ON MODEL ORDER DETERMINATION FOR COMPLEX EXPONENTIAL SIGNALS: PERFORMANCE OF AN FFT-INITIALIZED ML ALGORITHM

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ABSTRACT

We present an algorithm for model order determination and simultaneous maximum likelihood parameter estimation for complex exponential signal modeling. The algorithm exploits initial nonparametric (*i.e.*, FFT) frequency location estimates and Cramér-Rao Bound (CRB) resolution limits to significantly reduce the search space for the correct model order and parameter estimates. The algorithm initially overestimates the model order. After iterative minimization to obtain maximum likelihood (ML) parameter estimates for that order, a post-processing step eliminates the extraneous sinusoidal modes using CRB resolution limits and statistical detection tests. Because the algorithm searches on only a limited set of model orders and parameter regions, it is computationally tractable even for large data lengths and model orders. In this paper we analyze the performance of the algorithm and compare with other existing approaches.

1. INTRODUCTION

A parametric modeling problem can be divided into two parts: model order detection and parameter estimation. Usually, model order detection and parameter estimation are considered separately. However, it is well known that the performance of parameter estimation algorithms is greatly affected by the pre-determined model order. In particular, with a lower model order, the estimated parameters become biased and with a higher model order, the variances of the estimated parameters increase [2, 3]. Therefore, it is important to have an accurate estimate of the model order. In this paper, we consider the model order determination issue for the complex exponential modeling problem.

For array signal processing, model order determination using information-theoretic criteria has been addressed in [4, 5, 6]. These algorithms search the entire model order space in order to find the correct model order according to some information criterion (AIC, MDL, etc.). For nonparametric algorithms (*e.g.*, [4]), the computation cost is modest but performance suffers even for moderate signal to noise ratios (SNRs). In contrast, parametric algorithms [5, 6] give good results but are computationally expensive.

For time series analysis, several authors have used eigenvalue or singular value analysis in the detection problem [7, 8]. The basic idea there is to identify the dominant singular values (SVs) (or eigenvalues) by some criterion (for instance, a pre-chosen threshold). The number of the dominant SVs is the model order. The technique in [7] requires the selection of a pre-determined threshold; both [7] and [8] suffer from poor detection performance. With the help of perturbation analysis, Fuchs [9] developed a statistical criterion based on the data autocorrelation matrix to detect the number of sinusoids. The approach requires a pre-determined χ^2 threshold and is computationally intensive. More recently, Reddy and Biradar [10] utilized information-theoretic criteria on the data matrix for detecting the number of damped/undamped sinusoids. This method is computationally attractive but the performance suffers even for moderate SNRs (as we show below).

Recently, we proposed a computationally tractable alternative for model order determination in [1]. The algorithm uses an initial nonparametric frequency estimate, obtained using an FFT, to reduce considerably the set of candidate model order hypotheses. In addition, we significantly reduce the initial condition points for nonlinear minimization procedures by use of an FFT initialization which is combined with a mode splitting algorithm to account for the limited FFT resolution. The splitting algorithm is based on the CRB resolution limit. Because the initialization procedure tends to initially overestimate the model order, we include a post-processing step to test for and eliminate extraneous modes in the model. This process of intelligently searching for model order and parameter estimates in a reduced subset of the entire space results in significant computational savings. In this paper we analyze the performance of the algorithm through theoretical derivations and extensive simulations. We also compare our approach with other existing methods, in particular the ones in [9, 10].

An outline of this paper is as follows. In Section 2 we present the data model and the problem formulation. In Section 3 we summarize the proposed algorithm. In Section 4 we show the consistency of the algorithm. In Section 5 we present simulation studies which demonstrate the ability of the proposed algorithm. We also compare the algorithm with other existing model order determination algorithms. Finally, in Section 6 we conclude the paper.

2. DATA MODEL AND PROBLEM FORMULATION

Consider a data vector $\{y_q\}_{q=0}^{m-1}$ of complex exponential samples

$$y_q = \sum_{i=1}^{n} \alpha_i e^{j(\omega_i q + \phi_i)} + e_q \qquad (1)$$

where $\alpha_i \in R$, ω_i, $\phi_i \in (-\pi, \pi]$. We assume that the exponential modes are distinct, *i.e.*, $\omega_k \neq \omega_l$ for $k \neq l$. We also assume that $\{e_q\}$ is a zero mean complex white Gaussian noise sequence with variance σ^2.

There are two related problems in exponential modeling. One is the estimation problem, *i.e.*, to estimate the parameters, $\{\alpha_i\}_{i=1}^{n}$, $\{\phi_i\}_{i=1}^{n}$, and $\{\omega_i\}_{i=1}^{n}$ in the model. The other problem is to determine the number of modes in the signal. In this paper we consider a combined parameter/order estimation algorithm for the model in Equation 1.

To address the order determination problem, we use results of maximum likelihood (ML) estimation of the parameter vector:

$$\theta = [\ \{\alpha_i\}_{i=1}^n \quad \{\phi_i\}_{i=1}^n \quad \{\omega_i\}_{i=1}^n\ ]^T. \tag{2}$$

Under the assumptions made, the negative log-likelihood function for an estimate $\widehat{\theta}$ of θ is given as [11, 12]

$$L(\widehat{\theta}) = \sum_{q=0}^{m-1} \left| y_q - x_q(\widehat{\theta}) \right|^2, \tag{3}$$

where $x_q(\widehat{\theta})$ is an estimate of the noiseless version of y_q parameterized by $\widehat{\theta}$. The ML estimate is then given by

$$\widehat{\theta}_{ML} = \arg\min_{\widehat{\theta}} L(\widehat{\theta}). \tag{4}$$

Note that the ML estimate given in Equation 4 is only valid if the number of modes in the estimate is correct, *i.e.*, $\hat{n} = n$. By increasing the number of modes in the estimate, we can always reduce the loss function $L(\widehat{\theta})$.

We design an algorithm which estimates the correct model order and the corresponding ML parameter estimates. We first obtain initial estimates using FFT peaks. We then apply a splitting algorithm that is based on a criterion derived from CRB results to refine the initial estimates. We next apply an ML procedure (using Equation 4) to these refined initial estimates. After the ML step, we prune the extra modes in the estimate. We repeat the ML estimation and the pruning cycle until no mode is pruned. The number of modes in the final estimate is the estimate of the model order; the final estimate is also the ML estimate of the model parameters.

We note that the above proposed algorithm simultaneously estimates both the model order and model parameters. Moreover, the algorithm does not search the entire space for the possible model order. The algorithm only considers a small subset of the entire space. This results in significantly reduced computation as compared to algorithms that search over possible model orders using ML estimation.

3. SUMMARY OF THE PROPOSED ALGORITHM

As described in the previous section, the proposed model order determination algorithm is based on the FFT peaks of the given signal. However, the FFT has limited resolution and thus cannot detect (and estimate) closely spaced modes. To solve this resolution limitation problem, we present an algorithm to split the FFT peaks into some larger number of peaks based on the resolution limit obtained from the CRB. This splitting algorithm introduces additional modes. Consequently, we increase the model order in this step, and often initially overestimate the model order. To prune any extra modes, we design a pruning algorithm which is based on the resolution limit and statistical detection tests.

For the sake of brevity, we summarize the essential idea of the algorithm; details are presented in [1]. The model-order-revealing ML algorithm is summarized as follows.

1. Obtain the locations, magnitudes, and phases of the FFT peaks of the signal. Obtain the estimated noise power $\widehat{\sigma^2}$.
2. Using the estimated noise power, calculate the resolution limit and the maximum number of resolvable modes per Fourier bin (Fbin). Split the peak locations according to the splitting algorithm described in [1].
3. Perform ML estimation on the split estimates using nonlinear minimization of the negative log-likelihood function (Equation 4). We use the complex signals-generalized version of the algorithm in [12].
4. Discard possibly extraneous modes using the MDL information criterion (see [1] for details.)
5. If any modes are discarded in Step 4, go to Step 3. If not, the estimation is completed.

4. CONSISTENCY OF THE PROPOSED ALGORITHM

In this section we establish the consistency of the proposed model order determination algorithm in terms of SNR. First, we note that the MDL criterion for our case is given as

$$\text{MDL} = m \log\left(L(\widehat{\theta})\right) + \frac{3\widehat{n}}{2} \log(m), \tag{5}$$

where $L(\widehat{\theta})$, m, and $\widehat{n}$ are defined as above. To show the consistency, we want to show that as $\sigma^2 \to 0$,

$$Prob\left\{ \frac{L(\widehat{\theta}_n)}{L(\widehat{\theta}_{\hat{n}})} > e^{\frac{3(\hat{n}-n)\log(m)}{2m}} \right\} \to 0 \quad \forall\, \hat{n} \neq n. \tag{6}$$

It is noted that as $\sigma^2 \to 0$, $\frac{L(\widehat{\theta}_n)}{L(\widehat{\theta}_{\hat{n}})} \to 0$ for $\hat{n} < n$ and $\frac{L(\widehat{\theta}_n)}{L(\widehat{\theta}_{\hat{n}})} \to 1$ for $\hat{n} > n$. However, $\hat{n} < n \Rightarrow e^{\frac{3(\hat{n}-n)\log(m)}{2m}} > 0$, and $\hat{n} > n \Rightarrow e^{\frac{3(\hat{n}-n)\log(m)}{2m}} > 1$. Therefore, the probability of error (Equation 6) goes to zero as $\sigma^2 \to 0$. This is also verified in the following computer simulation studies.

5. SIMULATION STUDIES

In this section we provide computer simulations to demonstrate the performance of the proposed algorithm. We also compare the algorithm with existing model order selection algorithms. In particular, we compare with Fuchs' approach [9] and the SVD-ITC approach [10]. Note that the approaches in [9] and [10] were designed for real sinusoids, and here we generalize the approaches to be suitable for complex exponentials. For Fuchs' approach, the χ^2 threshold is chosen so that the probability of underestimating model order is equal to 0.1 percent. For the SVD-ITC approach, we only consider the backward linear prediction incorporated with the MDL criterion scenario. For both Fuchs' and the SVD-ITC approaches, the prediction order (size of the data matrix) is chosen to be about $m/3$ to improve estimation accuracy (see, *e.g.* [13]), where m is the number of data points.

5.1. Example One: Two Well Separated Modes

In the first example we use a data sequence which is composed of $n = 2$ exponentials at frequencies of $2\pi/m$ and $-2\pi/m$ where $m = 25$ is the data length. The magnitudes of the exponentials are chosen to be one. The phases are zero.

Figure 1 shows the correct model order detection rate versus SNR per mode for this example, using 100 independent Monte-Carlo simulations. Here, SNR per mode is defined as

$$\text{SNR per mode} = 10 \log_{10} \frac{\min_i(|\alpha_i^2|)}{\sigma^2},$$

where $\alpha_1 = \alpha_2 = 1$ for this case. From the plot, we can see that the proposed algorithm (curve Y&M) successfully detects the model order for SNR/mode above 1 dB; in particular, the algorithm is consistent. Below 1 dB SNR, the detection performance degrades quickly. The phenomenon is

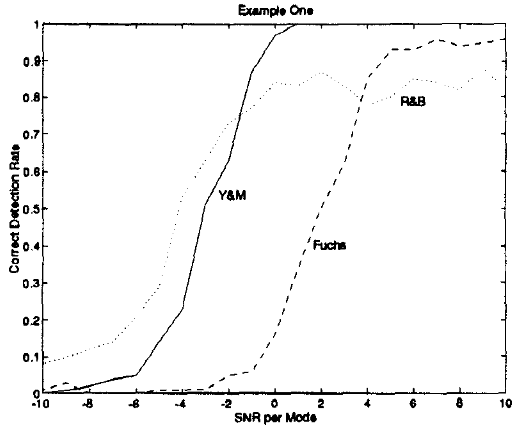

Figure 1. Detection results of Example One using the proposed (Y&M), Fuchs', and SVD-ITC (R&B) algorithms.

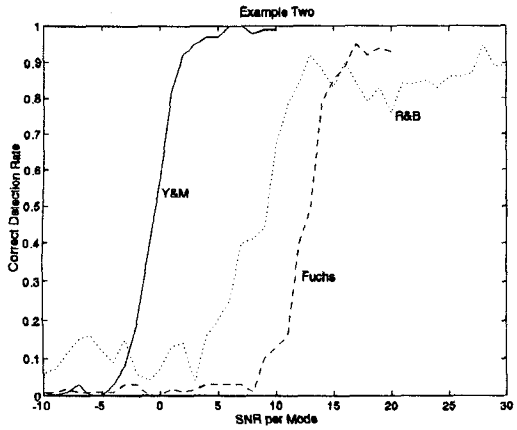

Figure 2. Detection results of Example Two using the proposed (Y&M), Fuchs', and SVD-ITC (R&B) algorithms.

the so-called threshold effect in [14]. The detection threshold for Fuchs' approach is 4 dB and that for the SVD-ITC approach (curve R&B) is about 0 dB. Note, however, that neither Fuchs' algorithm nor the SVD-ITC approach gives perfect detection rates even for SNRs above 10 dB; the consistency of the algorithms is not discussed in [9] and [10].

5.2. Example Two: Two Closely Spaced Modes

In this example we use the following data.

$$x_q = e^{j(2\pi(0.52)q+\pi/4)} + e^{j2\pi(0.5)q} \quad q = 0, 1, \ldots, 24.$$

This is the Kumaresan and Tufts example [15, 16].

Figure 2 shows the correct detection rate versus SNR per mode. Again, 100 independent Monte-Carlo simulations were run for the statistics, and SNR per mode is defined as before. From the figure, we see that the proposed algorithm can detect the correct model order for SNR/mode above 5 dB with false alarm rate below 2%. Again, we see the threshold effect for this example. The correct detection threshold is about 5 dB for the proposed algorithm. Fuchs' algorithm nor the SVD-ITC approach perfectly detect the model order for the SNR range we tried. The detection performance threshold is about 15 dB for Fuchs' approach and 13 dB for the SVD-ITC approach. The proposed algorithm requires an SNR/mode which is about 7-10 dB lower than that required by Fuchs' and the SVD-ITC approaches to achieve the same performance for this example.

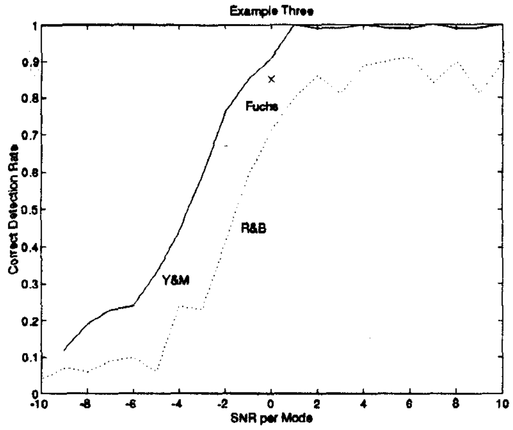

Figure 3. Detection results of Example Three using the proposed (Y&M), Fuchs', and SVD-ITC (R&B) algorithms.

5.3. Example Three

We also compare the three algorithms on an example similar to the one in [9, 10]. The data sequence is given as

$$x_q = \sqrt{20}\sin(2\pi f_1 q) + \sqrt{2}\sin(2\pi f_2 q + \phi) \quad q = 0, 1, \ldots, 63,$$

where $f_1 = 0.2$, $f_2 = 0.2 + 1/64$, and $\phi = 0$. Although the signal is real, we use complex representation in the experiments, which means we have 4 complex exponentials instead of 2 sinusoids.

Figure 3 shows the detection performance results. For compatibility with examples presented in [9, 10], we adopt prediction order $L = 32$ (not $L = m/3$) in computing results for Fuchs' and the SVD-ITC approaches. Because of the computation required, we show results for Fuchs' method at only one SNR. From the figure, we see that the proposed algorithm has the best performance in terms of detecting the model order. The three algorithms seem to have a similar detection performance threshold (0-1 dB).

The results for Fuchs' algorithm and for the SVD-ITC approach are slightly worse than those reported in [9, 10] because we use complex signals instead of real signals and did not randomize the phase. Simulations we performed using real signals (not reported here) gave performance results similar to those in [9, 10].

5.4. Discussion

From the above experiments, we find that the proposed algorithm is consistent in SNR. Fuchs' approach and the SVD-ITC approach seem to be consistent from the derivation of the algorithms. However, the experiments show significant misestimation of order even above 10 dB SNR.

For the detection performance threshold, the three algorithms have similar results for well separated signals (Example One and Three). However, for closely space signals (Example Two), the proposed algorithm outperforms the other two approaches by about 7-10 dB. Below the threshold SNR, we found that most of the misestimated cases for the proposed algorithm and Fuchs' approach are underestimated. However, for the SVD-ITC approach, the misestimated cases are overestimated for high SNRs and underestimated for low SNRs.

Although the SVD-ITC approach uses the MDL principle, the detection performance is in general worse than the one obtained from our proposed algorithm. We hypothesize the performance degradation is due to the data matrix formation, which does not fully use the whole data sequence. The problem is similar to the well-known windowing effect in FIR filtering. The effect is most pronounced when the signals are closely spaced and the data length is small.

Table 1. Computation required for the algorithms. Results were obtained from **MATLAB** flop counts (Unit: flop).

	Ex. 1	Ex. 2	Ex. 3
Y&M	4.52M	1.85M	10.27M
SVD-ITC	68.4K	68.4K	1.51M
Fuchs	5.85M	5.85M	7085.5M

Next we compare the computational cost of each order detection procedure. The computation required by each algorithm for each example is summarized in Table 1. From the table, we see that the SVD-ITC approach requires the least computation, the proposed algorithm more, and Fuchs' approach the most. We note that the computations for the SVD-ITC approach and Fuchs' approach are for detecting the model order only, and the computations for the proposed algorithm include both detecting the model order and estimating the ML parameter estimates. For Fuchs' algorithm, the computation for detection only is worse than that of the proposed algorithm for both detection and estimation. If an ML estimation procedure is used to estimate the model parameters after detecting the order, the computation difference will be even more significant. However, for the SVD-ITC approach, even if an ML estimation procedure is used to estimate the model parameters, the computation should still be about one third or one half of the computation needed for the proposed algorithm. This is because the proposed algorithm performs the ML estimation cycle for several times to determine the model order. Although the SVD-ITC approach is attractive in terms of computation, the detection performance is not promising, especially for closely spaced signals.

Also from the table, we see that the computation of the proposed algorithm for Example Two is lower than that of Example One although the data lengths are identical. This is because the average number of ML estimation cycles required in Example Two is about one-half of that for Example One. For the proposed algorithm, most of the computation required is due to the ML estimation step.

For Fuchs' approach, the computation is much higher than that of the SVD-ITC approach or the proposed algorithm. Most of the computation is in estimating the covariance matrix of the perturbed eigenvalues and inverting the covariance matrix. The approach is expected to have better detection performance for longer data lengths in keeping with the asymptotic assumption adopted in deriving the method. However, for longer data lengths (as in Example Three), the computational cost is unattractive.

6. CONCLUSIONS

We have presented an approach for determining the model order of complex exponential signals. The detection of model order and estimation of model parameters were treated simultaneously in the proposed algorithm.

We have shown that the proposed algorithm gives model order estimates that are consistent in terms of SNR. We have used computer simulations to verify the result.

Computer simulations demonstrated the model order detection performance of the proposed algorithm. We have also compared the algorithm with existing order selection algorithms in [9, 10]. We found that the algorithm generally outperforms the other two approaches in terms of detecting the model order, especially for closely spaced signals. The computation required for the proposed order determination algorithm is between the other two approaches with the advantage that the proposed algorithm also provides the ML parameter estimates.

7. ACKNOWLEDGMENT

This research was supported in part by ARPA under Grant MDA972-93-1-0015, and in part by Wright Laboratory.

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