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Optimal Array Geometries for Wideband DOA Estimation

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Abstract

We consider the role of array geometry on the Direction of Arrival (DOA) estimating performance of the array where the impinging signal is wideband. We concentrate on arrays that have isotropic performance. An isotropic array is one whose Cramér Rao bound (CRB) on the DOA of a single source is uniform for all angles. We derive the necessary and sufficient conditions on the location of array elements so that the array is isotropic. Both planar arrays and three dimensional arrays are considered. We also present several designs of isotropic planar and volume arrays and give example geometries.

1. Introduction

The number and location of the elements in an array strongly affects the Direction-of-Arrival (DOA) estimation performance of the array system. There is a considerable amount of work done on the design of the optimal array (optimal in terms of cost, space, error variance or resolution limits etc.). Most of the emphasis is devoted to linear arrays (or combination of linear arrays) as they are simple to analyze, provide the maximum aperture when the number of elements fixed and optimal DOA estimation algorithms are available for such arrays [1]–[5]. One of the main problems with the linear arrays is the nonuniformity of the performance: the DOA estimation performance degrades considerably near endfire. In this paper, we concentrate on arrays that have uniform performance over the whole field of view.

Several different performance and design criteria have been introduced to be used in obtaining optimal arrays. Performance comparisons of some common array geometries are presented in [6],[7]. In [8], the authors introduce a measure of similarity between array response vectors and show that the similarity measure can be tightly bounded below. The array with the highest bound is optimum in the sense that it has the best ambiguity resolution. In [9], a sensor locator polynomial is introduced for array design. A polynomial is constructed using prespecified performance levels, such as detection-resolution thresholds and Cramér-Rao Bounds (CRBs) on error variance, and its roots are the sensor locations of the desired linear or planar array. In [10], differential geometry is used to characterize the array manifold and an array design framework based on these parameters is proposed.

Studies regarding to arrays that have uniform performance with respect to a certain criterion can also be found in the literature. In [11], the asymptotic mean square angular error is used to define an isotropic array. The authors derive the angle CRB for a single far-field source and then derive conditions on the sensor locations to ensure the azimuth and elevation errors are uncoupled from each other in the bound. In [15] and [18] the authors consider conditions on the array geometry for which the single source azimuth and elevation CRBs are uncoupled, and in [15] a condition which gives isotropic array performance is derived. In [12], some popular array geometries are compared using the ratio of the number of sensor elements to the usable aperture as a performance measure; the Y-shaped array shown to achieve approximately uniform angle-of-arrival estimation performance with this criterion.

In this paper, we study planar and volume array geometries that have isotropic DOA estimation performance. For planar arrays, the arrays are isotropic in the sense that the CRB on the DOA estimation of a single source is uniform for all source arrival angles from 0 to 2π. For volume arrays we use the bound on the Mean Square Angular Error (MSAE) as the criterion. The MSAE is a scalar measure of the error between true and estimated unit bearing vectors pointing towards the source, and its bound is computed from the CRB. The array is said to be isotropic if the bound on the MSAE is constant for all azimuth and elevation angles in [0 2π]×[-$\frac{\pi}{2}$ $\frac{\pi}{2}$]. Since the CRB and bound on the MSAE are independent of any particular estimator and ML estimators asymptotically achieve these bounds, they are useful criteria for array design. Our results apply to both narrowband and wideband signals.

The organization of the paper is as follows. In Section II, we describe the system model and state our assumptions. Section III discusses the planar array scenario. First, we define the performance criterion and isotropy condition for planar arrays, and later we state the necessary and sufficient conditions on the array geometry for isotropy. We also propose different array design methods. In Section IV, we study the three dimensional arrays. We introduce the isotropy condition, and give the necessary and sufficient conditions on the three dimensional array for isotropic performance. Section V concludes the paper.

2. System Model

We assume an array of N identical sensors located in the space at locations r_i for $i \in [1, N]$. We will consider both planar arrays in which $r_i = [r_{x_i}, r_{y_i}]^T$ and volume arrays in which $r_i = [r_{x_i}, r_{y_i}, r_{z_i}]^T$.

Following [14], we adopt a system model describing a source impinging on the array. A single far-field source $s(t)$, which is in general wideband, impinges on the array from direction $\theta = [\phi, \psi]$, ϕ denoting the azimuth angle measured counterclockwise from the x-axis on the x-y plane, and ψ denoting the elevation angle measured counterclockwise from the x-axis on the x-z plane. The noise at the sensors are independent, zero mean Gaussian noise, and independent of the source signal. The observation time T is partitioned into K intervals of length T_d and a J-point discrete Fourier transform is applied to each interval. Assuming T_d is long enough, we say the discrete Fourier coefficients are uncorrelated. Then,

$$x_k(\omega_j) = A_\theta(\omega_j) s_k(\omega_j) + n_k(\omega_j), \quad j = 1, ..., J \quad k = 1, ..., K \tag{1}$$

where $x_k(\omega_j)$, $n_k(\omega_j)$ are $N \times 1$ vectors and $s_k(\omega_j)$ is a scalar. The elements of $x_k(\omega_j)$, $n_k(\omega_j)$ and $s_k(\omega_j)$ are the discrete Fourier coefficients of the sum of the sensor outputs, the noise and the signal source at the discrete frequency ω_j, respectively. $A_\theta(\omega_j)$ is given by

$$A_\theta(\omega_j) = [e^{j\omega_j d_1(\theta)}, e^{j\omega_j d_2(\theta)}, \ldots, e^{j\omega_j d_N(\theta)}]^T \tag{2}$$

where $d_k(\theta) = \frac{u^T \cdot r_k}{c}$ is the propagation delay associated with the k^{th} sensor, c is the speed of propagation and u denotes the unit bearing vector pointing towards the signal source. For a planar signal arriving from angle ϕ

$$u = [\cos(\phi),\ \sin(\phi)]^T \tag{3}$$

whereas for a three-dimensional case where a signal arrives from azimuth angle ϕ and elevation angle ψ,

$$u = [\cos(\phi)\cos(\psi),\ \sin(\phi)\cos(\psi),\ \sin(\psi)]^T \tag{4}$$

3. Planar Arrays

In this section we consider the special case of a planar array with elements at locations $r_i = [r_{x_i}, r_{y_i}]^T$. The array is used to estimate the DOA of a wideband signal $s(t)$ which is coplanar with the array. Here the signal arrives at an angle ϕ.

3.1 Single-Source CRB

For the system model described in Section II and under the planar array-coplanar signal assumption, the CRB for the source DOA estimate is given by [14]:

$$\begin{aligned} CRB(\phi) &= \frac{1}{2K}\Bigg[\sum_{j=1}^{J}\frac{1}{n_j}\Re\Bigg\{\Big(\dot{A}_\phi^H(\omega_j)P_{A_j}^\perp \dot{A}_\phi(\omega_j)\Big) \\ &\quad \odot\Bigg(R_s(\omega_j) - \Big(R_s^{-1}(\omega_j) + \frac{1}{n_j}A_\phi^H(\omega_j)\times A_\phi(\omega_j)\Big)^{-1}\Bigg)^T\Bigg\}\Bigg]^{-1} \end{aligned} \tag{5}$$

Here, $\dot{A}_\phi = \frac{\partial A}{\partial \phi}$, $P_A^\perp = I - A(A^H A)^{-1}A^H$ is the projection matrix onto the subspace orthogonal to the column space of A, $R_s(\omega_j)$ is the cross-spectral density matrix of the impinging signals at frequency ω_j and $\odot$ denotes the Hadamard product.

For the model given by (1) and (2), we can write the CRB as

$$CRB(\phi) = G(B,\phi)^{-1}\left(\frac{2KN}{c^2}\sum_{j=1}^{J}\frac{\omega_j^2}{n_j}p_j\left(1 - \frac{n_j}{p_j N + n_j}\right)\right)^{-1} \tag{6}$$

$$G(B,\phi) = \frac{\partial u}{\partial \phi}^T B \frac{\partial u}{\partial \phi} = \left[-\sin(\phi), \cos(\phi)\right] B \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix} \tag{7}$$

$$B = \frac{1}{N}\sum_{i=1}^{N}(r_i - r_c)(r_i - r_c)^T \tag{8}$$

where r_c is the centroid of the array, i.e.,

$$r_c = \frac{1}{N}\sum_{i=1}^{N} r_i \tag{9}$$

p_j is the signal power and n_j is the noise power at frequency interval j.

We see that the CRB is a product of a term $G(B, \phi)$ that depends only on the source DOA and the array geometry and a term that depends on source and noise powers as a function of frequency. This is an important property, because the impact of the array geometry on the CRB is the same regardless of whether the source spectrum in narrowband or broadband, and regardless of the source signal and noise spectral densities. Thus, the results that follow apply to a broad class of array signal processing scenarios. Moreover, the CRB depends on the array geometry only through the matrix B, which is the 2×2 "covariance" of the array points. Thus, any two geometries that have the same covariance matrix B will have identical single-source CRB performance.

We note that the array performance criterion we have chosen does not take into account potential array ambiguities that arise when the array manifold from two different DOAs are close to one another (see [8] for a discussion of this topic). For wideband arrays, ambiguities are not much of a problem because the frequency diversity eliminates most or all DOA ambiguities.

3.2 Geometric Interpretation

It is possible to write $G(B, \phi)$ explicitly in the following form,

$$G(B,\phi) = \sum_{i=1}^{N} \|r_i\|^2 \sin^2\left(\phi - \tan\left(\frac{r_{y_i}}{r_{x_i}}\right)\right) - \frac{1}{N}\left[\sum_{i=1}^{N} \|r_i\| \sin\left(\phi - \tan\left(\frac{r_{y_i}}{r_{x_i}}\right)\right)\right]^2 \tag{10}$$

The following geometrical interpretation can be derived from the above expression for the geometry-dependent term $G(B, \phi)$. Project the N sensor points onto a line that is perpendicular to the line passing through the origin and orthogonal to the direction vector u. (See Figure 1). Then $G(B, \phi)$ is the sample variance of these projected points.

Note also that $G(B, \phi)$ can be interpreted as a measure of the beamwidth of the array. Consider a fixed frequency ω_0 and corresponding wavelength $\lambda_0 = \frac{2\pi c}{\omega_0}$. If we choose the delay-and-sum array weights to steer the beam at an angle ϕ_0, then resulting complex-valued array response at angle ϕ at this frequency is given by

$$W_{\phi_0}(\phi) = a^H(\phi)a(\phi_0) \tag{11}$$

$$a(\phi) = [e^{j\frac{2\pi}{\lambda}u^T\cdot r_1}, \ldots, e^{j\frac{2\pi}{\lambda}u^T\cdot r_N}]^T \tag{12}$$

The array gain $|W_{\phi_0}(\phi)|$ can be approximated by a quadratic about the point $\phi = \phi_0$. A second order Taylor series expansion of $|W_{\phi_0}(\phi)|$ yields

$$|W_{\phi_0}(\phi)| \approx N - \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)^2 G(B, \phi_0)(\phi - \phi_0)^2 \tag{13}$$

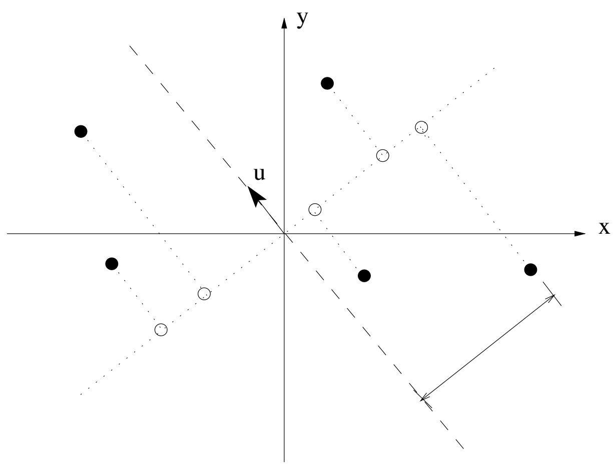

Figure 1: Geometrical interpretation of G(B,ϕ) as the sample variance of the array locations projected onto the line orthogonal to u

Using (13), the half-power beamwidth of the array is given by $\phi_{3dB} = \phi_0 + \frac{\lambda}{2\pi}\sqrt{\frac{1}{G(B,\phi_0)}}$. Thus, $\sqrt{G(B,\phi_0)}$ is inversely proportional to the half-power beamwidth of the array.

3.3 Isotropic Planar Arrays

From equation (8) we see that $G(B,\phi)$, and also the single-source CRB, is independent of translation of the array element locations, so without loss of generality we can assume that the array is centered at the origin, or equivalently, $r_c = [0,0]^T$. Under this assumption, B in (8) simplifies to

$$B = \frac{1}{N}\sum_{i=1}^{N} r_i r_i^T \tag{14}$$

We are interested in planar geometries whose single-source CRB is independent of signal arrival angle ϕ. We refer to such arrays as *isotropic arrays*. The following result characterizes the set of all isotropic arrays:

Theorem 1:
(a) An N element planar array which is centered at the origin ($r_c = [0,0]^T$) and represented by the 2-by-2 matrix $B = \frac{1}{N}\sum_{i=1}^{N} r_i r_i^T$, where $r_i = [r_{x_i}, r_{y_i}]^T$ is the location of the i^{th} sensor, is isotropic if and only if

$$B = kI_2 \tag{15}$$

where k is any positive constant and I_2 denotes the 2×2 identity matrix.

(b) If we parameterize the array geometry by an $N \times 1$ complex-valued vector g given by $g = [g_1, \ldots, g_N]^T$ where

$$g_i = r_{x_i} + j r_{y_i} \tag{16}$$

then condition stated in (15) is equivalent to

$$g^T g = 0 \tag{17}$$

(c) If the array is isotropic, then

$$G(B, \phi) = k = \frac{1}{2} \sum_{i=1}^{N} \|r_i\|^2 \qquad (18)$$

Proof: See [19].

3.4 Planar Array Design Examples

In this section we present four design methods for generating planar arrays and give examples for each method.

Circularly Symmetric Geometries:
The immediate solution where both equations (14) and (15) are satisfied is circularly symmetric geometries or any superposition of circularly symmetric geometries, where we define a circularly symmetric geometry as one in which $N \geq 3$ sensors are equally-spaced on a circle that has a radius greater than 0. A single sensor located at the origin is also in this class. An example geometry, where a 4 element and a 3 element circularly symmetric geometry are combined to form a 7 element array, is given in Figure 2. Notice that radii of the subarrays may be equal or unequal.

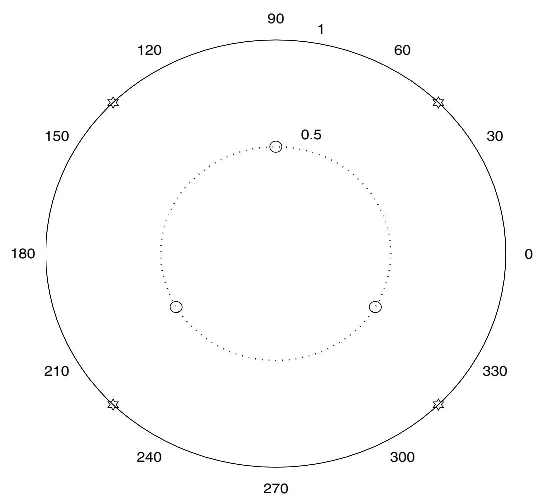

Figure 2: An isotropic planar array obtained as the superposition of a 4 element and a 3 element circularly symmetric array.

Rotated Geometries:

A second class of isotropic planar arrays can be obtained as follows. Consider an $\frac{N}{2}$ element subarray, where N is even, with arbitrary sensor locations. Define the origin as the centroid of these points, and define a second $\frac{N}{2}$ element subarray by rotating the first elements by either $90°$ or $-90°$. Consider the N-element array formed by the superposition of those two subarrays. Note that the centroid of the rotated elements is the origin, as is the centroid of the N-element array. Moreover, in complex coordinates, the N-element array vector g can be written as $g = [g_{\frac{N}{2}}^T \; e^{j\frac{\pi}{2}}, g_{\frac{N}{2}}^T]^T$, where $g_{\frac{N}{2}}$ parameterizes the $\frac{N}{2}$ element array. Then $g^T g = g_{\frac{N}{2}}^T g_{\frac{N}{2}} + e^{j\pi} g_{\frac{N}{2}}^T g_{\frac{N}{2}} = 0$, hence any geometry formed following this procedure is isotropic. An example array, generated by randomly selecting the locations of the first three elements, is shown in Figure 3.

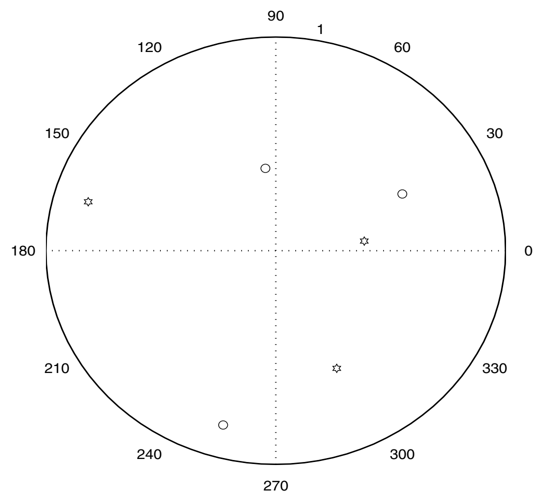

Figure 3: A 6 element isotropic array (circles and stars) formed from rotating a randomly generated 3-element subarray (stars) by $90°$.

More generally, one can take any arbitrary subarray of size N/m and combine its $0, 2\pi/m, \ldots, 2\pi(m-1)/m$ rotated versions about any arbitrary point (not just the center of gravity); the resulting N elements form an isotropic array. This array is also a superposition of m N/m-element circularly symmetric geometries, though, and covered in the previous example.

Completion of Arbitrary (N-2)-element Arrays:

It is possible to obtain an N element isotropic array from an arbitrary $(N-2)$ element array by adding two elements. Assume that locations of the first $(N-2)$ sensors are given and let r_{N-1} and r_N denote the locations of the remaining two sensors. Setting $\sum_{l=1}^{N} g_l = 0$ and $g^T g = 0$ (see (16)) gives

$$g_{N-1} + g_N = -\sum_{l=1}^{N-2} g_l \tag{19}$$

$$g_{N-1}^2 + g_N^2 = -\sum_{l=1}^{N-2} g_l^2 \tag{20}$$

The solutions to (19) and (20) uniquely determine the locations of the last two sensors so that the resulting N-element array is isotropic. An example geometry formed with this method is shown in Figure 4.

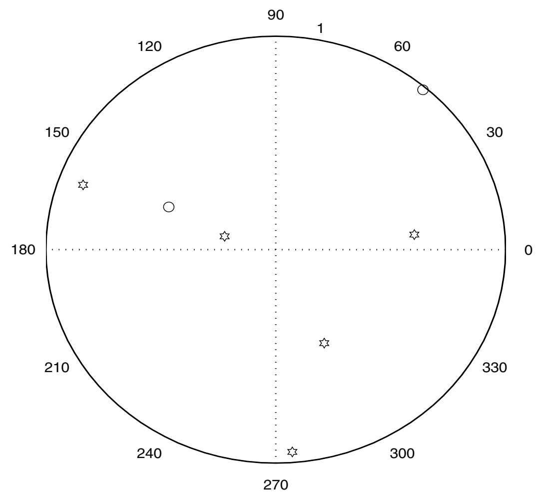

Figure 4: A 7-element isotropic array formed from an arbitrary 5-element subarray (stars) by adding 2 elements (circles).

X-shaped Isotropic Arrays:

We can combine two X-shaped geometries so that the resulting 8-element array is isotropic. An X-shaped geometry is a set of four sensors with radii $\|r\|$ and angles $\pm\alpha,\ \pm\pi - \alpha$. It can be shown that any pair of X-shaped geometries with parameters $(\|r_i\|, \alpha_i)$, $i = 1, 2$ that satisfy $\|r_1\|^2 \cos(2\alpha_1) + \|r_2\|^2 \cos(2\alpha_2) = 0$ is an isotropic array.

A special case is the superposition of X-shaped geometries are those whose elements lie along parallel two lines. For two superimposed X-shaped geometries, we constrain $\|r_1\| \cos(\alpha) = \|r_2\| \cos(\alpha)$. For example, the eight-element isotropic array with elements having x-values of ± 1 is shown in Figure 5.

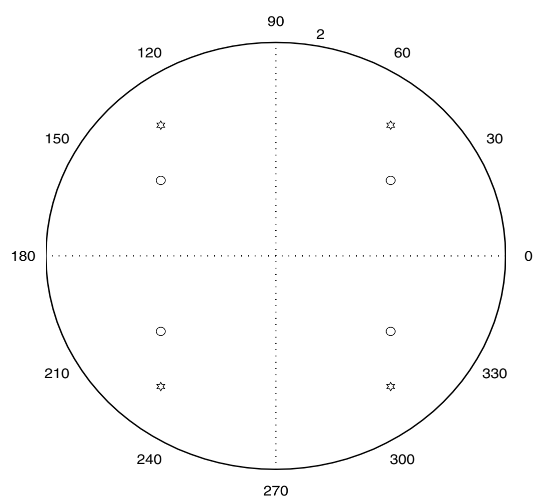

Figure 5: An 8-element isotropic array formed by combining two X-shaped geometries.

4. Three Dimensional Arrays

In this section, we consider an array that has elements located in R^3 and is used to estimate the DOA of a single wideband far-field source. The source direction is parameterized by $\theta = [\phi, \psi]^T$, where $\phi \in [0, 2\pi)$ and $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denote, respectively, the azimuth and the elevation of the source. The single source CRB for this scenario is a 2-by-2 matrix,

$$\begin{aligned} CRB(\theta) &= G(B,\theta)^{-1}\left(\frac{2KN}{c^2}\sum_{j=1}^{J}\frac{\omega_j^2}{n_j}p_j\left(1-\frac{n_j}{p_jN+n_j}\right)\right)^{-1} && (21)\\ G(B,\theta) &= J_\theta(u)^T B J_\theta(u) && (22)\end{aligned}$$

where B is given by (8) and $J_\theta(u)$ is the 3-by-2 Jacobian matrix of the u given in (4), and is equal to

$$J_\theta = \begin{bmatrix} \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \psi} \end{bmatrix} = \begin{bmatrix} -\sin(\phi)\cos(\psi) & -\cos(\phi)\sin(\psi) \\ \cos(\phi)\cos(\psi) & -\sin(\phi)\cos(\psi) \\ 0 & \cos(\psi) \end{bmatrix} \tag{23}$$

Once again, wee see that the CRB is the product of a term that depends only on the array geometry of source angle, and a term that depends only on the signal and noise parameters. We also see that the CRB depends on array geometry only through the 3×3 array covariance matrix B.

4.1 Performance Criterion

Estimating the azimuth and elevation corresponding to the DOA of the signal is equivalent to estimating the vector u. A direction $\theta = [\phi, \psi]^T$ uniquely specifies u via (4) and an estimate $\hat{\theta} = [\hat{\phi}, \hat{\psi}]$ uniquely specifies a vector $\hat{u}$. Let δ be the angle between the vectors u and $\hat{u}$; since both u and $\hat{u}$ are unit vectors, we have

$$\cos(\delta) = u^T \cdot \hat{u} \tag{24}$$

The Mean-Square Angular Error (MSAE) is introduced in [16] as a scalar measure of estimator performance in estimating a geometrical vector and defined as the expectation of δ^2. The MSAE enjoys two desired properties: it is independent of the choice of the reference coordinate frame and it does not suffer from the singularity inherent in spherical coordinates as $\psi \to \pm\frac{\pi}{2}$. The lower bound of the MSAE provides a performance criterion for a set of estimators that satisfies certain mild conditions, similar to those needed for the CRB. A derivation for the lower bound of the MSAE and a detailed discussion of the conditions for the applicability and tightness of the bound can be found in [17]. Assuming M is the number of observations, the asymptotic normalized MSAE is defined as

$$MSAE_\infty = \lim_{M\to\infty} ME\delta^2 \tag{25}$$

and when spherical coordinates are used, (25) is bounded below as

$$MSAE_\infty \geq MSAE_B = \cos^2(\psi)CRB_\infty(\phi) + CRB_\infty(\psi) \tag{26}$$

where CRB_∞ is the asymptotic CRB. For a geometrical interpretation of the MSAE_B see [11].

4.2 *Isotropic Three Dimensional Arrays*

We adopt MSAE_B as a performance criterion, and define a three dimensional array to be isotropic if the associated MSAE_B is constant for all $[\phi, \psi]^T \in [0, 2\pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. The following theorem defines the set of all isotropic three dimensional arrays:

Theorem 2:
(a) An N element array which is centered at the origin ($r_c = [0,0,0]^T$) and represented by the 3-by-3 matrix $B = \frac{1}{N}\sum_{i=1}^{N} r_i r_i^T$, where $r_i = [r_{x_i}, r_{y_i}, r_{z_i}]^T$ is the location of the i^{th} sensor, is isotropic if and only if

$$B = kI_3 \tag{27}$$

where k is any positive constant and I_3 is the 3-by-3 identity matrix.

(b) If (27) holds

$$G(B, \phi) = kI_2 \tag{28}$$

Proof: See [19].

We remark that (27) is also necessary and sufficient for the 2×2 CRB matrix to be independent of source arrival angle. In this case, the CRB in (22) is diagonal, and if it is scaled to remove the latitudinal scaling of azimuth, the CRB in the azimuthal and elevation directions are equal. That is, an uncertainty ellipse in spherical angle is a circle whose radius is independent of source arrival angle.

In [11], the authors give sufficient conditions on the array geometry so that MSAE_B is independent of the source signal DOA. The above theorem extends their results by proving that these conditions are also necessary.

4.3 *Three Dimensional Isotropic Array Example*

Analogously to the planar case, it can be shown that arrays formed by placing the sensor elements at vertices of any regular polyhedron[1], or a superposition of such arrays, result in three dimensional isotropic arrays. We conjecture that the result also holds for the 13 semiregular polyhedra.

It can also be shown that three elements can be added to an arbitrary $(N-3)$-element subarray to make the resulting N-element array isotropic. Simple design formulas for the locations of the three elements, similar to equations (19)–(20), can be derived.

5. Conclusion

In this paper, we studied planar and three dimensional arrays that have isotropic performance. For planar arrays, we adopted the single source wideband Cramér-Rao bound as the performance criterion and derived

[1] there are five regular polyhedra: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

the necessary and sufficient conditions on the location of sensor elements so that the CRB is constant for all arrival angles. These conditions are valid regardless of the source's frequency spectrum. We presented four methods to design isotropic planar arrays.

For three dimensional arrays, we chose the asymptotic Mean Square Angular Error as a measure for array isotropy. We derived necessary and sufficient conditions on the array geometry that ensure that the MSAE_B is independent of source azimuth and elevation arrival angle. When these conditions are satisfied, the azimuth and elevation are uncoupled in the CRB, and the CRB is independent of source signal arrival angle.

When designing isotropic arrays, an important practical issue that should be taken into account is the minimum allowable distance between sensors. We have assumed the noise components at the sensors are independent of each other; this assumption is violated when the sensor distances become small (see [3]). Our design methods do not guarantee that the resulting sensors are sufficiently well-separated, so if the designs produce closely-spaced sensors, they should be modified accordingly.

Acknowledgments

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