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# MODE-type Algorithm for Estimating Damped, Undamped or Explosive Modes (Paper Summary)

Mats Cedervall    Petre Stoica
Randolph Moses*
Systems and Control Group, Department of Technology
Uppsala University, P.O. Box 27, S-751 03 Uppsala, Sweden
*Phone:* +46-18-183079 *Fax:* +46-18-503611
*Email:* {mc,ps,rm}@syscon.uu.se
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**Abstract**

We propose a new algorithm for estimating the parameters of damped, undamped or explosive sinusoidal processes. The algorithm resembles the MODE algorithm which is commonly used for direction of arrival estimation in the array signal processing field. The algorithm is asymptotically (high SNR) optimal. Nevertheless it is computationally simple and easy to implement. Numerical examples are included to illustrate the performance of the proposed method.

## 1 Problem formulation

Let

$$\bar{y}(t) = \sum_{k=1}^{n} \alpha_k \rho_k^t + \bar{e}(t), \; t = 1, 2, \ldots N \tag{1.1}$$

be the equation describing the observed signal. In (1.1) $\alpha_k \in \mathcal{C}$; $\mid \rho_k \mid \gtreqless 1$; $\bar{e}(t)$ is circularly symmetric Gaussian distributed noise with variance $\sigma^2$, and $n$ is given. The number of data samples $N$ is typically small, and the signal to noise ratio

*On leave from the Department of Electrical Engineering, Ohio State University, USA.

(SNR) is usually assumed to be high. The SNR of the $k^{th}$ component in (1.1) is defined as follows:

$$\mathrm{SNR}_k = 10\log_{10}\left(\frac{E_k}{N\sigma^2}\right) \ [\mathrm{dB}]$$

where $E_k$ is the total energy of the $k^{th}$ mode.

$$E_k = |\alpha_k|^2 \sum_{l=0}^{N-1} |\rho_k|^{2l} = |\alpha_k|^2 \cdot \begin{cases} N \text{ if } |\rho_k| = 1 \\ \frac{1-|\rho_k|^{2N}}{1-|\rho_k|^2} \text{ otherwise} \end{cases}$$

The problem of interest herein is to estimate $\{\rho_k\}$ (and perhaps $\{\alpha_k\}$ as well, which is an easy task once $\{\rho_k\}$ has been obtained).

## 1.1 Applications

This problem has importance in a number of applications, including speech modeling [3], elctrocardiogram signal modeling [2], and radar scattering analysis from stepped frequency measurements [1]. In all of these applications, both damped modes and explosive modes may arise. For example, in radar scattering, damped or explosive modes can arise because the frequency response of different scattering centers may be decreasing or increasing as a function of frequency, respectively.

# 2 Solution by using MODE

Let $y(t)$ be defined as

$$y(t) = \begin{bmatrix} \bar{y}(t) \\ \vdots \\ \bar{y}(t+m-1) \end{bmatrix}$$

for some $m > n$, and define

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & & \vdots \\ \rho_1^{m-1} & \rho_2^{m-1} & \cdots & \rho_n^{m-1} \end{bmatrix}; \quad x(t) = \begin{bmatrix} \alpha_1\rho_1^t \\ \vdots \\ \alpha_n\rho_n^t \end{bmatrix}; \quad e(t) = \begin{bmatrix} \bar{e}(t) \\ \vdots \\ \bar{e}(t+m-1) \end{bmatrix}$$

With the above definitions we can write

$$y(t) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & & \vdots \\ \rho_1^{m-1} & \rho_2^{m-1} & \cdots & \rho_n^{m-1} \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1\rho_1^t \\ \\ \alpha_k\rho_n^t \end{bmatrix}}_{x(t)} + e(t) = Ax(t) + e(t).$$

The key equation here is (see above):

$$y(t) = Ax(t) + e(t)$$

which resembles the "standard" model used in sensor array signal processing. We form the following "sample covariance matrix"

$$\hat{R}_d = \sum_{K=1}^{M} y\left((k-1)d+1\right) y^*\left((k-1)d+1\right) \tag{2.1}$$

where $d > 0$ is an integer which measures the degree of overlapping between adjacent snapshots, (the smaller the $d$ the more overlapped those vectors are; for $d \geq m$ there is no overlapping), and the total number of snapshots $M$ is defined as

$$M = \left\lfloor \frac{N-m}{d} \right\rfloor + 1$$

where $\lfloor \cdot \rfloor$ means rounding to the nearest smaller integer. The role of the $d$ will be discussed in the full paper.

For sufficiently large SNR values $\hat{R}_d$ in (2.1) is close to the matrix

$$R_d = AP_dA^* \tag{2.2}$$

where

$$P_d = \sum_{i=1}^{M} x\left((i-1)d+1\right) x^*\left((i-1)d+1\right).$$

Let $\{b_k\}_{k=0}^{n}$ be the coefficients of the following polynomial

$$b_0 z^n + \cdots + b_{n-1}z + b_n = b_0 \prod_{k=1}^{n} (z - \rho_k)$$

and let

$$B^* = \begin{bmatrix} b_n & \cdots & b_0 & & 0 \\ & \ddots & & \ddots & \\ 0 & & b_n & \cdots & b_0 \end{bmatrix} \quad (m-n) \times m.$$

Also define the eigenvalue decomposition of $R_d$ as

$$R_d = \begin{bmatrix} S & G \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^* \\ G^* \end{bmatrix} = S\Lambda S^* \tag{2.3}$$

where $S$ is the matrix whose columns are the $n$ principal eigenvectors of $R_d$ and $\Lambda$ is a diagonal matrix with the corresponding eigenvalues on the diagonal (note from (2.2) that $\text{rank}(R_d) = n$). It is well known that $\mathcal{R}(S) = \mathcal{R}(A)$, where $\mathcal{R}$ denotes the range-operator. Consequently, because $B^*A = 0$ (as is readily verified) we have

$$B^*S = 0. \tag{2.4}$$

This is a key property whose potential for parameter estimation is clear. Let $\hat{S}$ and $\hat{\Lambda}$ denote the sample counterparts of the above $S$ and $\Lambda$. In view of (2.4), we can expect that the following equation in the unknowns $\{b_k\}$ (which are used to reparameterize the estimation problem under discussion) holds approximately (for SNR $\gg 1$):

$$B^* \hat{S} \simeq 0. \tag{2.5}$$

Based on (2.5) we can estimate the parameter vector by minimization of the following MODE-like[5] criterion

$$\operatorname{tr}\left(BW_1B^*\hat{S}W_2\hat{S}^*\right). \tag{2.6}$$

where

$$b^T = \left[\begin{array}{ccc} b_n & \cdots & b_0 \end{array}\right].$$

In the *full paper* we introduce some intuitively appealing weights, and we also show that they in fact are asymptotically (for SNR $\gg 1$) optimal. The derivation of the high SNR optimal weighting matrices $W_1$ and $W_2$ is in fact the main theoretical contribution of our (full version) paper. Plugging these weights into (2.6) we arrive at the following cost function, that is to be minimized.

$$f(b) = \operatorname{tr}\left(B\left(B^*B\right)^{-1}B^*\hat{S}\hat{\Lambda}\hat{S}^*\right). \tag{2.7}$$

We note that, in view of (2.4), minimization of (2.7) with $B^*B$ replaced by any positive definite matrix (such as $I$) gives *consistent estimates* of $\{b_k\}$ (consistent in SNR). Furthermore, it can be shown that, asymptotically in SNR, replacement of $B^*B$ in (2.7) by a consistent estimate has no effect on the asymptotic accuracy. Hence the following two-step procedure appears suitable to use as for the minimization of (2.7).

Step 1. Compute the $n$ principal eigenpairs of $\hat{R}_d$. Let

$$f_W(b) = \operatorname{tr}\left(BW^{-1}B^*\hat{S}\hat{\Lambda}\hat{S}^*\right).$$

Compute the $n$ principal eigenpairs of $\hat{R}_d$. Obtain initial (consistent) estimates of $b$ by minimizing $f_W(b)$ with $W = I$.

Step 2. Derive enhanced estimates of $\{b_k\}$ as the minimizers of $f_W(b)$, with $W = \hat{B}^*\hat{B}$, where $\hat{B}$ is made from the estimates obtained in Step 1. Obtain $\{\rho_k\}$ from $\{b_k\}$.

Both the minimization steps above can be efficiently minimized by the algorithm outlined in the following subsection.

## 2.1 Minimization of $f_W(b)$

We start out from the following form of the general cost function in (2.6):

$$\begin{aligned} f(b) &= \operatorname{vec}(B)^*\left(W_1^T \otimes \hat{S}W_2\hat{S}^*\right)\operatorname{vec}(B) && (2.8)\\ &= \tilde{b}^*\left(W_1^T \otimes \hat{S}W_2\hat{S}^*\right)\tilde{b}. && (2.9)\end{aligned}$$

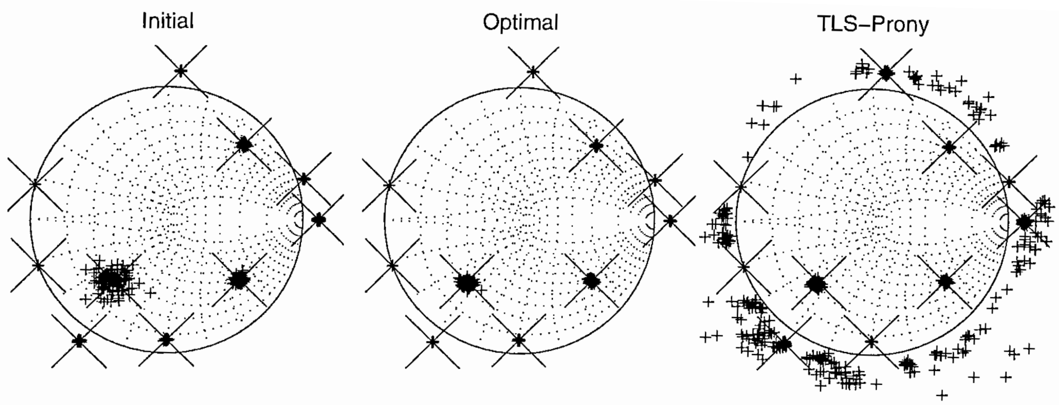


Figure 1:

where vec denotes the vectorization operator

$$\tilde{b}^* = \text{vec}\,(B)^* = \left[\begin{array}{ccccc} b^T & 0^T & b^T & 0^T & \cdots \end{array}\right], \qquad (2.10)$$

and where $\otimes$ denotes the Kronecker matrix product. If we use $\bar{\Omega}$ to denote the matrix $W_1^T \otimes \hat{S} W_2 \hat{S}^*$ from which the *rows and columns corresponding to the zeros in* $\tilde{b}$ *are eliminated*, and also denote by $\Omega$ the following matrix

$$\Omega^T = \left[\begin{array}{ccc} I & \cdots & I \end{array}\right] \bar{\Omega} \left[\begin{array}{c} I \\ \vdots \\ I \end{array}\right]$$

then (2.9) can be written as

$$f(b) = b^* \Omega b. \qquad (2.11)$$

The function (2.11) is to be minimized with respect to $b$, under an appropriate constraint. If we choose a unit norm constraint on $b$ we obtain the Total Least Squares Solution (TLS) which is easily obtained as the eigenvector of $\Omega$ corresponding to the smallest eigenvalue. In summary

$$\hat{b} = \text{the smallest eigenvector of } \Omega.$$

# 3 Numerical example

Figure 1 compares proposed method with the TLS-Prony [4] method. The example is adopted from [4]. There are ten exponential modes, which are selected in such a way that the scenario is a rather general one. The true pole locations are indicated with large '×'s in Figure 1. There were $N = 100$ data points and $\sigma = 0.01$. The amplitude coefficients, $\{\alpha_k\}_{k=1}^{10}$, are chosen so that each mode energy is unity. This corresponds to an SNR of 20 dB per mode. In Figure 1 the '+' signs show the pole-estimates obtained from 100 independent Monte-Carlo simulations with the proposed algorithm (both initial and optimal estimates) and the TLS-Prony algorithm discussed in [4]. The proposed algorithm produces more reliable estimates and in addition it is computationally simpler and more straightforward to implement.

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Systems and Control Group
Department of Technology
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Randolph Moses
Department of Electrical Engineering
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