Effects of Nonzero Bandwidth on Direction of Arrival Estimators in Array Signal Processing*

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ABSTRACT

We consider the impact of bandwidth on narrowband direction-of-arrival (DOA) estimation using an array of sensors. We derive expressions for the DOA bias for three array processing algorithms: MUSIC, ESPRIT, and Weighted Subspace Fitting. The bias expressions are found by a perturbation analysis of these algorithms for small relative bandwidths of the sources. We compare the perturbation-based bias predictions to actual bias for some cases of interest.

1. INTRODUCTION

We consider the effect of bandwidth on the direction of arrival (DOA) estimates using an array of sensors. Our problem is motivated by communications and sensor problems in which the relative bandwidths of the source signals, while small, may not be negligibly small. Our interest is to quantify the bias (and variance) of narrowband DOA estimators due to nonzero source bandwidths.

An important and popular class of narrowband DOA estimation algorithms is based on decomposing the array covariance matrix into a low-rank signal subspace and an orthogonal noise subspace. The low-rank structure arises from a zero-bandwidth assumption; when the signals have non-zero, bandwidth, the low-rank structure of the signal subspace is lost. Correspondingly, the statistical properties of DOA estimates, and in particular the DOA biases, are effected.

One alternative to DOA estimation for sources with nonzero bandwidth is to use wideband source location algorithms (see, *e.g.*, [1]). However, these algorithms are more complex than their narrowband counterparts, so the use of narrowband algorithms is preferred when the bandwidths are small enough that the DOA bias is negligible or tolerable.

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In this paper we quantify the bias in DOA estimates for three popular subspace-based DOA estimators: MUSIC, ESPRIT and weighted subspace fitting (WSF). Specifically, we assume multiple signals of non-zero bandwidth impinging on an arbitrary array configuration and give analytical expressions for the resulting bias of the estimates of the directions of arrival.

The bias analysis is based on a series expansion of the signal and noise subspaces of the array covariance matrix as a function of the relative bandwidths of the source signals; as such, the analysis applies to cases where the relative bandwidths are "small". We compare our expressions to the true bias resulting from sources with nonzero bandwidth for some cases of interest.

Our analysis is a perturbation analysis of the array covariance matrix, and is similar in principle to several related perturbation analyses on, *e.g.*, sensor positioning errors and uncertainties in the sensor gain and phases (see *e.g.* [2] and the references therein). Many such perturbation analyses assume a perturbation that retains the low-rank signal subspace structure; in contrast, the non-zero bandwidth of the source signals destroys this low-rank property. In related earlier work [3], which is in part based on the perturbation analyses in [4, 5], we studied the effect of source angular spread on DOA estimation, another problem in which the low-rank subspace structure is destroyed. In the present paper we show that the perturbation due to non-zero bandwidth can be formulated in a way similar to that resulting from multipath. Thus, our work is is partly an extension of the results in [3].

2. MODEL AND ASSUMPTIONS

We assume an arbitrarily configured array of m sensors, whose sampled output at time t is the complex m-vector $\mathbf{x}(t) = [x_1(t), \ldots, x_m(t)]$. A number of uncorrelated signals $\{s_k(t)\}_{k=1}^{n}$ impinge on the array. Each source has total power q_k and arrives from an angle θ_k. Each signal is random, and its spectral density function $S_k(\omega)$ is a shifted and scaled version of a normalized "shape" spectral density $\widetilde{S}_k(\omega)$, whose center frequency is zero, whose bandwidth is one, and whose total power is one. That is,

$$S_k(\omega) = q_k \frac{1}{\beta_k \omega_k} \widetilde{S}_k\left(\frac{\omega - \omega_k}{\beta_k \omega_k}\right). \quad (1)$$

We see that $S_k(\omega)$ has relative bandwidth β_k, with $0 \leq \beta_k \leq 1$.

The "spectral shape" autocorrelation function $\tilde{r}_k(\tau)$, given by the inverse Fourier transform of $\widetilde{S}_k(\omega)$, is normalized such that $\tilde{r}_k(0) = 1$. In addition, since $\widetilde{S}_k(\omega)$ has finite second moment, $\tilde{r}_k(\tau)$ is continuously differentiable at $\tau = 0$. The corresponding signal autocorrelation, $r_k(\tau)$, is found from equation (1) to be

$$r_k(\tau) = q_k e^{i\omega_k \tau} \cdot \tilde{r}_k(\beta_k \omega_k \tau). \quad (2)$$

Note that as $\beta_k \to 0$, $r_k(\tau) \to q_k e^{i\omega_k \tau}$ and $S_k(\omega) \to 2\pi q_k \delta(\omega - \omega_k)$ as desired.

We note that $\widetilde{S}_k(\omega)$ (and consequently, $S_k(\omega)$) need not have finite support as long as its second moment is finite; see [6] for details.

The received signal at the μth sensor can be expressed as (for $\mu = 1, \ldots, m$)

$$x_\mu(t) = \sum_{k=1}^{n} \tilde{a}_\mu(\theta_k) s_k(t - T_{k\mu}) + n_\mu(t), \quad (3)$$

where $T_{k\mu}$ is the time taken for the kth signal to propagate from an arbitrary reference point to the μth element of the array. The complex array gain at the angle θ is given by the vector

$$\tilde{\mathbf{a}}(\theta) = [\tilde{a}_1(\theta), \tilde{a}_2(\theta), \ldots, \tilde{a}_m(\theta)]^T \quad (4)$$

and is assumed to be constant with respect to frequency over the interval corresponding to the bandwidth of $s_k(t)$. This assumption, used also for bandwidth performance of adaptive antenna systems [7], valid for small relative bandwidths, which is the case of interest here. Even if $\tilde{\mathbf{a}}$ is not independent of frequency, it can be corrected via calibration . The noise vector $\mathbf{n}(t) = [n_1(t) \ldots n_m(t)]^T$ is a zero mean, circularly complex random vector with $E\{\mathbf{n}(t)\mathbf{n}^*(s)\} = \lambda^2 \mathbf{I}_m \delta_{t,s}$ and $E\{\mathbf{n}(t)\mathbf{n}^T(s)\} = 0$.

If the source signals have zero bandwidth, *i.e.* if $\beta_k = 0$, then the array covariance matrix $\mathbf{R} = E\{\mathbf{x}(t)\mathbf{x}^*(t)\}$ is given by the standard "nominal" expression:

$$\mathbf{R}_0 = \mathbf{A}(\omega_0, \theta_0)\mathbf{Q}\mathbf{A}^*(\omega_0, \theta_0) + \lambda^2 \mathbf{I}_m \quad (5)$$

where $\theta_0 = [\theta_1 \ldots \theta_n]^T$, $\omega_0 = [\omega_1 \ldots \omega_n]^T$, $\mathbf{A}(\omega_0, \theta_0) = [\mathbf{a}(\omega_1, \theta_1) \ldots \mathbf{a}(\omega_n, \theta_n)]$ is the $(m \times n)$ array manifold matrix, and $\mathbf{Q} = \mathrm{diag}\{q_1 \ldots q_n\}$. Note that

$$\mathbf{a}(\omega_k, \theta_k) \triangleq [\tilde{a}_1(\theta_k) e^{-i\omega_k T_{k1}} \ldots \tilde{a}_m(\theta_k) e^{-i\omega_k T_{km}}]^T. \quad (6)$$

For nonzero signal bandwidths, it is shown in [6] that the covariance matrix $\mathbf{R}$ becomes

$$\mathbf{R} = \sum_{k=1}^{n} \mathbf{a}(\omega_k, \theta_k) q_k \mathbf{a}^*(\omega_k, \theta_k) \odot \mathbf{B}_k + \lambda^2 \mathbf{I}_m \quad (7)$$

where $\odot$ denotes the Hadamard (element wise) product. The $(m \times m)$ matrices $\{\mathbf{B}_k\}_{k=1}^n$ are defined by their (μ, ν)th elements:

$$\mathbf{B}_k(\mu, \nu) = \tilde{r}_k(\beta_k \omega_k (T_{k\nu} - T_{k\mu})). \quad (8)$$

For $\beta_k = 0$, $\mathbf{B}_k$ is a matrix whose elements are all ones, and $\mathbf{R}$ reduces to $\mathbf{R}_0$ in equation (5).

3. SMALL PERTURBATION PROPERTIES OF R

We now assume that the fractional bandwidths β_k are "small" and find the Taylor series expansion of $\mathbf{R}$ about the nominal covariance $\mathbf{R}_0$ and retain terms to second order in β_k. It is straightforward to show that

$$\mathbf{R} \approx \mathbf{R}_0 + \sum_{k=1}^{n} \left[\beta_k \widetilde{\mathbf{C}}_k + \beta_k^2 \widetilde{\mathbf{B}}_k\right] \quad (9)$$

where $\widetilde{\mathbf{C}}_k \equiv 0$ and the (μ, ν)th element of $\widetilde{\mathbf{B}}_k$ is given by

$$\begin{aligned} \widetilde{\mathbf{B}}_k(\mu, \nu) &= a_\mu(\omega_k, \theta_k) q_k a_\nu^*(\omega_k, \theta_k) \\ &\quad \cdot (T_{k\nu} - T_{k\mu})^2 \omega_k^2 \tilde{r}_k''(0). \end{aligned} \quad (10)$$

Note that

$$\begin{aligned} \tilde{r}_k''(0) &= \lim_{\tau \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} -\omega^2 \widetilde{S}_k(\omega) e^{i\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} -\omega^2 \widetilde{S}_k(\omega) d\omega. \end{aligned} \quad (11)$$

The integral in (11) is well defined since $\widetilde{S}_k(\omega)$ is assumed to have a finite second moment.

4. BIAS ANALYSIS OF DOA ESTIMATORS

We analyze the bias and variance of DOA estimators when the source signals have small but nonzero bandwidth. The approach we take is to perform a small perturbation analysis of the estimation algorithms, using the perturbation results on $\mathbf{R}$ obtained above (and on its corresponding sample estimate $\widehat{\mathbf{R}}$).

The analysis is similar to the one in [3] for sources that are spread in arrival angle, and we are able to use the results in [3] on signal and noise subspace perturbation.

We note that, similarly to [3], the finite-sample variance of the DOA estimates are the same as the nominal (*i.e.*, zero bandwidth) DOA variances; these variance expressions are well-studied in the literature. Below we present an analysis for the DOA bias terms.

4.1. Signal and Noise Subspaces

In practice, the true covariance matrix $\mathbf{R}$ is not available and must be estimated from N samples of the array output, $\{\mathbf{x}(t)\}_{t=1}^{N}$, according to

$$\widehat{\mathbf{R}} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^*(t). \quad (12)$$

Consider the following subspace decompositions of $\mathbf{R}_0$ and $\widehat{\mathbf{R}}$:

$$\mathbf{R}_0 = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^* + \lambda^2\mathbf{G}\mathbf{G}^* \quad (13)$$
$$\widehat{\mathbf{R}} = \widehat{\mathbf{S}}\widehat{\mathbf{\Lambda}}\widehat{\mathbf{S}}^* + \widehat{\mathbf{G}}\widehat{\mathbf{\Sigma}}\widehat{\mathbf{G}}^* \quad (14)$$

where $\mathbf{\Lambda} = \text{diag}\{\lambda_1 \ldots \lambda_n\}$ contains the n largest eigenvalues of $\mathbf{R}_0$, $\mathbf{S}$ is the corresponding matrix of the n associated orthonormal eigenvectors, and $\mathbf{G}$ is the matrix of the remaining $m-n$ orthonormal eigenvectors. We assume $\{\lambda_i\}_{i=1}^{n}$ are distinct and greater than λ^2. The matrices $\widehat{\mathbf{S}}, \widehat{\mathbf{\Lambda}}, \widehat{\mathbf{G}}$ and $\widehat{\mathbf{\Sigma}}$ are similarly defined. Then as $\beta_k \to 0$ and $N \to \infty$, we have [4] $\widehat{\mathbf{R}} \to \mathbf{R}$, $\widehat{\mathbf{S}} \to \mathbf{S}$, $\widehat{\mathbf{\Lambda}} \to \mathbf{\Lambda}$, $\widehat{\mathbf{\Sigma}} \to \lambda^2\mathbf{I}_{m-n}$, and $\widehat{\mathbf{G}}\widehat{\mathbf{G}}^* \to \mathbf{G}\mathbf{G}^*$. We also define $\tilde{\mathbf{\Lambda}} \triangleq \mathbf{\Lambda} - \lambda^2\mathbf{I}_n$, $(n \times n)$, and

$$\mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^* = \mathbf{A}\mathbf{A}^{\dagger}, \quad (15)$$

the projection matrix onto the range space of $\mathbf{A}$ (note that $\mathbf{A}^{\dagger} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$ designates the pseudo-inverse of $\mathbf{A}$). To simplify the notation, in what follows we will often write $\mathbf{a}_k$ instead of $\mathbf{a}(\omega_k, \theta_k)$ and $\mathbf{d}_k$ instead of $d\mathbf{a}_k/d\theta_k$.

4.2. The MUSIC Algorithm

The weighted MUSIC algorithm gives the DOA estimates $\{\hat{\theta}_k^M\}_{k=1}^{n}$ as the n largest maxima of the scalar function

$$V(\theta) = \text{tr}\left\{\mathbf{P}_{\mathbf{a}(\theta)}\widehat{\mathbf{S}}\mathbf{W}\widehat{\mathbf{S}}^*\right\} \quad (16)$$

where $\mathbf{W}$ is a non-negative definite matrix, chosen by the user. Following the approach of [3] it can be shown that the MUSIC bias, to second order in β_k, is given by

$$E\{\hat{\theta}_k^M\} - \theta_k \approx \frac{\text{Re}\left\{\mathbf{a}_k^{\dagger}\mathbf{S}\mathbf{W}\tilde{\mathbf{\Lambda}}^{-1}\mathbf{S}^*\left[\sum_{i=1}^{n}\beta_i^2\widetilde{\mathbf{B}}_i\right]\mathbf{G}\mathbf{G}^*\mathbf{d}_k\right\}}{\mathbf{a}_k^{\dagger}\mathbf{S}\mathbf{W}\mathbf{S}^*\mathbf{a}_k^{\dagger*}\cdot\mathbf{d}_k^*\mathbf{G}\mathbf{G}^*\mathbf{d}_k} \quad (17)$$

4.3. The ESPRIT Algorithm

The ESPRIT algorithm uses the fact that the array can be partitioned into two subsets. The two sub-arrays are identical except for a translational shift of Δ wavelengths. Define the matrices

$$\mathbf{S}_1 = [\mathbf{I}_m \quad 0]\mathbf{S} \quad (m \times n) \quad (18)$$
$$\mathbf{S}_2 = [0 \quad \mathbf{I}_m]\mathbf{S} \quad (m \times n) \quad (19)$$
$$\phi = (\mathbf{S}_1^*\mathbf{S}_1)^{-1}\mathbf{S}_1^*\mathbf{S}_2 \quad (n \times n) \quad (20)$$

and similarly $\widehat{\mathbf{S}}_1$, $\widehat{\mathbf{S}}_2$ and $\hat{\phi}$. If $\{\rho_k\}_{k=1}^{n}$ and $\{\hat{\rho}_k\}_{k=1}^{n}$ are the eigenvalues of the matrices ϕ and $\hat{\phi}$ respectively, then the DOA estimates of the ESPRIT algorithm are given by

$$\hat{\theta}_k^E = \sin^{-1}\left(\frac{\arg\hat{\rho}_k}{2\pi\Delta}\right), \quad k = 1, \ldots, n, \quad (21)$$

where the shift-invariance of the array has been used. Let γ_k and η_k denote the left and right eigenvectors of ϕ, normalized so that $\gamma_k^*\eta_k = 1$. Introduce

$$\mu_k^* = \gamma_k^*(\mathbf{S}_1^*\mathbf{S}_1)^{-1}\mathbf{S}_1^*\{[0 \quad \mathbf{I}_m] - \rho_k[\mathbf{I}_m \quad 0]\}. \quad (22)$$

Then, to second order in β_k, the ESPRIT bias is given by [3]

$$E\{\hat{\theta}_k^E\} - \theta_k \approx 1/(2\pi\Delta\cos\theta_k) \cdot \text{Im}\left\{\frac{1}{\rho_k}\mu_k^*\mathbf{G}\mathbf{G}^*\left[\sum_{i=1}^{n}\beta_i^2\widetilde{\mathbf{B}}_i\right]\mathbf{S}\tilde{\mathbf{\Lambda}}^{-1}\eta_k\right\} \quad (23)$$

4.4. The WSF Algorithm

The WSF algorithm computes the DOA estimates as the vector $\hat{\boldsymbol{\theta}}^W$ that maximizes a scalar loss function (of a vector variable), *i.e.*,

$$\hat{\boldsymbol{\theta}}^W = \arg\max V(\boldsymbol{\theta}), \quad (24)$$

where $\hat{\boldsymbol{\theta}}^W = [\hat{\theta}_1^W, \ldots, \hat{\theta}_n^W]^T$ is the vector of WSF DOA estimates and

$$V(\boldsymbol{\theta}) = \text{tr}\left[\mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}\widehat{\mathbf{S}}\mathbf{W}\widehat{\mathbf{S}}^*\right]. \quad (25)$$

According to [3], the WSF bias is (to second order in β_k) given by

$$E\{\hat{\boldsymbol{\theta}}^W\} - \boldsymbol{\theta}_0 \approx \left\{2\,\text{Re}[(\mathbf{A}^{\dagger}\mathbf{S}\mathbf{W}\mathbf{S}^*\mathbf{A}^{\dagger*} \odot \mathbf{D}^*\mathbf{G}\mathbf{G}^*\mathbf{D}]\right\}^{-1}\mathbf{V}'(\boldsymbol{\theta}_0) \quad (26)$$

with $\mathbf{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_2]$ and where $\mathbf{V}'(\boldsymbol{\theta})$, the gradient vector of $V(\boldsymbol{\theta})$, has the kth element

$$\mathbf{V}_k'(\boldsymbol{\theta}_0) \approx 2\text{Re}\left\{\mathbf{e}_k^T\mathbf{A}^{\dagger}\mathbf{S}\mathbf{W}\tilde{\mathbf{\Lambda}}^{-1}\mathbf{S}^*\left[\sum_{i=1}^{n}\beta_i^2\widetilde{\mathbf{B}}_i\right]\mathbf{G}\mathbf{G}^*\mathbf{d}_k\right\} \quad (27)$$

5. NUMERICAL EXAMPLES

We first note that for a single source impinging a ULA, it can be shown that the DOA bias is zero for all three methods considered [6]. We therefore consider a two-source case. We have a uniform linear array (ULA) with m sensors at half-wavelength spacing relative to the source center frequency. We consider two sources, each with flat spectrum over its frequency band; *i.e.*, $\tilde{S}_k(\omega) = \pi$ for $|\omega| \leq 1$, so $\tilde{r}_k(\beta_k\omega_k\tau) = \mathrm{sinc}\,(\beta_k\omega_k\tau)$. The signals arrive at angles θ_1 and θ_2 measured from the broadside of the array. The center frequencies ω_1 and ω_2 are equal, as are the relative bandwidths β_1 and β_2.

Figure 1 shows the MUSIC DOA estimation bias for two sources at $\theta_1 = 20°$ and $\theta_2 = 50°$, as a function of the relative bandwidth, for $m = 5$ sensors. The cases of equal ($q_2/q_1 = 1$) and different ($q_2/q_1 = 5$) source powers are shown. In the figures, we show both the "true" bias computed using the exact expression for $\mathbf{R}$, and the second-order predicted bias using equation (17). We show results for MUSIC, but the results for ESPRIT and WSF are very similar. We see that for this case the bias is well-predicted using the analysis in the paper for relative bandwidths below about 20%. Note that the bias of a weak source can become significant even for small relative bandwidths, an indication that bias may be a significant issue when sources have different powers (such as in the near-far problem in wireless communications).

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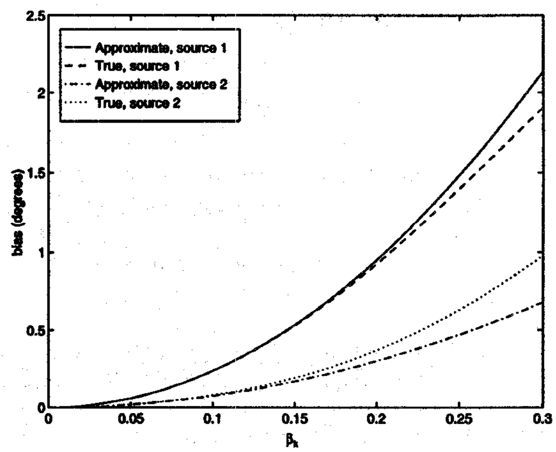
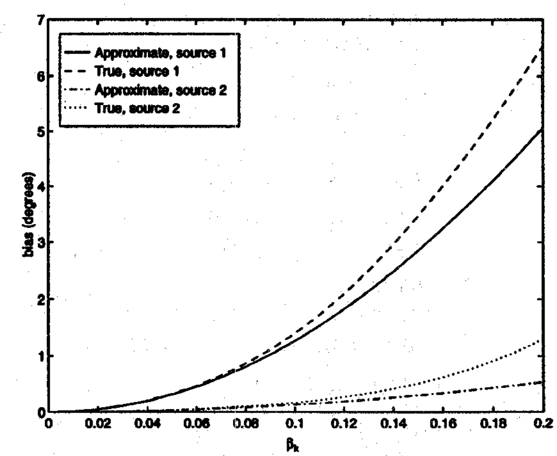


(a) Sources of equal power



(b) Sources of different power ($q_2/q_1 = 5$)

Figure 1. MUSIC bias as a function of relative bandwidth for $m = 5$ sensors and two sources at $\theta_1 = 20°$ and $\theta_2 = 50°$. Note the different scales in the two figures.