ON THE GEOMETRY OF ISOTROPIC WIDEBAND ARRAYS

Ülkü Baysal, Randolph L. Moses

Department of Electrical Engineering
The Ohio State University, Columbus, OH 43210 USA
{baysalu, randy}@ee.eng.ohio-state.edu

ABSTRACT

We consider the role of array geometry on the Direction of Arrival (DOA) estimating performance of the array where the impinging signal is wideband. We define the criteria under which the planar and volume arrays are isotropic, and derive the necessary and sufficient conditions on the location of array elements so that the array has isotropic performance. We also present several designs of isotropic planar and volume arrays and give example geometries.

1. INTRODUCTION

The number and location of the elements in an array strongly affects the DOA estimation performance of the array system. There is a considerable amount of work done on the design of the optimal array (optimal in terms of cost, space, error variance or resolution limits etc.). Most of the emphasis is devoted to linear arrays (or combination of linear arrays) as they are simple to analyze, provide the maximum aperture when the number of elements fixed and optimal DOA estimation algorithms are available for such arrays. One of the main problems with the linear arrays is the nonuniformity of the performance: the DOA estimation performance degrades considerably near endfire. In this paper, we concentrate on arrays that have uniform performance over the whole field of view.

Several different performance and design criteria have been introduced to be used in obtaining optimal arrays. In [1], the authors introduce a measure of similarity between array response vectors and show that the similarity measure can be tightly bounded below. The array with the highest bound is optimum in the sense that it has the best ambiguity resolution. In [2], a sensor locator polynomial is introduced for array design. A polynomial is constructed using prespecified performance levels, such as detection-resolution thresholds and Cramér-Rao Bounds (CRBs) on error variance, and its roots are the sensor locations of the desired linear or planar array. In [3], differential geometry is used to characterize the array manifold and an array design framework based on these parameters is proposed. Studies regarding to arrays that have uniform performance with respect to a certain criterion can also be found in the literature. In [4], the asymptotic mean square angular error is used to define an isotropic array. The authors derive the angle CRB for a single far-field source and then derive conditions on the sensor locations to ensure the azimuth and elevation errors are uncoupled from each other in the bound.

In this paper, we study planar and volume array geometries that have isotropic DOA estimation performance. For planar arrays, the arrays are isotropic in the sense that the CRB on the DOA estimation of a single source is uniform for all source arrival angles from 0 to 2π. For volume arrays we use the bound on the Mean Square Angular Error (MSAE) as the criterion. The MSAE is a scalar measure of the error between true and estimated unit bearing vectors pointing towards the source, and its bound is computed from the CRB. The array is said to be isotropic if the bound on the MSAE is constant for all azimuth and elevation angles in $[0\ 2\pi]\times[-\frac{\pi}{2}\ \frac{\pi}{2}]$. Since the CRB and bound on the MSAE are independent of any particular estimator and ML estimators asymptotically achieve these bounds, they are useful criteria for array design.

2. SYSTEM MODEL

We assume an array of N identical sensors located in space at locations r_i for $i \in [1, N]$. Following [5], we adopt a system model describing a source impinging on the array. A single far-field source $s(t)$, which is in general wideband, impinges on the array from direction $\theta = [\phi, \psi]$, ϕ denoting the azimuth angle measured counterclockwise from the x-axis on the x-y plane, and ψ denoting the elevation angle measured counterclockwise from the x-axis on the x-z plane. The noise at the sensors is independent, zero mean Gaussian noise, and independent of the source signal. The observation time T is partitioned into K intervals of length T_d and a J-point discrete Fourier transform is applied to each interval. Assuming T_d is long enough, we say the dis-

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crete Fourier coefficients are uncorrelated. Then,

$$x_k(\omega_j) = A_\theta(\omega_j)s_k(\omega_j) + n_k(\omega_j), \quad \begin{array}{l} j = 1, ..., J \\ k = 1, ..., K \end{array} \quad (1)$$

where $x_k(\omega_j)$, $n_k(\omega_j)$ are $N \times 1$ vectors and $s_k(\omega_j)$ is a scalar. The elements of $x_k(\omega_j)$, $n_k(\omega_j)$ and $s_k(\omega_j)$ are the discrete Fourier coefficients of the sum of the sensor outputs, the noise and the signal source at the discrete frequency ω_j, respectively. $A_\theta(\omega_j)$ is given by

$$A_\theta(\omega_j) = [e^{j\omega_j d_1(\theta)}, e^{j\omega_j d_2(\theta)}, \ldots, e^{j\omega_j d_N(\theta)}]^T \quad (2)$$

where $d_k(\theta) = \frac{u^T \cdot r_k}{c}$ is the propagation delay associated with the k^{th} sensor, c is the speed of propagation and u denotes the unit vector pointing towards the signal source (bearing vector).

3. PLANAR ARRAYS

In this section we consider the special case of a planar array with elements at locations $r_i = [r_{x_i}, r_{y_i}]^T$. The array is used to estimate the DOA of a wideband signal $s(t)$ which is coplanar with the array or equivalently the bearing vector $u = [\cos(\phi), \sin(\phi)]^T$. Here the signal arrives at an angle ϕ.

For the system model described in Section II and under the planar array-coplanar signal assumption, the CRB for the source DOA estimate is given by eqn (2.24) in [5]. For the model given by (1) and (2), we can write the CRB explicitly as

$$CRB(\phi) = G(B,\phi)^{-1}\left(\frac{2KN}{c^2}\sum_{j=1}^{J}\frac{\omega_j^2}{n_j}p_j\left(\frac{p_jN}{p_jN+n_j}\right)\right)^{-1} \quad (3)$$

$$G(B,\phi) = \frac{\partial u}{\partial \phi}^T B \frac{\partial u}{\partial \phi} = [-\sin(\phi), \cos(\phi)]\, B \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix} \quad (4)$$

$$B = \frac{1}{N}\sum_{i=1}^{N}(r_i - r_c)(r_i - r_c)^T \quad (5)$$

where r_c is the centroid of the array, i.e.,

$$r_c = \frac{1}{N}\sum_{i=1}^{N} r_i \quad (6)$$

p_j is the signal power and n_j is the noise power at frequency interval j.

We see that the CRB is a product of a term $G(B, \phi)$ that depends only on the source DOA and the array geometry and a term that depends on source and noise powers as a function of frequency. This is an important property, because the impact of the array geometry on the CRB is the same regardless of whether the source spectrum in narrowband or broadband, and regardless of the source signal and noise spectral densities. Thus, the results that follow apply to a broad class of array signal processing scenarios. Moreover, the CRB depends on the array geometry only through the matrix B, which is the 2×2 "covariance" of the array points. Thus, any two geometries that have the same covariance matrix B will have identical CRB performance.

3.1. Isotropic Planar Arrays

Without loss of generality we can assume that the array is centered at the origin, or equivalently, $r_c = [0, 0]^T$. Under this assumption, B in (5) simplifies to

$$B = \frac{1}{N}\sum_{i=1}^{N} r_i r_i^T \quad (7)$$

We are interested in planar geometries whose single-source CRB is independent of signal arrival angle ϕ. We refer to such arrays as *isotropic arrays*. The following result characterizes the set of all isotropic arrays:

Theorem 1 :

(a) An N element planar array which is centered at the origin ($r_c = [0, 0]^T$) and represented by the 2-by-2 matrix $B = \frac{1}{N}\sum_{i=1}^{N} r_i r_i^T$, where $r_i = [r_{x_i}, r_{y_i}]^T$ is the location of the ith sensor, is isotropic if and only if

$$B = kI_2 \quad (8)$$

where k is any positive constant and I_2 denotes the 2×2 identity matrix.

(b) If the array is isotropic, then

$$G(B,\phi) = k = \frac{1}{2}\sum_{i=1}^{N}\|r_i\|^2 \quad (9)$$

Proof: See [8].

3.2. Planar Array Design Examples

In this section we present four design methods for generating planar arrays and give examples for each method.

Circularly Symmetric Geometries :

The immediate solution where both equations (7) and (8) are satisfied is circularly symmetric geometries or any superposition of circularly symmetric geometries, where we define a circularly symmetric geometry as one in which $N \geq 3$ sensors are equally-spaced on a circle that has a radius greater than 0. A single sensor located at the origin is also in this class. An example geometry is shown Figure 1 (a)

Rotated Geometries :

Consider an $\frac{N}{2}$ element subarray, where N is even, with arbitrary sensor locations. Define the origin as the centroid of these points, and define a second $\frac{N}{2}$ element subarray by rotating the first elements by either 90° or −90°. Superposition of the original and rotated subarrays forms an N-element isotropic planar array. For details and proofs see [8]. An example array generated with this method is shown in Figure 1 (b).

Completion of Arbitrary (N-2)-element Arrays :

Suppose locations of the sensors of an arbitrary $(N-2)$-element array are given. By properly locating the remaining two sensors along one of the eigenvectors of the B matrix that represents the $(N-2)$-element array, we can form an isotropic planar array. For details and proofs see [8]. An example geometry formed with this method is shown in Figure 1 (c).

X-shaped Isotropic Arrays

We can combine two or more X-shaped geometries so that the resulting 4n-element array (n=2,3..) is isotropic. An X-shaped geometry is a set of four sensors with radius $\|r\|$ and angles $\pm\alpha$, $\pm\pi-\alpha$. A special case is the superposition of X-shaped geometries are those whose elements lie along two lines. For example, the eight-element isotropic array with elements having x-values of ± 1 is shown in Figure 1 (d). See [8] for a more detailed explanation.

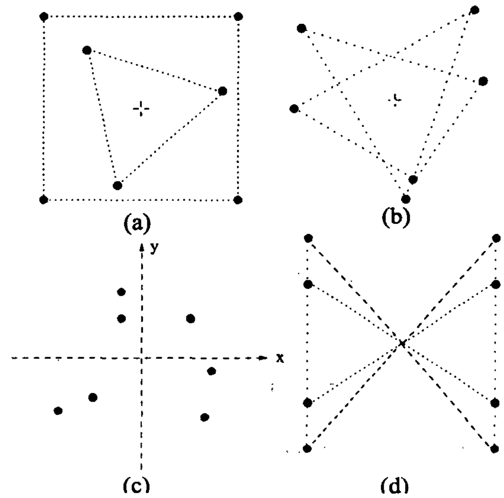

Fig. 1. (a) An isotropic planar array obtained as the superposition of a 4 element (black spheres) and 3 element (gray spheres) circularly symmetric array. (b) 6 element isotropic array (black and gray spheres) formed from rotating a randomly generated 3-element subarray (black spheres) by 90°. (c) A 7-element isotropic array formed from an arbitrary 5-element subarray (black spheres) by adding 2 elements (gray spheres). (d) An 8-element isotropic array formed by combining two X-shaped geometries.

4. THREE DIMENSIONAL ARRAYS

In this section, we consider an array that has elements located in R^3 and is used to estimate the DOA of a single wideband far-field source or equivalently the bearing vector $u = [\cos(\phi)\cos(\psi), \sin(\phi)\cos(\psi), \sin(\psi)]^T$. The source direction is parameterized by $\theta = [\phi, \psi]^T$, where $\phi \in [0, 2\pi)$ and $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denote, respectively, the azimuth and the elevation of the source. The single source CRB for this scenario is a 2-by-2 matrix ,

$$CRB(\theta) = G(B,\theta)^{-1}\left(\frac{2KN}{c^2}\sum_{j=1}^{J}\frac{\omega_j^2}{n_j}p_j\left(\frac{p_jN}{p_jN+n_j}\right)\right)^{-1} \tag{10}$$

$$G(B,\theta) = J_\theta(u)^T B J_\theta(u) \tag{11}$$

where $J_\theta(u)$ is the 3-by-2 Jacobian matrix for u and is equal to

$$J_\theta = \begin{bmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \end{bmatrix} = \begin{bmatrix} -\sin(\phi)\cos(\psi) & -\cos(\phi)\sin(\psi) \\ \cos(\phi)\cos(\psi) & -\sin(\phi)\cos(\psi) \\ 0 & \cos(\psi) \end{bmatrix} \tag{12}$$

Once again, we see that the CRB is the product of a term that depends only on the array geometry of source angle, and a term that depends only on the signal and noise parameters. We also see that the CRB depends on array geometry only through the 3×3 array covariance matrix B.

4.1. Performance Criterion

Estimating the azimuth and elevation corresponding to the DOA of the signal is equivalent to estimating the vector u. The Mean-Square Angular Error (MSAE) is introduced in [6] as a scalar measure of estimator performance in estimating a geometrical vector. A derivation for the lower bound of the MSAE and a detailed discussion of the conditions for the applicability and tightness of the bound can be found in [7]. The lower bound of the MSAE provides a performance criterion for a set of estimators that satisfies certain mild conditions, similar to those needed for the CRB. When spherical coordinates are used, MSAE is bounded below by $MSAE_B$ as

$$MSAE \geq MSAE_B = \cos^2(\psi)CRB(\phi) + CRB(\psi) \tag{13}$$

For a geometrical interpretation of the MSAE_B see [4].

4.2. Isotropic Three Dimensional Arrays

We adopt MSAE_B as a performance criterion, and define a three dimensional array to be *isotropic* if the associated MSAE_B is constant for all $[\phi, \psi]^T \in [0, 2\pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. The following theorem defines the set of all isotropic three dimensional arrays:

Theorem 2 :

(a) An N element array which is centered at the origin ($r_c = [0, 0, 0]^T$) and represented by the 3-by-3 matrix $B = \frac{1}{N}\sum_{i=1}^{N} r_i r_i^T$, where $r_i = [r_{x_i}, r_{y_i}, r_{z_i}]^T$ is the location of the i^{th} sensor, is isotropic if and only if

$$B = kI_3 \tag{14}$$

where k is any positive constant and I_3 is the 3-by-3 identity matrix.

(b) If (14) holds

$$G(B, \phi) = kI_2 \tag{15}$$

Proof: See [8].

We remark that (14) is also necessary and sufficient for the 2×2 CRB matrix to be independent of source arrival angle. In this case, the CRB in (10) is diagonal, and if it is scaled to remove the latitudinal scaling of azimuth, the CRB in the azimuthal and elevation directions are equal. That is, an uncertainty ellipse in spherical angle is a circle whose radius is independent of source arrival angle.

In [4], the authors give sufficient conditions on the array geometry so that MSAE_B is independent of the source signal DOA. The above theorem extends their results by proving that these conditions are also necessary.

4.3. Three Dimensional Isotropic Array Example

Regular Polyhedron :

It can be shown that arrays formed by placing the sensor elements at vertices of any regular polyhedra, or a superposition of such arrays, result in three dimensional isotropic arrays. We conjecture that the result also holds for the 13 semiregular polyhedra.

Completion of Arbitrary (N-3)-element Arrays :

Three elements can be added to an arbitrary $(N-3)$-element subarray to make the resulting N-element array isotropic. By properly locating the remaining three sensors in the plane spanned by the two eigenvectors of the B matrix of the $(N-3)$-element subarray, we can form an N-element isotropic array. For details and proofs see [8].

5. CONCLUSION

In this paper, we studied planar and three dimensional arrays that have isotropic performance. For planar arrays, we adopted the single source wideband Cramér-Rao bound as the performance criterion and derived the necessary and sufficient conditions on the location of sensor elements so that the CRB is constant for all arrival angles. These conditions are valid regardless of the source's frequency spectrum. We presented four methods to design isotropic planar arrays.

For three dimensional arrays, we chose the asymptotic Mean Square Angular Error as a measure for array isotropy. We derived necessary and sufficient conditions on the array geometry that ensure that the MSAE_B is independent of source azimuth and elevation arrival angle. When these conditions are satisfied, the azimuth and elevation are uncoupled in the CRB, and the CRB is independent of source signal arrival angle.

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