Fast Recursive AR Estimation from an Overdetermined System of Extended Yule Walker Equations

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ABSTRACT

This paper presents a fast recursive least squares algorithm for estimating the AR coefficients in an AR or ARMA model. The algorithm is based on solving an overdetermined system of "t" Extended Yule Walker equations for the "p" AR coefficients. Experimental results indicate that the proposed algorithm is able to provide improved estimates over those of similar recursive algorithms, at a computational cost of an additional 2t multiplies and 2t adds.(1)

INTRODUCTION

In the past several years there has been a growing interest in developing online procedures for identifying the essential attributes of a time series $\{x(n)\}$. Often this identification takes the form of an ARMA (p,q) model, or of two special cases, the AR(p) or MA(q) models. The ARMA (p,q) model arises from assuming the time series to be the output of a causal linear filter of the form

$$x(k) = -\sum_{i=1}^{p} a_i x(k-i) + \sum_{j=0}^{q} b_j w_{k-j} \tag{1}$$

where the series $\{w(n)\}$ is zero mean, unit variance white noise. It is well known that a time series $\{x(n)\}$ satisfying (1) has its power spectral density function $S_x(e^{j\omega})$ given by

$$S_x(e^{j\omega}) = \left| \frac{b_0 + b_1 e^{-j\omega} + \ldots + b_q e^{-jq\omega}}{1 + a_1 e^{-j\omega} + \ldots + a_p e^{-jp\omega}} \right|^2 \tag{2}$$

The a_i and b_j coefficients are referred to as the autoregressive (AR) coefficients and the moving average (MA) coefficients, respectively.

Several procedures have been developed for estimating the ARMA model's a_i and b_j coefficients from a given finite set of observations $x(1)$, $x(2),\ldots,x(n)$. On-line, or recursive procedures, form this estimate in such a way that when a next observation $x(n+1)$ is measured, the coefficient estimates based on the first n data observations are updated in a computationally efficient manner. Usually a tradeoff exists between the number of computations required per update and the speed at which the coefficient estimates converge.

One popular strategy for obtaining coefficient estimates is to first estimate the AR coefficients, then to use these estimates to in turn obtain the MA coefficient estimates(1) [1],[3],[4],[5]. One advantage of this technique is that spectral estimates can be obtained by solving only linear equations. If this technique is to be used, one must first determine a suitable procedure for recursive AR coefficient estimation.

An important subset of recursive AR coefficient estimators are the so-called "fast recursive least squares" algorithms. These algorithms are computationally efficient recursive implementations of a set of off-line algorithms that are based on the Extended Yule Walker (EYW) equations. It is well known that the AR coefficients of an ARMA(p,q) time series $\{x(n)\}$ satisfy the equations

$$r_x(k) + \sum_{i=1}^{p} a_i r_x(k-i) = 0 \qquad k>q \tag{3}$$

(1) In some problems, such as spectral estimation, the MA coefficients are implicitly determined by estimating a set of related parameters, c.f. [1].

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where $r_x(k) = E\{x(n)x(n-k)\}$ is the autocorrelation function associated with the time series. The first "t" of the equations (3) may be written in matrix form as

$$\begin{bmatrix} r_x(q+1) & r_x(q) & \ldots r_x(q-p+1) \\ r_x(q+2) & r_x(q+1) & \ldots r_x(q-p+2) \\ \cdot & \cdot & \cdot \\ \cdot \quad = & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ r_x(q+t) & r_x(q+t-1) & \ldots r_x(q-p+t) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

or more compactly as

$$\underline{r} + R\,\underline{a} = \underline{0} \tag{4}$$

The given data set can be used to form the estimates $\underline{r}_n$ and R_n of $\underline{r}$ and R resulting in the equations

$$\underline{r}_n + R_n\underline{a}_n = \underline{e}_n \tag{5}$$

where the right hand side of (4) has been replaced by an error vector $\underline{e}_n$ since the equations (5) are not, in general, consistent for $t>p$.

There are several ways to form the estimates $\underline{r}_n$ and R_n. The estimates chosen here are the "prewindow" estimates

$$\begin{aligned} r_n &= Y_n^T\underline{x}_n \\ R_n &= Y_n^T X_n \end{aligned} \tag{6}$$

where

$$X_n = \begin{bmatrix} 0 & 0 & 0 \\ x(q+1) & 0 & \cdot \\ x(q+2) & 0 & \cdot \\ \cdot & x(q+1) & \cdot \\ \cdot & \cdot & x(q+1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x(n-1) & x(n-2) & x(n-p) \end{bmatrix} \tag{7a}$$

$$Y_n = \begin{bmatrix} 0 & 0 & 0 \\ x(1) & 0 & 0 \\ x(2) & x(1) & \cdot \\ \cdot & \cdot & x(1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x(n-q-1) & x(n-q-2) & x(n-q-t) \end{bmatrix} \tag{7b}$$

$$\underline{x}_n = [x(q+1) \quad x(q+2) \ \ldots \ x(n)]' \tag{7c}$$

For the special case $t=p$, fast recursive algorithms have been developed to find exact solutions to equation (5) for both the AR case ($q=0$) [2] and the ARMA case ($q\geq 0$) [1]. However, it has been noted [1],[3],[5] that by choosing t greater than the minimal value of p and by finding the least-squared error solution for $\underline{a}_n$ in (5), one can often obtain more accurate AR coefficient estimates. This paper extends the applicability of the $t>p$ estimator by presenting a fast recursive algorithm that is based on an approximation to equation (5) for the case $t>p$.

<u>Derivation of the Algorithm</u>

For $t>p$, the least squared error solution to (5) is given by

$$\underline{a}_n = -[(Y_n'X_n)'(Y_n'X_n)]^{-1}(Y_n'X_n)'(Y_n'\underline{x}_n) \tag{8}$$

where X_n, Y_n, and $\underline{x}_n$ are given in equation (7). It is desired to determine $\underline{a}_{n+1}$ from $\underline{a}_n$ in a computationally efficient manner. This can be accomplished by considering the <u>pth</u> order forward prediction error vector, defined by

$$\underline{f}^x_{p,n} = \underline{x}_n + X_n\underline{a}_n \tag{9}$$

The (k-q)<u>th</u> element of $\underline{f}^x_{p,n}$ can be interpreted as the error that results from estimating $x(k)$ by a linear combination of the p most recent observations $x(k-1),\ldots,x(k-p)$. When the AR coefficient vector (8) is substituted in equation (9), the resulting "optimal" forward prediction error vector becomes

$$\begin{aligned} \underline{f}^x_{p,n} &= \{I-X(YY'X)'X]^{-1}(YY'X)'\}\underline{x}_n \\ &= \{I-X[Z'X]^{-1}Z'\}\underline{x}_n \end{aligned} \tag{10}$$

where $Z = Z_n = Y_nY_n'X_n$ and $X = X_n$.

It was shown in [1] that a prediction error of this form results from projecting the vector $\underline{x}_n$ onto the subspace $M_x = \text{span}\{\text{columns of } X_n\}$, where the projection is orthogonal to $M_z = \text{span}\{\text{columns of } Z_n\}$.

A recursive AR coefficient estimation algorithm that is based on forming updates of the prediction error vector is given in [1], but this algorithm requires the nonsquare matrices X_n and Z_n to be "lower triangular" and "Toeplitz", i.e.

$$\begin{aligned} (X_n)_{i,j} &= (X_n)_{i-j} \text{ for } i>j \\ & \quad 0 \text{ for } i\leq j \end{aligned} \tag{11}$$

and similarly for Z_n. Unfortunately, the matrix $Z_n = Y_nY_n'X_n$ is not in the form of equation (11). In order to obtain a computationally efficient recursive algorithm, the matrix Z_n is approximated by one which does satisfy (11). In particular, define Z_n as

$$\begin{aligned} (Z_n)_{i,j} &= z(q+i-j)\ , \ i>j \\ & \quad 0,\ i\leq j \end{aligned} \tag{12}$$

where

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$$z(k) = [0,0,\ldots 0,1]Y_k Y_k \underline{x}_k \quad (13)$$

Substituting Z_n for $Y_n Y_n' X_n$ in eq. (8) yields the AR coefficient vector estimate

$$\underline{a}_n = -[Z_n' X_n]^{-1}[Z_n' \underline{x}_n] \quad (14)$$

The structure of Z_n, now permits the derivation of a recursive algorithm that computes the exact solution to (14) at every time n. This derivation follows the one in [1], with Z_n replacing Y_n. The resulting set of order update and time update equations are

$$f^x_{m+1,n} = f^x_{m,n} - b^x_{m,n-1}\sigma_{m,n}/\omega_{m,n-1} \quad (15)$$

$$f^z_{m+1,n} = f^z_{m,n} - b^z_{m,n-1}\tau_{m,n}/\omega_{m,n-1} \quad (16)$$

$$b^x_{m+1,n} = b^x_{m,n-1} - f^x_{m,n}\tau_{m,n}/\mu_{m,n} \quad (17)$$

$$b^z_{m+1,n} = b^z_{m,n-1} - f^z_{m,n}\sigma_{m,n}/\mu_{m,n} \quad (18)$$

$$\mu_{m+1,n} = \mu_{m,n} - \tau_{m,n}\tau_{m,n}/\omega_{m,n-1} \quad (19)$$

$$\omega_{m+1,n} = \omega_{m,n-1} - \sigma_{m,n}\tau_{m,n}/\mu_{m,n} \quad (20)$$

$$\gamma_{m+1,n} = \gamma_{m,n} - b^x_{m,n-1}b^z_{m,n-1}/\omega_{m,n-1} \quad (21)$$

$$\sigma_{m,n} = \sigma_{m,n-1} + f^x_{m,n}b^z_{m,n-1}/\gamma_{m,n} \quad (22)$$

$$\tau_{m,n} = \tau_{m,n-1} + b^x_{m,n-1}f^z_{m,n}/\gamma_{m,n} \quad (23)$$

$$\mu_{m,n} = \mu_{m,n-1} + f^x_{m,n}f^z_{m,n}/\gamma_{m,n} \quad (24)$$

$$\omega_{m,n} = \omega_{m,n-1} + f^x_{m,n}b^z_{m,n}/\gamma_{m+1,n+1} \quad (25)$$

The algorithm is initialized by

$$\sigma_{m,q+1} = \tau_{m,q+1} = 0 \quad m=0,1,\ldots,p-1$$

$$f^x_{0,n} = b^x_{0,n} = x(n) \quad (26)$$

$$f^z_{0,n} = b^z_{0,n} = z(n) \quad n=q+2,q+3,\ldots$$

$$\gamma_{0,n} = 1$$

where $z(n)$ is given by (12) and (13). Note that $z(n)$ may be efficiently computed if the elements of the vector $(Y_n'\underline{x}_n)$ are stored and updated. The update equation for the ith element of $(Y_n'\underline{x}_n)$ is

$$(Y_{n+1}'\underline{x}_{n+1})_i = (Y_n'\underline{x}_n)_i + x(n+1)x(n+1-q-i) \quad (27)$$

The element $z(n+1)$ is then calculated by

$$z(n+1) = \sum_{i=1}^{t} (Y_{n+1}'\underline{x}_{n+1})_i\, x(n+1-q-i) \quad (28)$$

A total of 2t multiplies and 2t adds are needed to perform the computations in (27) and (28).

The recursive algorithm may be implemented using the lattice filter preceded by a preprocessor that computes $z(n)$. This preprocessor section implements equations (27) and (28). The preprocessor represents the only difference between this algorithm and the fast recursive t=p algorithm [1] (in the latter algorithm, the preprocessor was a qth order time delay). The total number of computations required to update the proposed algorithm is 18p + 2t multiplies and 9p + 2t adds, as compared with 18p multiplies and 9p adds for the fast recursive t=p algorithm.

The autoregressive coefficients can be recovered from the lattice coefficients by setting $A_{0,n}(z) = \tilde{A}_{0,n}(z) = 1$ and by recursively calculating for m=1, 2,...,p the polynomials

$$A_{m,n}(z) = A_{m-1,n}(z) + z^{-1}\frac{\sigma_{m,n}}{\omega_{m,n-1}}\tilde{A}_{m-1,n}(z) \quad (29)$$

$$\tilde{A}_{m,n}(z) = z^{-1}\tilde{A}_{m-1,n}(z) + \frac{\tau_{m,n}}{\mu_{m,n}}A_{m-1,n}(z) \quad (30)$$

The coefficients of $A_{m,n}(z)$ are the desired AR coefficients at time n for the AR model order m. Thus, the recursive algorithm determines AR coefficient estimates for the desired order p as well as for all lower orders. This added advantage can be helpful in choosing the "best" denominator order for the ARMA model when it is not known a priori.

NUMERICAL EXAMPLE

In order to illustrate the performance of the proposed AR coefficient estimator, a spectral estimation example is presented. For this example the data are generated by

$$x(k) = 20\sin(0.4\pi k) + 2\sin(0.5\pi k) + w(k) \quad (31)$$

where $\{w(n)\}$ is zero mean, unit variance white noise. These data are recognized as being two sinusoids in additive white noise. The signal to noise ratios of the sinusoids are 10dB and 0dB, respectively.

For comparative purposes, three separate algorithms are used: 1) the solution to equation (5) with t=p (Method A) 2) the solution to equation (5) with t>p (Method B) 3) the proposed algorithm (Method C). Methods A and C have fast recursive implementations, and Method B is the estimator that Method C attempts to approximate. For each method, ten separate spectral estimates (each using a different noise realization) are generated to give some indication of the average performance. 100 data points are used for each estimate, the model orders are p=q=4, and for Methods B and C the parameter t is set to 10. Numerator spectrum estimates are obtained by using a technique proposed by Kay [4], omitting the backward estimates.

The resulting spectral estimates are shown in Figure 1. It can be seen that Methods B and C provide substantially better estimates than does Method A. Methods A and C have fast recursive implementations, and the improvement in performance for Method C over Method A is obtained at a cost of 20 additional multiplies and 20 additional adds per time interval.

Other tests performed to date indicate that the proposed algorithm generally provides more accurate parameter estimates than Method A if t is chosen greater than p. However, unlike Method B, whose estimates generally improve as t is increased, the proposed algorithm seems to perform no better for t above around 3p. In fact, coefficient estimates sometimes become worse if t is made too large.

CONCLUSIONS

An algorithm for recursively estimating the AR coefficients in an ARMA model has been presented. Unlike other algorithms of this type, the proposed method is based on solving an overdetermined system of Extended Yule Walker equations. The algorithm may be implemented using a lattice filter, and its computational requirement is 18p + 2t multiplies and 9p + 2t adds per update.

Simulation examples indicate that the proposed algorithm is able to provide more accurate spectral estimates than the t=p fast recursive algorithm (Method A) at a cost of only 2t extra multiplies and 2t extra adds per update. It should be cautioned, however, that extensive further testing along with theoretical investigation are needed to make general claims. Some topics that warrant further investigation include: 1) dependence of the algorithm on t and the possible optimum choice for this parameter, 2) the addition of an exponential forgetting factor and its effect on performance, and 3) a theoretical analysis of the convergence properties, 4) the derivation of an effective MA coefficient estimator to use in conjunction with this or other recursive AR coefficient estimators.

It is important to note that while this algorithm is presented for ARMA models, setting q=0 causes it to become an estimator for the autoregressive model. It was pointed out in [1] that improved AR coefficient estimation can be obtained by using an overdetermined set of Normal Equations. The proposed algorithm with q=0 may enable this improvement to be realized for recursive AR identification problems as well.

REFERENCES

1 J. A. Cadzow and R. L. Moses, "An Adaptive ARMA Spectral Estimator", Proc. Spectral Estimation Workshop, August 17-18, 1981, Hamilton, Ontario.

2 D. Lee and M. Morf, "Recursive Square Root Ladder Estimation Algorithms", Proc. 1980 IEEE Conf. Acoust., Speech, and Sig. Proc., Denver, CO, pp. 1005-1017.

3 K. Ogino, "Computationally Fast Algorithms for ARMA Spectral Estimation", Ph.D. Dissertation, VA Poly. Inst. & State Univ., June 1981.

4 S. Kay, "A New ARMA Spectral Estimator", IEEE Trans. Acoust., Speech, and Sig. Proc., Vol. ASSP 28, No. 5, October 1980, pp. 585-588.

5 M. Kaveh and S. Bruzzone, "A Comparitive Overview of ARMA Spectral Estimation", Proc. Spectral Estimation Workshop, August 17-18, 1981, Hamilton, Ontario.

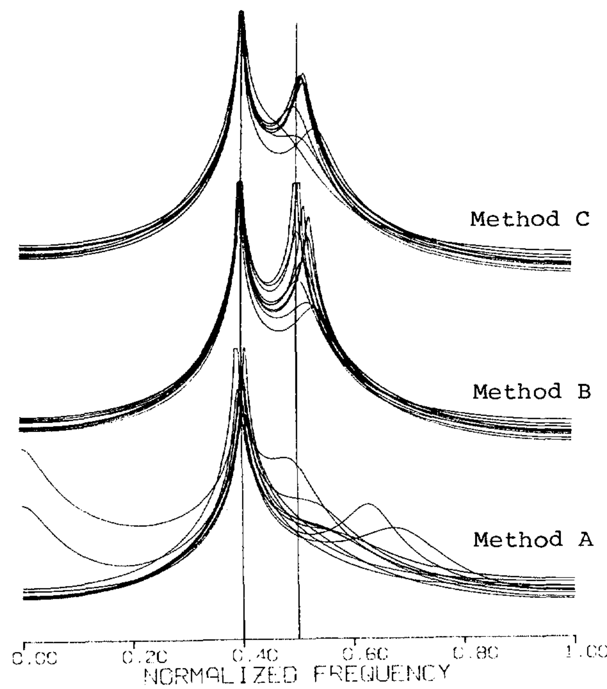

Figure 1: Spectral Estimates for Two Sinusoids in Additive White Noise