Extension of Decoupled ML to Coherent Multipath Signals*

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Summary

We present an algorithm for estimating the directions of arrival (DOAs) and signal amplitudes of known, possibly coherent signals impinging on an array of sensors. The algorithm is an extension to the DEML method of Li, *et.al.*, to handle coherent multipath which may be present in the signals. We derive a large-sample Maximum Likelihood estimator for the signal parameters. The algorithm is computationally efficient because the nonlinear minimization step decouples into a set of minimizations of smaller dimension. We also derive the asymptotic statistical variance of the parameter estimates, develop an analytical expression for the CR bound for this signal scenario, and compare the two both theoretically and numerically.

1 Introduction

Array signal processing has been a topic of considerable interest. A number of high resolution DOA estimation algorithms have been developed, including MUSIC, ESPRIT and Weighted Subspace Fitting (WSF). (see, *e.g.*, [1, 2, 3] and their references). There has also been considerable developments on the accuracy of these techniques [4, 5].

More recently, there has been interest in developing algorithms that assume some *a priori* signal knowledge to improve DOA estimation capability [6, 7, 8]. This interest is motivated by applications in which partial knowledge of the incoming signals is a reasonable assumption. One such application is mobile telecommunications, where incoming signals of interest have known preamble sequences that can be exploited to improve DOA estimation accuracy and/or decrease computational cost.

One attractive algorithm for DOA estimation of known signals is the Decoupled Maximum Likelihood (DEML) method [7]. The DEML method is a large sample ML algorithm which is computationally efficient because the nonlinear minimization step in the algorithm decouples into a set of one-dimensional minimizations.

The DEML algorithm in [7] is based on the assumption that the desired signals are uncorrelated with one another, and the algorithm breaks down when the signals are strongly correlated. In this paper we extend the DEML algorithm to handle coherent signals impinging on the array. The modification, which we term Coherent DEcoupled

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Maximum Likelihood (CDEML), is also a large sample ML algorithm, and its nonlinear minimization step also decouples into a set of minimizations of smaller dimension.

2 Signal Model and Problem Formulation

The array output vector $\mathbf{x}(t)$ is modeled as

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t) \tag{2.1}$$

where $\mathbf{x}(t) \in \mathcal{C}^{m\times 1}$ is the received data vector, $\mathbf{s}(t) \in \mathcal{C}^{d\times 1}$ is the incident signal vector and $\mathbf{n}(t) \in \mathcal{C}^{m\times 1}$ is an additive noise vector term. The matrix $\mathbf{A}(\boldsymbol{\theta})$ $(m \times d)$ is the array manifold describing the array transfer response as a function of the signal parameter vector $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_d] \in \mathcal{R}^{d\times 1}$. Each column of $\mathbf{A}(\boldsymbol{\theta})$ is a steering vector $\mathbf{a}(\theta_k)$.

We make the following assumptions in the derivation of the algorithm.

Assumption 1. The array manifold $\mathbf{A}(\boldsymbol{\theta})$ is unambiguous; *i.e.*, the vectors $\{\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_{m-1})\}$ are linearly independent for any set of distinct $\{\theta_1, \dots, \theta_{m-1}\}$. $\square$

Assumption 2. The noise $\mathbf{n}(t)$ is circularly symmetric zero-mean Gaussian with second-order moments

$$\mathrm{E}[\mathbf{n}(t)\mathbf{n}^*(s)] = \mathbf{Q}\delta_{t,s}, \qquad \mathrm{E}[\mathbf{n}(t)\mathbf{n}^T(s)] = \mathbf{0} \tag{2.2}$$

where $(\cdot)^*$ denotes complex conjugate transpose. The noise covariance matrix $\mathbf{Q}$ is assumed to be positive definite, but is otherwise unknown. $\square$

Assumption 3. The impinging signals $\mathbf{s}(t)$ are scaled versions of a set of c known sequences $\{y_1(t), \dots, y_c(t)\}$. In other words

$$\mathbf{s}(t) = \boldsymbol{\Gamma}\mathbf{y}(t) \tag{2.3}$$

where $\mathbf{y}(t) = [y_1(t), \dots, y_c(t)]^T$ and $\boldsymbol{\Gamma}$ is a $(d \times c)$ matrix. The source signals $y_k(t)$ are assumed to be "quasi-stationary" [9]; that is, the "covariance matrix" of $\mathbf{y}(t)$ given by

$$\mathbf{R}_{yy} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \mathbf{y}(t)\mathbf{y}^*(t), \tag{2.4}$$

is well-defined. We assume $\mathbf{R}_{yy} > 0$, and that the source signals and noise vectors are uncorrelated, so that, $\mathbf{R}_{yn} = 0$, with $\mathbf{R}_{yn}$ defined similarly to $\mathbf{R}_{yy}$. $\square$

Assumption 4. The matrix $\boldsymbol{\Gamma}$ in (2.3) has the following structure:

$$\boldsymbol{\Gamma} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1d_1} & 0 & & & \cdots & & & 0 \\ 0 & \cdots & 0 & \gamma_{21} & \cdots & \gamma_{2d_2} & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & \ddots & & \vdots & \\ 0 & & & \cdots & & & 0 & \gamma_{c1} & \cdots & \gamma_{cd_c} \end{bmatrix}^T . \tag{2.5}$$

Each index $\{d_k\}_{k=1}^{c}$ denotes the (known) number of incoming signals corresponding to the k^{th} source signal $y_k(t)$. $\square$

Since there are only d unknown elements of $\mathbf{\Gamma}$, we parameterize $\mathbf{\Gamma}$ as $\mathbf{\Gamma}(\boldsymbol{\gamma})$, where the $(d \times 1)$ vector $\boldsymbol{\gamma}$ is defined as

$$\boldsymbol{\gamma} = \left[\boldsymbol{\gamma}_1^T, \boldsymbol{\gamma}_2^T, \ldots, \boldsymbol{\gamma}_c^T\right]^T, \quad (d \times 1), \tag{2.6}$$

and where each $\boldsymbol{\gamma}_k^T = [\gamma_{k1}, \ldots, \gamma_{kd_k}]$. We correspondingly partition $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T, \ldots, \boldsymbol{\theta}_c^T]. \tag{2.7}$$

where each $\boldsymbol{\theta}_k^T = [\theta_{k1}, \ldots, \theta_{kd_k}]^T$. Thus, each incident signal $s_{kl}(t) = \gamma_{kl} y_k(t)$ and arrives at angle θ_{kl}, for $k = 1, \ldots, c$ and $l = 1, \ldots, d_k$. The case $d_k > 1$ corresponds to coherent multipath from the $y_k(t)$ source.

The CDEML algorithm we present is derived for signal scenarios satisfying Assumptions 1–4. The DEML algorithm in [7] is a special case, imposing the additional assumption that $\mathbf{\Gamma}$ is square and diagonal, or, equivalently, that $\mathbf{d}_k \equiv 1$. Both CDEML and DEML are large sample ML estimators when $\mathbf{R}_{yy}$ is diagonal.

3 Derivation of the Algorithm

In this Section we derive a large-sample Maximum Likelihood (ML) estimator for $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$. The negative log-likelihood function of the array output vectors $\mathbf{x}(t)$, $t = 1, \ldots, N$; to within an additive constant it is given by

$$L(\boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{Q}) = \ln|\mathbf{Q}| + \operatorname{tr}\left\{\mathbf{Q}^{-1}\frac{1}{N}\sum_{t=1}^{N}[\mathbf{x}(t) - \mathbf{B}\mathbf{y}(t)]\,[\mathbf{x}(t) - \mathbf{B}\mathbf{y}(t)]^*\right\}, \tag{3.1}$$

where $|\cdot|$ denotes the determinant of a matrix and $\mathbf{B}(\boldsymbol{\theta}, \boldsymbol{\gamma}) \stackrel{\triangle}{=} \mathbf{A}(\boldsymbol{\theta})\mathbf{\Gamma}(\boldsymbol{\gamma})$. In the following we suppress the explicit dependence of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{\Gamma}$ on $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ to simplify notation.

It can be shown that the $\mathbf{Q}$ that minimizes L in (3.1) is given by

$$\widehat{\mathbf{Q}}(\mathbf{B}) = \frac{1}{N}\sum_{n=1}^{N}[\mathbf{x}(t) - \mathbf{B}\mathbf{y}(t)]\,[\mathbf{x}(t) - \mathbf{B}\mathbf{y}(t)]^*. \tag{3.2}$$

Inserting $\widehat{\mathbf{Q}}(\mathbf{B})$ into (3.1) and taking the exponential, we obtain the following cost function (to within a constant)

$$\begin{aligned} F_1(\boldsymbol{\theta}, \boldsymbol{\gamma}) &= \left|\widehat{\mathbf{R}}_{xx} + \mathbf{B}\widehat{\mathbf{R}}_{yy}\mathbf{B}^* - \mathbf{B}\widehat{\mathbf{R}}_{yx} - \widehat{\mathbf{R}}_{yx}^*\mathbf{B}^*\right| \\ &= \left|\widehat{\mathbf{R}}_{xx} - \widehat{\mathbf{R}}_{yx}\widehat{\mathbf{R}}_{yy}^{-1}\widehat{\mathbf{R}}_{yx}^* + (\mathbf{B} - \widehat{\mathbf{R}}_{yx}\widehat{\mathbf{R}}_{yy}^{-1})\widehat{\mathbf{R}}_{yy}(\mathbf{B} - \widehat{\mathbf{R}}_{yx}\widehat{\mathbf{R}}_{yy}^{-1})^*\right| \\ &= \left|\widehat{\mathbf{Q}}\right|\left|\mathbf{I} + \widehat{\mathbf{Q}}^{-1}(\mathbf{B} - \hat{\mathbf{B}})\widehat{\mathbf{R}}_{yy}(\mathbf{B} - \hat{\mathbf{B}})^*\right| \\ &= \left|\widehat{\mathbf{Q}}\right|\left|\mathbf{I} + \widehat{\mathbf{R}}_{yy}(\mathbf{B} - \hat{\mathbf{B}})^*\widehat{\mathbf{Q}}^{-1}(\mathbf{B} - \hat{\mathbf{B}})\right| \end{aligned} \tag{3.3}$$

where $\widehat{\mathbf{Q}} = \widehat{\mathbf{R}}_{xx} - \widehat{\mathbf{R}}_{yx}\widehat{\mathbf{R}}_{yy}^{-1}\widehat{\mathbf{R}}_{yx}^*$, $\hat{\mathbf{B}} = \widehat{\mathbf{R}}_{yx}\widehat{\mathbf{R}}_{yy}^{-1}$, $\widehat{\mathbf{R}}_{yy} = 1/N\sum_{t=1}^{N}\mathbf{y}(t)\mathbf{y}(t)^*$, and $\widehat{\mathbf{R}}_{xx}$ and $\widehat{\mathbf{R}}_{yx}$ similarly defined. Note that both $\widehat{\mathbf{Q}}$ and $\hat{\mathbf{B}}$ are consistent estimates of $\mathbf{Q}$ and $\mathbf{B}$, respectively. By using Taylor series expansion of $\ln(\mathbf{F}_1)$ about the true $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ vector coefficients, a straightforward extension to the derivation in [7] shows that minimizing F_1 is asymptotically equivalent to minimizing

$$F_2(\boldsymbol{\theta}, \boldsymbol{\gamma}) = \operatorname{tr}\left[\mathbf{R}_{yy}(\mathbf{B} - \widehat{\mathbf{B}})^* \widehat{\mathbf{Q}}^{-1}(\mathbf{B} - \widehat{\mathbf{B}})\right]. \tag{3.4}$$

Equation (3.4) is a large sample ML estimator for a general $\mathbf{R}_{yy}$ matrix, and involves a nonlinear minimization of dimension $2d$. If we further assume that $\mathbf{R}_{yy}$ is diagonal, the minimization of (3.4) decouples into the c minimization problems

$$\widehat{\boldsymbol{\theta}}_k, \gamma_k = \arg \min_{\boldsymbol{\theta}_k, \boldsymbol{\gamma}_k} \left[\mathbf{A}(\boldsymbol{\theta}_k)\boldsymbol{\gamma}_k - \widehat{\mathbf{b}}_k\right]^* \widehat{\mathbf{Q}}^{-1} \left[\mathbf{A}(\boldsymbol{\theta}_k)\boldsymbol{\gamma}_k - \widehat{\mathbf{b}}_k\right] \quad k = 1, \ldots, c, \tag{3.5}$$

where $\widehat{\mathbf{b}}_k$ denotes the k^{th} column of $\widehat{\mathbf{B}}$ and $\mathbf{A}(\boldsymbol{\theta}_k)$ is the part of $\mathbf{A}$ corresponding to $\boldsymbol{\theta}_k$. The minimization with respect to $\boldsymbol{\gamma}_k$ is

$$\boldsymbol{\gamma}_k(\boldsymbol{\theta}_k) = \left\{\widehat{\mathbf{Q}}^{-1/2}\mathbf{A}(\boldsymbol{\theta}_k)\right\}^{\dagger} \widehat{\mathbf{Q}}^{-1/2}\widehat{\mathbf{b}}_k \tag{3.6}$$

where $(\cdot)^{\dagger}$ is the Moore-Penrose of a matrix. Substituting (3.6) into (3.5), we arrive at the following cost-function for estimating $\widehat{\boldsymbol{\theta}}_k$

$$\widehat{\boldsymbol{\theta}}_k = \arg \min_{\boldsymbol{\theta}_k} \left\{\widehat{\mathbf{b}}_k^* \left[\widehat{\mathbf{Q}}^{-1} - \widehat{\mathbf{Q}}^{-1}\mathbf{A}(\boldsymbol{\theta}_k)\left(\mathbf{A}^*(\boldsymbol{\theta}_k)\widehat{\mathbf{Q}}^{-1}\mathbf{A}(\boldsymbol{\theta}_k)\right)^{-1}\mathbf{A}^*(\boldsymbol{\theta}_k)\widehat{\mathbf{Q}}^{-1}\right]\widehat{\mathbf{b}}_k\right\}. \tag{3.7}$$

Once $\widehat{\boldsymbol{\theta}}_k$ is found from (3.7), the amplitude estimates $\boldsymbol{\gamma}_k$ are obtained from (3.6).

We remark that the above algorithm is consistent; this follows from the consistency of the exact ML and the asymptotic equivalence of the CDEML and ML methods.

4 Statistical Analysis

In this section we state some results on the statistical properties of the CDEML algorithm; the proofs are given in the full version of the paper. The asymptotic statistical properties of the parameter estimates are stated in Theorem 1. Theorem 2 gives the CRB for the corresponding signal model. Theorem 3 states that the CDEML algorithm is asymptotically efficient for diagonal $\mathbf{R}_{yy}$.

Theorem 1 *Let $\boldsymbol{\alpha} = [\boldsymbol{\theta}^T \;\; \mathrm{Re}\{\boldsymbol{\gamma}\}^T \;\; \mathrm{Im}\{\boldsymbol{\gamma}\}^T]^T$ be the $(3d \times 1)$ parameter vector, and let $\widehat{\boldsymbol{\alpha}} =$ be the corresponding CDEML estimate obtained using equations (3.7) and (3.6). If $\mathbf{R}_{yy}$ is diagonal, then the normalized asymptotic (large N) covariance matrix of $\widehat{\boldsymbol{\alpha}}$ is given by:*

$$\mathrm{E}\left((\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})^T\right) = \frac{1}{2N}\mathbf{H}^{-1}\mathbf{V}\mathbf{H}^{-1}, \tag{4.1}$$

$$\mathbf{H} = \begin{bmatrix} \mathrm{Re}(\mathbf{H}_1) & \mathrm{Re}(\mathbf{H}_2^T) & \mathrm{Im}(\mathbf{H}_2^T) \\ \mathrm{Re}(\mathbf{H}_2) & \mathrm{Re}(\mathbf{H}_3) & -\mathrm{Im}(\mathbf{H}_3) \\ \mathrm{Im}(\mathbf{H}_2) & \mathrm{Im}(\mathbf{H}_3) & \mathrm{Re}(\mathbf{H}_3) \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} \mathrm{Re}(\mathbf{V}_1) & \mathrm{Re}(\mathbf{V}_2^T) & \mathrm{Im}(\mathbf{V}_2^T) \\ \mathrm{Re}(\mathbf{V}_2) & \mathrm{Re}(\mathbf{V}_3) & -\mathrm{Im}(\mathbf{V}_3) \\ \mathrm{Im}(\mathbf{V}_2) & \mathrm{Im}(\mathbf{V}_3) & \mathrm{Re}(\mathbf{V}_3) \end{bmatrix}$$

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{D}^*\mathbf{Q}^{-1}\mathbf{D} \odot (\boldsymbol{\Gamma}\boldsymbol{\Gamma}^*)^T & \qquad \mathbf{V}_1 &= \mathcal{D}^*\left(\mathbf{R}_{yy}^{-T} \otimes \mathbf{Q}\right)\mathcal{D} \\ \mathbf{H}_2 &= \mathbf{A}^*\mathbf{Q}^{-1}\mathbf{D} \odot (\boldsymbol{\Gamma}\mathbf{E}_{\boldsymbol{\Gamma}}^*)^T & \qquad \mathbf{V}_2 &= \mathcal{D}^*\left(\mathbf{R}_{yy}^{-T} \otimes \mathbf{Q}\right)\mathcal{A} \\ \mathbf{H}_3 &= \mathbf{A}^*\mathbf{Q}^{-1}\mathbf{A} \odot (\mathbf{E}_{\boldsymbol{\Gamma}}\mathbf{E}_{\boldsymbol{\Gamma}}^*)^T & \qquad \mathbf{V}_3 &= \mathcal{A}^*\left(\mathbf{R}_{yy}^{-T} \otimes \mathbf{Q}\right)\mathcal{A} \end{aligned}$$

$\mathcal{D} = \mathrm{diag}\{\mathbf{Q}^{-1}\mathbf{D}_k\gamma_k\}_{k=1}^{c}$, $\mathcal{A} = \mathrm{diag}\{\mathbf{Q}^{-1}\mathbf{A}_k\}_{k=1}^{c}$, $\mathbf{D} = [\mathbf{D}_1, \ldots, \mathbf{D}_c] = [\mathbf{d}_{11}, \ldots, \mathbf{d}_{1d_1}, \ldots, \mathbf{d}_{cd_c}]$, *where* $\mathbf{d}_{kl} \triangleq \frac{\partial \mathbf{a}(\theta_{kl})}{\partial \theta_{kl}}$, $\mathbf{E}_{\mathbf{\Gamma}}$ *denotes a matrix of the same dimensions as* $\mathbf{\Gamma}$ *in (2.5), but with the* γ_{kl} *replaced by ones, and* $\otimes$ *denotes Kronecker product.*

Theorem 2 *For the signal model in Section 2 under Assumptions 1–4, and for* $\widehat{\mathbf{R}}_{yy} > 0$, *the CRB of* $\boldsymbol{\alpha}$ *is given by:*

$$\mathrm{CRB}(\boldsymbol{\alpha}) = \frac{1}{2N} \begin{bmatrix} \mathrm{Re}(\mathbf{F}_1) & \mathrm{Re}(\mathbf{F}_2^T) & \mathrm{Im}(\mathbf{F}_2^T) \\ \mathrm{Re}(\mathbf{F}_2) & \mathrm{Re}(\mathbf{F}_3) & -\mathrm{Im}(\mathbf{F}_3) \\ \mathrm{Im}(\mathbf{F}_2) & \mathrm{Im}(\mathbf{F}_3) & \mathrm{Re}(\mathbf{F}_3) \end{bmatrix}^{-1}, \tag{4.2}$$

$$\begin{aligned} \mathbf{F}_1 &= \mathbf{D}^*\mathbf{Q}^{-1}\mathbf{D} \odot \left(\mathbf{\Gamma}\widehat{\mathbf{R}}_{yy}\mathbf{\Gamma}^*\right)^T, \qquad \mathbf{F}_2 = \mathbf{A}^*\mathbf{Q}^{-1}\mathbf{D} \odot \left(\mathbf{\Gamma}\widehat{\mathbf{R}}_{yy}\mathbf{E}_{\mathbf{\Gamma}}^T\right)^T, \\ \mathbf{F}_3 &= \mathbf{A}^*\mathbf{Q}^{-1}\mathbf{A} \odot \left(\mathbf{E}_{\mathbf{\Gamma}}\widehat{\mathbf{R}}_{yy}\mathbf{E}_{\mathbf{\Gamma}}^T\right)^T, \end{aligned}$$

and where $\mathbf{D}$ *and* $\mathbf{E}_{\mathbf{\Gamma}}$ *are defined as in Theorem 1.*

If $\mathbf{R}_{yy}$ is diagonal, it can be shown that the right-hand sides of equations (4.1) and (4.2) are asymptotically equivalent, giving:

Theorem 3 *When* $\mathbf{R}_{yy}$ *is diagonal, the CDEML algorithm is asymptotically statistically efficient.*

5 Numerical Example

We examine the performance of CDEML for a uniform linear array with half wavelength spacing and 10 elements. There are two known source signals; one arrives at 5°, and the other arrives from two directions, 0° and 10°. The source signals are random Gaussian sequences and are uncorrelated. The SNR of each received signal is 0 dB, and

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T.$$

Figure 1a shows the RMSE and the CRB of the DOA estimates obtained from the CDEML algorithm for different numbers of snapshots, N. The solid lines are the CRB standard deviations for the three received signals; the lowest curve is for the single source from 5°, and the two upper curves are for the multipath signals arriving at 0° and 10°. Since the sources are uncorrelated, these curves also represent the asymptotic performance of the CDEML algorithm. The circles and 'x's are the DOA RMSE obtained from 100 Monte-Carlo simulations. This figure numerically verifies that the algorithm is asymptotically efficient.

Figure 1b shows the RMSE of the CDEML estimates and the CRBs as a function of array size m. As the array size increases, the CRB of the two multipath signals approach that of the single source. Again, the simulation performance agrees closely with the statistical theory.

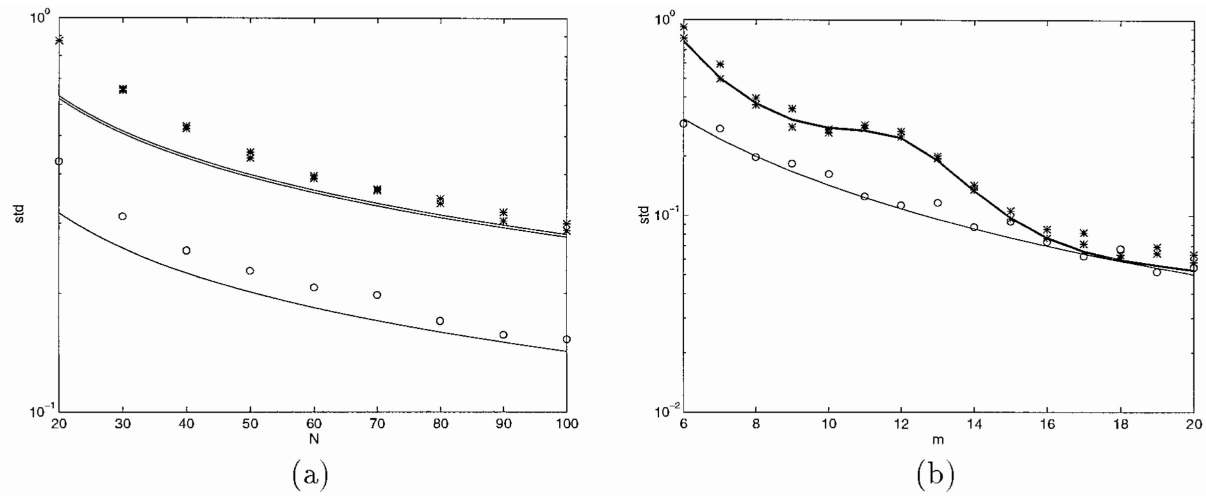

Figure 1: CRB (solid lines) and RMSEs (x and $\circ$) of DOA estimates for the CDEML algorithm, as a function of: (a) number of data samples N, and (b) array size m.

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