z)$ of the form (4.9), which may cocur during the recursive computation of (3.2).

THEOREM 4. Let $A_p(z)$ be determined from $A_n(z)$ using (3.2) for $k=n, n-1, \ldots, p+1$. Assume that $A_p(z)$ is of the form (4.9) (or, equivalently, that $A_{p-1}(z)\equiv 0$). Define $B_{p-1}(z) \triangleq A_p'(z)$. Then

$$\begin{aligned} &n_{A_n}^- = n_{B_{p-1}}^+ + s^-, \quad n_{A_n}^+ = n_{B_{p-1}}^+ + s^+, \\ &n_{A_n}^0 = p - 2n_{B_{p-1}}^+, \end{aligned} \tag{4.12}$$

where s^- and s^+ are as defined in Theorem 2.

PROOF. See Appendix F.

We illustrate the use of Theorem 4 for handling Singular case II by means of a numerical example.

EXAMPLE 4.2. Let

$$\begin{aligned} A_5(z) &= z^5 - z^4 - 2z^3 - z^2 + z + 2 \\ &= (z^2-1)(z-2)(z^2+z+1). \end{aligned}$$

Note that $n_{A_5}^- = 0$, $n_{A_5}^+ = 1$ and $n_{A_5}^0 = 4$. The Schur–Cohn procedure, combined with the method previously described for resolving Singular case II, leads to the results shown in Table 4. It follows from Table 4 that $n_{B_3}^+ = 0$, $s^+ = 1$ and $s^- = 0$ which, when inserted into (4.12), leads to the correct zero distribution of $A_5(z)$.

It is worth remarking that for some $A_n(z)$ polynomials, both Singular cases I and II may occur, possibly more than once. In such situations we have to repeat the procedures described above for handling the singular cases and the so obtained results should be interpreted with care.

We end this section by a numerical example where both Singular cases I and II occur.

EXAMPLE 4.3. Let

$$\begin{aligned} A_8(z) &= 2z^8 - 4z^7 - z^6 + 6z^5 - 9z^4 \\ &\quad + 0z^3 + 4z^2 - 2z + 4 \\ &= 2(z-2)(z^2-1)(z^3+2)(z^2+\tfrac{1}{2}). \end{aligned}$$

Note that $n_{A_8}^- = 2$, $n_{A_8}^+ = 4$ and $n_{A_8}^0 = 2$. By applying the Schur–Cohn test to $A_8(z)$, together with the techniques for handling the singular cases introduced above, we obtain the results shown in Table 5. From this table we get $n_{B_1}^+ = 0$, $s_{2,7}^- = 3$ and $s_{2,7}^+ = 2$, which, when inserted into (4.12), give $n_{A_7}^- = 3$, $n_{A_7}^+ = 2$ and $n_{A_7}^0 = 2$. We also note that $a_{7,7}a_{8,8} < 0$, $s_{7,8}^- = 0$ and $s_{7,8}^+ = 1$, which, when used in (3.8a–c), give the correct zero distribution.

4.3. Wide-sense stability test

The results established so far can be readily used to derive a wide-sense stability test (a polynomial

Table 4
Schur–Cohn table for Example 4.2, illustrating Singular case II

A_5	**1**	−1	−2	−1	1	2	$\phi_5 = 2$
$-\phi_5 A_5^*$	−4	−2	2	4	2	−2	
A_4	**−3**	−3	0	3	3		$\phi_4 = -1$
$-\phi_4 A_4^*$	3	3	0	−3	−3		
A_3	0	0	0	0	0		Singular case II
B_3	**−12**	−9	0	3			$\phi_3 = -0.25$
$-\phi_3 B_3^*$	0.75	0	−2.25	−3			
B_2	**−11.25**	−9	−2.25				$\phi_2 = 0.2$
$-\phi_2 B_2^*$	0.45	1.8	2.25				
B_1	**−10.8**	−7.2					$\phi_1 = 0.666$
$-\phi_1 B_1^*$	4.8	7.2					
B_0	**−6**						

Table 5

Schur–Cohn table for Example 4.3

A_8	**2**	−4	−1	6	−9	0	4	−2	4	$\phi_8=2$
$-\phi_8 A_8^*$	−8	4	−8	0	18	−12	2	8	−4	
A_7	**−6**	0	−9	6	9	−12	6	6		$\phi_7=-1$
$-\phi_7 A_7^*$	6	6	−12	9	6	−9	0	−6		
zA_6	0	6	−21	15	15	−21	6	0		Singular case I
$\frac{1}{3}A_6$	0	0	2	−7	5	5	−7	2		
$\bar{A}_7$	−6	0	−7	−1	14	−7	−1	8		$\bar{\phi}_7=-1.333$
$-\bar{\phi}_7 \bar{A}_7^*$	10.6667	−1.3333	−9.3333	18.6667	−1.333	−9.3333	0	−8		
$\bar{A}_6$	**4.6667**	−1.3333	−16.3333	17.6667	12.6667	−16.3333	−1			$\bar{\phi}_6=$ −0.2143
$-\bar{\phi}_6 \bar{A}_6^*$	−0.2143	−3.5000	2.7143	3.7857	−3.5000	−0.2857	1			
$\bar{A}_5$	**4.4524**	−4.8333	−13.6190	21.4524	9.1667	−16.6190				$\bar{\phi}_5=$ −3.7326
$-\bar{\phi}_5 \bar{A}_5^*$	−62.0326	34.2157	80.0736	−50.8347	−18.0410	16.6190				
$\bar{A}_4$	**−57.5802**	29.3824	66.4545	−29.3824	−8.8743					$\bar{\phi}_4=0.1514$
$-\bar{\phi}_4 \bar{A}_4^*$	1.3677	4.5284	−10.2421	−4.5284	8.8743					
$\bar{A}_3$	**−56.2125**	33.9108	56.2125	−33.9108						$\bar{\phi}_3=0.6033$
$-\bar{\phi}_3 \bar{A}_3^*$	20.4571	−33.9108	−20.4571	33.9108						
$\bar{A}_2$	**−35.7554**	0	35.7554							$\bar{\phi}_2=-1$
$-\bar{\phi}_2 \bar{A}_2^*$	35.7554	0	−35.7554							
$\bar{A}_1$	0	0								Singular case II
B_1	**−71.5109**	0								$\phi_1=0$
$-\phi_1 B_1^*$	0	0								
B_0	**−71.5109**									

$A_n(z)$ is called wide-sense stable if $n^+_{A_n}=0$). In doing so, we rediscover the test introduced in the interesting paper [3]. This test proceeds as follows:

(i) Use (3.2) to compute $A_{n-1}(z)$, $A_{n-2}(z), \ldots$. If one encounters a $|\phi_p|>1$ then $A_n(z)$ is unstable (this follows from Theorem 2 since in the above case $s^+>1$).

(ii) If one encounters a $|\phi_p|=1$ and the corresponding polynomial $A_p(z)$ is in Singular case I, then again $A_n(z)$ is unstable. This is so since such an $A_p(z)$ cannot have all its zeros on C_1 (in that case it would be symmetric or skew symmetric) and, therefore, it must have zeros both inside and outside C_1 (since $|\phi_p|=|$product of the zeros of $A_p(z)|=1$); thus $n^+_{A_p}>1$, $n^-_{A_p}>1$ in Theorem 2, and the assertion follows.

(iii) If one encounters a $|\phi_p|=1$ and the polynomial $A_p(z)$ is in Singular case II then $A_n(z)$ is wide-sense stable if and only if $B_{p-1}(z)=A_p'(z)$ is wide-sense stable (c.f. Theorem 4). Thus the test proceeds on $B_{p-1}(z)$.

Wide-sense stability tests, such as the above one, find applications in systems and circuits problems. For example, the procedure of [22] for determining the stable polynomial which is closest, in the Euclidean norm sense, to an unstable one, requires a wide-sense stability test to check whether a given coefficient vector belongs to the stability set (including its boundary).

5. The inverse problem in the singular case

The inverse problem in the singular case is, by analogy with the regular case, to synthesize a polynomial $A_n(z)$ which does not satisfy (3.5) and which has a certain specified zero distribution with respect to C_1. For such a polynomial, the corresponding sequence of $\{\phi_k\}$ must contain some elements of unit modulus. Then the forward recursion (3.12) cannot be used to generate $A_n(z)$. Thus it appears difficult to derive general solutions to the inverse problem in the singular case. In the following, we present some specialized results on zero distribution of symmetric polynomials (which belong to the singular class). Such polynomials occur in spectral estimation and system identification problems, and the results which we present are relevant there.

To be more specific, let us introduce the following symmetric polynomial:

$$\begin{aligned}A(z)&=g_0+\cdots+g_{m-1}z^{m-1}+g_mz^m\\&\quad+g_{m-1}z^{m+1}+\cdots+g_0z^{2m}\\&=G(z)+z^mG^*(z),\end{aligned}\tag{5.1}$$

where

$$G(z)=g_0+g_1z+\cdots+g_{m-1}z^{m-1}+\tfrac{1}{2}g_mz^m.$$

Polynomials of the form (5.1) appear in several signal processing applications. Two such applications are

- Estimation of the frequencies of multiple sinusoids from noise-corrupted data [21].
- Estimation of the spectral density of a moving-average or mixed autoregressive moving-average process (see, e.g., [6, 23, 24]).

In the first application, we wish to correct as necessary a given $G(z)$ polynomial such that the corresponding $A(z)$ polynomial has *all* its zeros on C_1. The angular positions of the zeros of $A(z)$ are then used as estimates of the sinusoidal frequencies [21]. In the second application, the problem is different: a given $G(z)$ polynomial is to be corrected such that the corresponding $A(z)$ has *no* zero on C_1, and thus corresponds to a valid spectral density function [6].

5.1. Ensuring all zeros lie on the unit circle

The following corollary of Lemma B provides some guidelines for solving the first inverse problem stated above.

COROLLARY 1. The polynomial $A(z)$ defined by (5.1) has all its zeros on C_1 if $n_G^-=0$.

PROOF. Note that (5.1) is in the form of (2.1) with $P(z)=A(z)$, $B(z)=G(z)$, $K(z)=1$ and $L(z)=z^m$. It follows from Lemma B, part (i) that for $n_G^-=0$ we have

$$\begin{aligned}n_A^0&\geq n_G^0+|n_G^++m-n_G^-|\\&=n_G^0+n_G^++m=2m=n_A,\end{aligned}\tag{5.2}$$

which concludes the proof. □

The condition of the above corollary is sufficient, but not necessary. To see this, consider the following example: For $m=2$ and $G(z)=1+4z+3z^2$ (with zeros at $z=-1, -\frac{1}{3}$) we get $A(z)=(z+1)^4$. Thus, a procedure based on the condition $n_G^-=0$ to ensure that $n_A^0=2m$ may be overly restrictive, and must be used with care. A less restrictive procedure is described in the following.

Consider the decomposition (4.11) of $A(z)$, instead of the decomposition (5.1). Define

$$B(z)=\left(\frac{1}{2m}\right)A'(z),\tag{5.3a}$$

and note from (4.11) that

$$A(z)=zB(z)+B^*(z).\tag{5.3b}$$

It follows from Theorem 4 that $n_A^0=2m$ if and only if $n_B^+=0$. The perturbed $A(z)$ polynomial frequently dealt with in applications is close to a polynomial with all zeros on C_1. We may therefore think of ensuring $n_A^0=2m$ by slightly correcting the (perturbed) polynomial $B(z)$. This way to proceed has a subtle aspect which should be clarified: by altering $B(z)$, this polynomial may no longer be proportional to the derivative of a symmetric polynomial (the polynomial $A(z)$ in (5.3b) corresponding to an arbitrary $B(z)$ is still symmetric, but the decomposition in (5.3b) is not unique); therefore Theorem 4 cannot be invoked to prove that the corrected $A(z)$ polynomial satisfies $n_A^0=2m$. However, by applying Lemma B, part (i) to (5.3b) for a general $B(z)$ and under the assumption $n_B^+=0$ we get

$$n_A^0\geq n_B^0+|n_B^+-n_B^--1|=n_B^0+n_B^-+1=2m.$$

This shows that a sufficient condition for $n_A^0=2m$ is $n_B^+=0$.

The condition $n_B^+=0$ is not necessary for $n_A^0=2m$. For example, $B(z)=z^3+\alpha z^2-\alpha$ (with $n_B^+>1$ for $\alpha>1$) gives $A(z)=z^4+1$. In spite of this comment, since only small perturbations on the coefficients of $B(z)$ are usually required (c.f. the discussion above) one can expect that the corrected $A(z)$ polynomial is close to the original polynomial.

Procedures for enforcing $n_B^+=0$ have been discussed earlier in Section 3.2. Using these procedures, a method for correcting $A(z)$ so that all its zeros lie on C_1 is as follows:

- From $A(z)$ construct $B(z)$ using (5.3a).
- Test whether $n_B^+=0$. If so, then $\bar{A}(z)=A(z)$. If not, use one of the stabilization methods outlined in Section 3.2 to form a corrected polynomial $\bar{B}(z)$ with $n_{\bar{B}}^+=0$.
- Reconstruct $\bar{A}(z)$ from $\bar{B}(z)$ using (5.3b).

The above procedure is computationally very simple, and could be included as an added step in many frequency estimation algorithms.

EXAMPLE 5.1. Consider the polynomial

$$A_4(z)=z^4-2.0836z^3+3.0674z^2 -2.0836z+1 \tag{5.4}$$

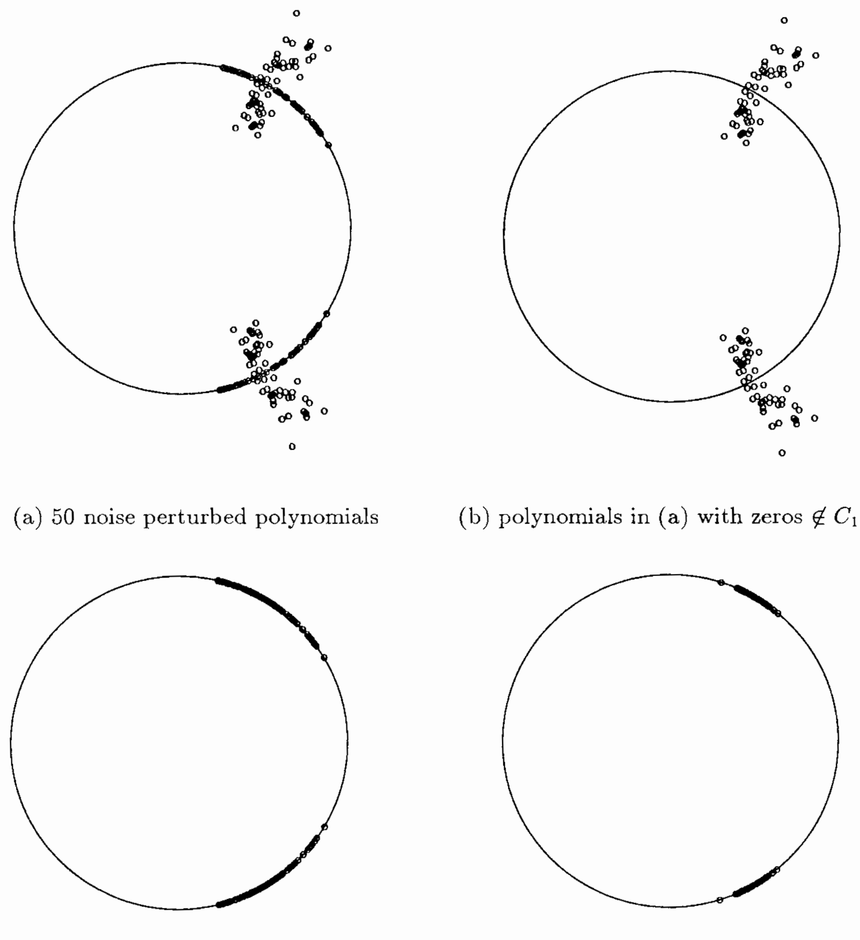
with zeros at $1\,e^{\pm j0.3\pi}$ and $1\,e^{\pm j0.35\pi}$. We generate 50 perturbed polynomials, each one constructed by adding zero mean white Gaussian noise with standard deviation $\sigma=0.1$ to the first three coefficients of $A_4(z)$, and setting the z^0 and z^1 coefficients equal to the perturbed z^4 and z^3 coefficients, respectively; thus, the perturbed polynomials retain coefficient symmetry. The zeros of these perturbed polynomials are shown in Fig. 3(a). Of the 50 sets of zeros, 26 have the property that not all zeros are on the unit circle; these 26 sets are shown in Fig. 3(b).

One application of the zero moving method discussed above concerns sinusoidal frequency estimation using prediction polynomial techniques [21]. In this application one is primarily interested in the angles of the zeros, as these correspond to the frequencies to be estimated. If the zeros are closely spaced, then errors in the prediction polynomial estimate can cause the zeros of the polynomial to be off the unit circle. It can be seen that in these 26 sets the two frequencies were not resolved (there is a pair of zeros with the same angle, but off the unit circle instead of two zeros with different angles on the unit circle).

The method described in this section for ensuring all zeros of a polynomial lie on the unit circle was applied to these 50 polynomials; for this example, Method C was used to stabilize $B(z)$. The zeros of the resulting polynomials are shown in Fig. 3(c). The zeros in Fig. 3(c) differ from Fig. 3(a) for only 26 of the 50 sets; the 26 changed polynomial zeros are shown in Fig. 3(d) and can be compared with Fig. 3(b).

In the sinusoidal frequency estimation problem, the angles of the polynomial zeros are of interest. Table 6 shows the means and standard deviations of the zero angles (frequencies) for the 26 uncorrected and corrected cases corresponding to Figs. 3(b) and 3(d) (the remaining 24 sets of zeros in Figs. 3(a) and 3(c) are the same for both the uncorrected and corrected polynomials, so their contributions to the error are equal.) It can be seen that the uncorrected polynomials which have zeros off the unit circle give a single frequency estimate at about the average of the two true frequencies. The correction procedure results in two resolved frequencies which are closer to the original frequencies than are the uncorrected frequencies.

Several points can be noted from this example. If the zeros of $A(z)$ are not close to each other, then small perturbations on the coefficients often do not move the zeros off the unit circle, and the stabilization procedure has no effect. Even when zeros are closely spaced, coefficient perturbations may not result in zeros off the unit circle (this occurred about half the time in the example shown). When the zeros are off the unit circle, the two frequencies not resolved generally result in zeros whose angles are close to the average of the two true pole angles. For these cases, the



(a) 50 noise perturbed polynomials (b) polynomials in (a) with zeros $\notin C_1$

(c) 50 corrected polynomials (d) corrected polynomials in (b)

Fig. 3. Perturbed and corrected polynomial zeros corresponding to Example 5.1.

Table 6

Means and standard deviations of polynomial zero angles for uncorrected and corrected polynomials in Example 5.1

	Frequency 1 mean (std)	Frequency 2 mean (std)
True	0.3 (0)	0.35 (0)
Uncorrected	0.3274 (0.0186)	
Corrected	0.3168 (0.0178)	0.3530 (0.0177)

stabilization method is successful at moving the zeros onto the unit circle to resolve the frequencies, and results in angle estimates which are closer to the angles of the unperturbed polynomial zeros.

5.2. Ensuring no zeros lie on the unit circle

Here we require $A(z)$ to be modified so that none of its zeros lie on the unit circle. Note from (5.1) that if $\bar{z} \neq \pm 1$ is a zero of $A(z)$, then $1/\bar{z}$ is also a (distinct) zero. If we know that no zero of the corrected $A(z)$ has modulus 1, then all the zeros must occur in reciprocal pairs. Therefore, the corrected $A(z)$ can be factored in the form $\lambda z^m H(z) H(z^{-1})$, where $\lambda = \pm 1$ and $H(z)$ is a polynomial of degree m. If $\lambda = +1$, then $z^{-m} A(z)$ represents a valid spectral density function.

We first present a result which is relevant to the application introduced above.

COROLLARY 2. A necessary condition for the polynomial $A(z)$ to have no zero on C_1 is $n_G^- = m$, where $G(z)$ is defined in (5.1).

PROOF. From Lemma B, part (i) with $n_G^- = m$ we have $n_A^0 \geqslant n_G^0 + |n_G^+ + m - n_G^-| \geqslant 2(m - n_G^-)$. Thus if $n_G^- < m$ then $A(z)$ has at least two zeros on C_1. □

The condition in the above corollary is necessary, but not sufficient, as we show below. From (5.1), $A(z)$ can be written as

$$A(z) = z^m D(z), \qquad D(z) \triangleq [G^*(z) + G^*(z^{-1})]. \tag{5.5}$$

Note that the spectral factorization $D(z) = H(z)H(z^{-1})$ exists when $D(z) \geqslant 0$ on C_1. From (5.5) we see that $D(z) \geqslant 0$ on C_1 if and only if $\mathrm{Re}[G^*(z)] \geqslant 0$ on C_1. Define the sets

$$D = \{g = [g_0, g_1, \ldots, g_m/2] \mid n_G^- = m\},$$

$$\tilde{D} = \{g \mid \mathrm{Re}[G^*(z)] \geqslant 0 \text{ on } C_1\}.$$

It follows that $\tilde{D} \subset D$. However, $\tilde{D} \neq D$ in general (as the following example shows), and thus the condition in Corollary 2 is not a sufficient condition.

EXAMPLE 5.2. Let $m = 2$ and $g_m = 2$. The corresponding sets D and $\tilde{D}$ are shown in Fig. 4 (see, e.g., [31] for a derivation). As expected $\tilde{D} \subset D$, but $\tilde{D} \neq D$.

Since the condition of Corollary 2 is not a sufficient one, the polynomial $G(z)$ which satisfies it may not be a solution to the spectral estimation problem mentioned earlier. However, it can be shown that $\tilde{D}$ is a closed, convex cone [6, 23]. Therefore, to obtain a solution in $\tilde{D}$, we may proceed in the following way.

- Correct $G(z)$ using the technique of Section 3.2 to ensure that $n_G^- = m$. Denote the corrected polynomial as $\bar{G}(z)$.
- Test whether the coefficient vector $\bar{g}$ of $\bar{G}(z)$ is in the set $\tilde{D}$. (To test whether $\bar{g} \in \tilde{D}$ one can use the decomposition (5.3) of $A(z)$ and apply Theorem 4 to $B(z)$. It follows from this theorem that $g \in \tilde{D}$ if and only if $n_B^+ = m$. Other tests of the condition $g \in \tilde{D}$ can be found in [14, 30].)
- If so, then stop.
- If not, then use a bisecting procedure on the line connecting $\bar{g}$ and the point $g^0 \triangleq [0, \ldots, 0, \bar{g}_m/2]$ to determine a point in $\tilde{D}$ which is of minimum distance from $\bar{g}$.

The convexity of $\tilde{D}$ is a nice property which makes it possible to use the bisecting procedure. We expect the point $\bar{g}$ provided by the first step will in general be close to $\tilde{D}$. Note also that the point g^0 is situated well inside $\tilde{D}$. Thus the procedure introduced above may be expected to produce satisfactory solutions in most cases.

It might be though that the condition $n_B^+ = m$ can be used to conceive a procedure for ensuring $n_A^0 = 0$, similar to the procedure described in Section 5.1. For such a procedure to be valid we need to show that the polynomial $A(z)$ obtained by (5.3b) from a general $B(z)$ with $n_B^+ = m$ (and $n_B^- = m - 1$), has no zero on C_1. But this is not generally true:

$$B(z) = z^3 + z - 0.625 = (z - 0.5)(z^2 + 0.5z + 1.25)$$

with $n_B^+ = 2$, $n_B^- = 1$ gives $A(z) = z^4 - 0.625z^3 + 2z^2 - 0.625z + 1 = (z^2 + 1)(z^2 - 0.625z + 1)$. Hence the condition ($n_B^+ = m$, $n_B^- = m - 1$) is not sufficient for $n_A^0 = 0$ when $B(z)$ in (5.3) is a general polynomial. However this condition is necessary: assuming $n_A^0 = 0$ and using Lemma B, part (i) we get

$$\begin{aligned} & 0 \geqslant n_B^0 + |n_B^+ - n_B^- - 1| \\ \Leftrightarrow\;& n_B^0 = 0, \quad n_B^+ - n_B^- = 1 \\ \Leftrightarrow\;& n_B^+ + n_B^- = 2m - 1, \quad n_B^+ - n_B^- = 1 \\ \Leftrightarrow\;& n_B^+ = m, \quad n_B^- = m - 1. \end{aligned}$$

Thus, the conditions $n_B^+ = m$ and $n_B^- = m - 1$ can be used to devise a 'necessary' procedure followed by a correction step, similar to the one described above in this section.

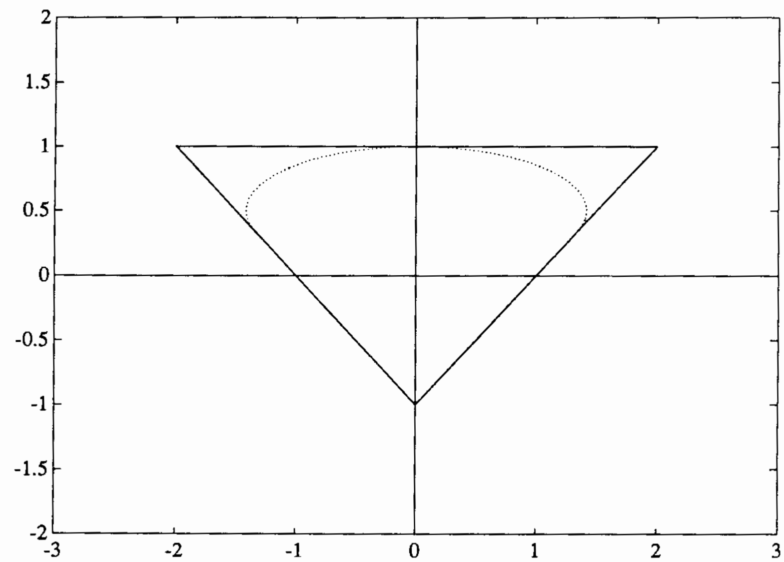


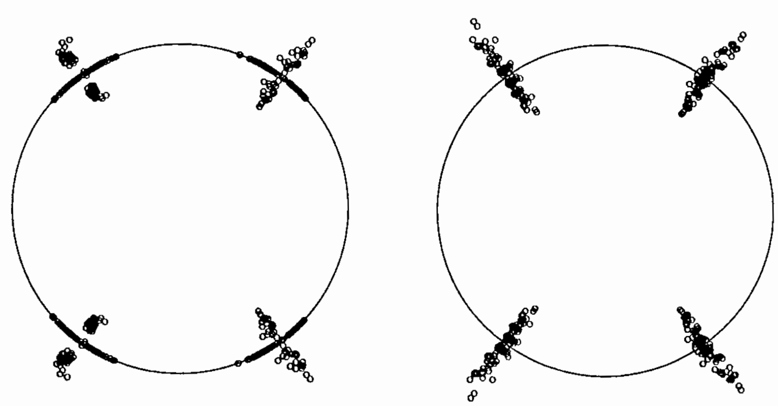
Fig. 4. The sets D (solid line) and $\tilde{D}$ (dotted line) for $m=2$.

EXAMPLE 5.3. Consider the polynomial

$$A_8(z) = (z^4 + 0.618z^2 + 1)^2, \tag{5.6}$$

with a pair of zeros at $1e^{\pm j0.3\pi}$ and $1e^{\pm j0.7\pi}$. We generate 50 perturbed polynomials, each one constructed by adding zero mean white Gaussian noise with standard deviation $\sigma = 0.1$ to the first five polynomial coefficients of $A_8(z)$, and setting the z^{8-k} coefficient equal to the z^k coefficient for $k = 0, 1, 2, 3$. In this way, the perturbed polynomials satisfy the coefficient symmetry property. The zeros of these polynomials are shown in Fig. 5(a).

The method described in this section for ensuring that no zeros of a polynomial lie on the unit circle was applied to these 50 polynomials; for this example, Method C was used to stabilize $G(z)$. The zeros of the corrected polynomials are shown in Fig. 5(b). Note that the angles of the corrected zeros



(a) perturbed polynomials (b) corrected polynomials

Fig. 5. Perturbed and corrected polynomial zeros corresponding to Example 5.3.

are much closer to the true zero angles than in the uncorrected case.

As mentioned earlier, this stabilization procedure has an application in spectral estimation, where an estimate of a moving average spectral density function is required to be non-negative definite, but its estimate may not be. In this case, $A(e^{j\omega})$ corresponds to the spectrum; this spectrum is non-negative definite if the zeros of $A(z)$ do not lie on the unit circle. Figure 6 shows the true spectrum and the 50 uncorrected and corrected spectral estimates corresponding to the zeros in Fig. 5. The uncorrected and corrected spectra exhibit similar variation among the experiments, but the corrected spectra are all nonnegative as is desired.

6. Summary

We have considered several aspects of the Schur–Cohn procedure, and of its application to the inverse problem. We first established two fundamental lemmas on the zero distribution of a polynomial of a certain form. These lemmas facilitated

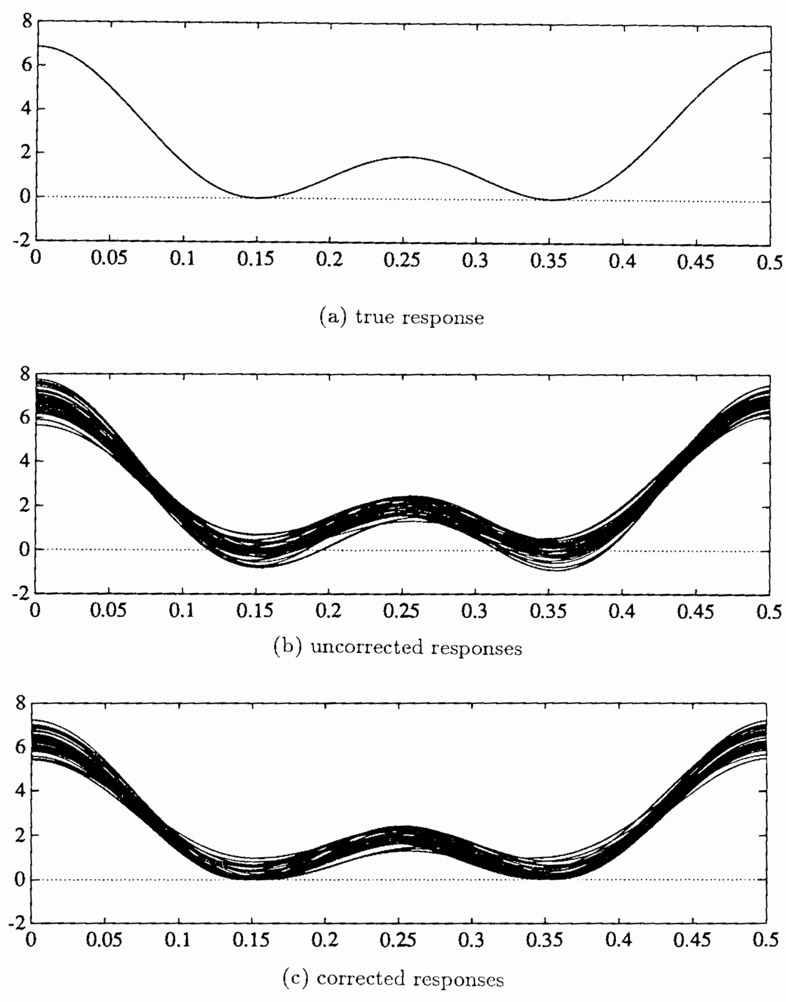


Fig. 6. Frequency response plots corresponding to $A(z)$ polynomials in Example 5.3. (a) Original polynomial. (b) Uncorrected polynomials. (c) Corrected polynomials.

simple proofs of the Schur–Cohn procedure in both the regular and singular cases, and led to new methods for handling one of the singular cases and for solving various inverse problems. We presented algorithms for three inverse problems, namely stabilizing an unstable polynomial in the regular case, correcting a symmetric polynomial so that all of its zeros lie on the unit circle, and correcting a symmetric polynomial so that none of its zeros lie on the unit circle. These algorithms are computationally efficient and easily programmed, and can be readily incorporated into relevant spectral estimation or system identification procedures.

Acknowledgment

This work was started and a first draft completed during our visit in 1984 to the Measurement and Control Group of the Technical University of Eindhoven, Eindhoven, The Netherlands. We are grateful to the members of the group, particularly to Professor Pieter Eykhoff for offering a most stimulating research environment.

Appendix A. Proof of Lemma A

First consider the equalities (2.4) and (2.7). Observe that the zeros of $B(z)$ on C_1 are also zeros of $B^*(z)$ (with the same multiplicity) and, therefore, zeros of $P(z)$ as well. Cancel these zeros from $P(z)$ and $B(z)$ and denote the resulting polynomials by $\bar{P}(z)$ and $\bar{B}(z)$. Then $\bar{B}(z)$ has no zeros on C_1. It follows from (2.1) that $\bar{P}(z)$ and $\bar{B}(z)$ satisfy

$$\bar{P}(z) = K(z)\bar{B}(z) - L(z)\bar{B}^*(z). \qquad \text{(A.1)}$$

To show that the equality $n_P^0 = n_B^0$ holds, let $\tilde{z}$ denote a possible zero of $\bar{P}(z)$ on C_1. Then, from (A.1),

$$K(\tilde{z})\bar{B}(\tilde{z}) = -L(\tilde{z})\bar{B}^*(\tilde{z}),$$

which implies $|K(\tilde{z})| = |L(\tilde{z})|$ since $|\bar{B}(z)| = \bar{B}^*(z)|$ on C_1. But this contradicts the assumptions in both (i) and (ii). Therefore no such zero $\tilde{z}$ can exist, so $n_P^0 = n_B^0$.

Next consider the equalities (2.2) and (2.5). These equalities follow directly from Rouché's Theorem applied to (A.1). Finally, (2.3) and (2.6) follow immediately from (2.2), (2.4) and (2.5), (2.7), respectively. □

Appendix B. Proof of Lemma B

Let ε denote a small number. Define

$$P_\varepsilon(z) = K(z)B(z) + (1+\varepsilon)L(z)B^*(z). \qquad \text{(B.1)}$$

Application of Lemma A to $P_\varepsilon(z)$ for $\varepsilon < 0$ gives

$$n_{P_\varepsilon}^- = n_B^- + n_K^-, \qquad \text{(B.2a)}$$

$$n_{P_\varepsilon}^+ = \max(n_K, n_L) + n_B^+ - n_K^-, \qquad \text{(B.2b)}$$

$$n_{P_\varepsilon}^0 = n_B^0. \qquad \text{(B.2c)}$$

For $\varepsilon > 0$, Lemma A gives

$$n_{P_\varepsilon}^- = n_B^+ + n_L^-, \qquad \text{(B.3a)}$$

$$n_{P_\varepsilon}^+ = \max(n_K, n_L) + n_B^- - n_L^-, \qquad \text{(B.3b)}$$

$$n_{P_\varepsilon}^0 = n_B^0. \qquad \text{(B.3c)}$$

Next note that

$$P_\varepsilon(z) = P(z) + \varepsilon L(z)B^*(z). \qquad \text{(B.4)}$$

For sufficiently small ε, the zeros of $P(z)$ which lie inside or outside C_1 remain in those regions. However, the zeros of $P(z)$ which lie on C_1 may leave C_1 and become outside or inside zeros for $P_\varepsilon(z)$. Thus, the number of zeros of P_ε which lie outside (inside) C_1 is greater than or equal to the number of outside (inside) zeros of $P(z)$. This observation, together with (B.2a)–(B.3c), imply that

$$\begin{aligned}
n_P^- &\leqslant \min(n_B^+ + n_L^-, n_B^- + n_K^-),\\
n_P^+ &\leqslant \max(n_K, n_L) + \min(n_B^+ - n_K^-, n_B^- - n_L^-),\\
n_P^0 &\geqslant n_B + \max(n_K, n_L)\\
&\quad -\min(n_B^+ + n_L^-, n_B^- + n_K^-)\\
&\quad -\max(n_K, n_L) - \min(n_B^+ - n_K^-, n_B^- - n_L^-)\\
&= n_B + n_K^- + n_L^- - 2\min(n_B^+ + n_L^-, n_B^- + n_K^-)\\
&= n_B + n_K^- + n_L^-\\
&\quad +2\max(-n_B^+ - n_L^-, -n_B^- - n_K^-)\\
&= n_B^0 + \max(n_B^- + n_K^- - n_L^- - n_B^+,\\
&\qquad -(n_B^- + n_K^- - n_L^- - n_B^+))\\
&= n_B^0 + |n_B^+ + n_L^- - n_B^- - n_K^-|,
\end{aligned}$$

which concludes the proof of part (i).

To prove part (ii), note that for sufficiently small ε, $P_\varepsilon(z)$ must have a zero $\bar{z}_i$ which is close to z_i. A Taylor series expansion of $P_\varepsilon(z)$ around z_i, evaluated at $\bar{z}_i$, gives

$$\begin{aligned}
0 = P_\varepsilon(\bar{z}_i) &\approx \underbrace{P(z_i)}_{0} + \varepsilon L(z_i)B^*(z_i)\\
&\quad + [P'(z_i) + \varepsilon\{L'(z_i)B^*(z_i)\\
&\quad + L(z_i)B^{*\prime}(z_i)\}](\bar{z}_i - z_i),
\end{aligned}$$

which implies that

$$\begin{aligned}
\bar{z}_i &\approx z_i - \frac{\varepsilon L(z_i)B^*(z_i)}{P'(z_i)}\\
&= z_i\left[1 - \varepsilon\frac{L(z_i)B^*(z_i)}{z_iP'(z_i)}\right]. \qquad \text{(B.5)}
\end{aligned}$$

A first-order approximation of $|\bar{z}_i|^2$ readily follows from (B.5):

$$\begin{aligned}
|\bar{z}_i|^2 &\approx 1 - 2\varepsilon\,\mathrm{Re}\left\{\frac{L(z_i)B^*(z_i)}{z_iP'(z_i)}\right\}\\
&= 1 + 2\varepsilon\,\mathrm{Re}\left\{\frac{K(z_i)B(z_i)}{z_iP'(z_i)}\right\}, \qquad \text{(B.6)}
\end{aligned}$$

where use has been made of the fact that

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{L(z_i)B^*(z_i)}{z_iP'(z_i)}\right\} + \mathrm{Re}\left\{\frac{K(z_i)B(z_i)}{z_iP'(z_i)}\right\}\\
&= \mathrm{Re}\left\{\frac{P(z_i)}{z_iP'(z_i)}\right\} = 0.
\end{aligned}$$

Thus, if (2.9) holds and if $\varepsilon > 0$ ($\varepsilon < 0$) is sufficiently small, then all the zeros $z_i \in C_1$ of $P(z)$ move inside (outside) C_1; in other words, under (2.9) $P(z)$ and $P_\varepsilon(z)$ have the same number of zeros outside (inside) C_1 if $\varepsilon > 0$ ($\varepsilon < 0$). Combining this property with (B.2a)–(B.3c) concludes the proof of part (iia). The proof of (iib) follows similarly. □

Appendix C. Proof of Theorem 2

The proof proceeds by induction. A straightforward application of Lemma A to (3.2) shows that:
If $|a_{0,k}/a_{k,k}| < 1$, then

$$n_{A_{k-1}}^- + 1 = n_{A_k}^-, \qquad \text{(C.1)}$$

$$n_{A_{k-1}}^+ = k - n_{A_k}^0 - n_{A_k}^- = n_{A_k}^+. \qquad \text{(C.2)}$$

If $|a_{0,k}/a_{k,k}| > 1$, then

$$n_{A_{k-1}}^- + 1 = n_{A_k}^+, \qquad \text{(C.3)}$$

$$n_{A_{k-1}}^+ = k - n_{A_k}^0 - n_{A_k}^+ = n_{A_k}^-. \qquad \text{(C.4)}$$

Also, we have from (3.4)

$$\begin{aligned}
a_{k-1,k-1}a_{k,k} > 0 \quad &\Leftrightarrow \quad |a_{0,k}/a_{k,k}| < 1,\\
a_{k-1,k-1}a_{k,k} < 0 \quad &\Leftrightarrow \quad |a_{0,k}/a_{k,k}| > 1.
\end{aligned}$$

Using this observation, (C.1)–(C.4) can be written as follows:
If $a_{k-1,k-1}a_{k,k} > 0$,

$$n_{A_k}^- = n_{A_{k-1}}^- + s_{k-1,k}^-, \qquad \text{(C.5a)}$$

$$n_{A_k}^+ = n_{A_{k-1}}^+ + s_{k-1,k}^+. \qquad \text{(C.5b)}$$

If $a_{k-1,k-1}a_{k,k} < 0$,

$$n_{A_k}^- = n_{A_{k-1}}^+ + s_{k-1,k}^-, \qquad \text{(C.6a)}$$

$$n_{A_k}^+ = n_{A_{k-1}}^- + s_{k-1,k}^+. \qquad \text{(C.6b)}$$

This establishes (3.7a), (3.7b) and (3.8a), (3.8b) for $n=p+1$. Next assume that these equations hold for some $n=l>p$. Using this assumption and (C.5a)–(C.6b), we can write

(a) If $a_{p,p}a_{l+1,l+1}>0$, then

(a1) if $a_{l,l}a_{l+1,l+1}>0 \Leftrightarrow a_{p,p}a_{l,l}>0$,

$$n^-_{A_{l+1}}=n^-_{A_l}+1=n^-_{A_p}+s^-_{p,l}+1$$
$$=n^-_{A_p}+s^-_{p,l+1},$$
$$n^+_{A_{l+1}}=n^+_{A_l}=n^+_{A_p}+s^+_{p,l}=n^+_{A_p}+s^+_{p,l+1}.$$

(a2) if $a_{l,l}a_{l+1,l+1}<0 \Leftrightarrow a_{p,p}a_{l,l}<0$,

$$n^-_{A_{l+1}}=n^+_{A_l}=n^-_{A_p}+s^+_{p,l}=n^-_{A_p}+s^-_{p,l+1},$$
$$n^+_{A_{l+1}}=n^-_{A_l}+1=n^+_{A_p}+s^-_{p,l}+1$$
$$=n^+_{A_p}+s^+_{p,l+1}.$$

(b) If $a_{p,p}a_{l+1,l+1}<0$, then

(b1) if $a_{l,l}a_{l+1,l+1}>0 \Leftrightarrow a_{p,p}a_{l,l}<0$,

$$n^-_{A_{l+1}}=n^-_{A_l}+1=n^+_{A_p}+s^-_{p,l}+1$$
$$=n^+_{A_p}+s^-_{p,l+1},$$
$$n^+_{A_{l+1}}=n^+_{A_l}=n^-_{A_p}+s^+_{p,l}=n^-_{A_p}+s^+_{p,l+1}.$$

(b2) if $a_{l,l}a_{l+1,l+1}<0 \Leftrightarrow a_{p,p}a_{l,l}>0$,

$$n^-_{A_{l+1}}=n^+_{A_l}=n^+_{A_p}+s^+_{p,l}=n^+_{A_p}+s^-_{p,l+1},$$
$$n^+_{A_{l+1}}=n^-_{A_l}+1=n^-_{A_p}+s^-_{p,l}+1$$
$$=n^-_{A_p}+s^+_{p,l+1}.$$

Thus, (3.7a), (3.7b), (3.8a) and (3.8b) hold for the index $n=l+1$, which completes the proof by induction.

Finally, (3.7c) and (3.8c) follow directly from (3.7a)–(3.7b) and (3.8a)–(3.8b), respectively. □

Appendix D. Proof of Theorem 3

First note that $z^{-1}[A_p(z)-\beta A_p^*(z)]$ is the polynomial given by (4.3), where $a_{m,p-1}=a_{m+1}-\beta a_{p-m-1}\neq 0$ (c.f. (4.2)). Thus, $\bar{A}_p(z)$ is a polynomial of degree p, with

$$\bar{\phi}_p=\frac{a_0+\rho[a_{m+1}-\beta a_{p-m-1}]}{a_0\beta}$$
$$=\beta\left[1+\frac{\rho}{a_0}(a_{m+1}-\beta a_{p-m-1})\right].$$

It is clear from the above equation that if $\rho\neq -2a_0/[a_{m+1}-\beta a_{p-m-1}]$, then $|\bar{\phi}|_p\neq 1$. Next observe that we have

$$z^{m+1}\bar{A}_p(z)=(\rho+z^{m+1})A_p(z)-\rho\beta A_p^*(z) \tag{D.1}$$

and that for $z=e^{j\omega}$, $\omega\in[-\pi,\pi]$,

$$|\rho+e^{j\omega(m+1)}|^2=1+2\rho\cos(m+1)\omega+\rho^2>\rho^2$$
$$=|-\beta\rho|^2 \quad \text{for } |\rho|<\tfrac{1}{2}.$$

Applying Lemma A to (D.1) with $K(z)=\rho+z^{m+1}$ and $L(z)=-\beta\rho$, we get

$$(m+1)+n^-_{\bar{A}_p}=n^-_{A_p}+(m+1) \Rightarrow n^-_{\bar{A}_p}=n^-_{A_p},$$
$$n^+_{\bar{A}_p}=(m+1+p)-n^0_{A_p}-n^-_{A_p}-(m+1)=n^+_{A_p},$$
$$n^0_{\bar{A}_p}=n^0_{A_p},$$

which completes the proof. □

Appendix E. Proof of Lemma 2

First observe that (4.11) is in the form of (2.1) with $P(z)=pA_p(z)$, $K(z)=z$, $L(z)=\beta$ and $B(z)=A_p'(z)$. With these definitions,

$$\operatorname{Re}\left\{\frac{K(z)B(z)}{zP'(z)}\right\}=\frac{1}{p}>0$$

for all $z\neq 0$ which are not zeros of $P'(z)$. Also, $P'(z)=pB(z)$, so if $z_i\in C_1$ and z_i is not a zero of $B(z)$, then $P'(z_i)\neq 0$. Thus, by Lemma B, part (ii) it follows that

$$n^+_{A_p}=1+n^+_{A_p'}-1=n^+_{A_p'},$$

which concludes the proof. □

Appendix F. Proof of Theorem 4

By Lemma 2, $n^+_{A_p} = n^+_{B_{p-1}}$. Next note that the zeros of a polynomial of the form of (4.9) occur in reciprocal pairs. In other words, if $\bar{z}$ ($\bar{z} \neq 0$ and ± 1) is a zero of $A_n(z)$, then $1/\bar{z}$ is also a zero. Thus

$$n^-_{A_p} = n^+_{A_p} = n^+_{B_{p-1}}, \qquad n^0_{A_p} = p - 2n^+_{B_{p-1}}$$

and the assertion of the theorem follows immediately from Theorem 2. □

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