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DETERMINING THE CLOSEST STABLE POLYNOMIAL TO AN UNSTABLE ONE

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Abstract

This paper considers the following problem: given a polynomial whose zeros do not all lie on or inside the unit circle, find the "closest" polynomial whose zeros are all on or inside the unit circle. The measure of closeness used is the Euclidean distance in coefficient space. The direct formulation of this problem leads to a minimization problem with nonlinear constraints, and direct solution is difficult. We approach the problem by considering a related minimization problem with linear constraints. We then hypothesize that only a finite number of solutions to the linear problem are candidate solutions to the given nonlinear problem. While a general proof of the hypothesis has not been found, numerical examples indicate that it may hold for a large number of cases.

I. Introduction

In AR and ARMA modeling problems, one often obtains an estimate of an autoregressive (denominator) polynomial. Depending on the particular estimator used, this polynomial may or may be "stable"; that is, it may or may not have all its zeros inside the unit circle. Examples of AR estimators which do not guarantee stability include the covariance and prewindow methods [1], and most singular value decomposition-based methods [2]. In addition, nearly all noniterative methods of ARMA modeling first estimate the AR coefficients by using some form of the extended Yule-Walker equations; these methods almost never guarantee that the estimated AR polynomial is stable [3,2,1].

Many applications require that the estimated denominator polynomial (either from AR or ARMA modeling) be stable. Such is the case in speech synthesis problems and system identification applications, for example. If one must use an algorithm which does not ensure stability, the following problem is of interest: given a polynomial whose zeros are not all inside the unit circle, find a "close" polynomial whose zeroes are all inside the unit circle. We call this the stabilization problem.

There are several ways to stabilize an unstable polynomial. One method is to find the zeros of the unstable polynomial, and if any zero has magnitude greater than one, change it to have magnitude equal to (or slightly less than) one. In this case the stable polynomial is "close" to the original one in the sense of minimizing a distance measure based on the zero locations of the polynomials. Another method based on the Schur parameters associated with a polynomial could be used: find the Schur parameters of the given polynomial (using the Levinson-Durbin recursions), and change any Schur parameter with magnitude greater than one to one which is (slightly less than) one in magnitude [4].

This paper considers solutions to the stabilization problem that minimize an error in coefficient space. The reason for working in coefficient space is that most algorithms which estimate these polynomials actually estimate the coefficients of the polynomials. Since the coefficients are being estimated, it is natural to stabilize the polynomial by perturbing the coefficients as little as possible. Moreover, asymptotic variance expressions for these coefficient estimates have been obtained for several algorithms [5,1]; the asymptotic covariance matrix can be used as a weighting matrix in a weighted coefficient norm to form the distance measure. A stable polynomial whose (weighted) distance from the given polynomial is minimum has the interpretation of a minimum variance solution to the stabilization problem.

II. Problem Statement

Assume we are given the real vector $b = [b_1, \ldots, b_n]^T$, and that its associated polynomial

$$B(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n \tag{1}$$

has at least one zero z_0 satisfying $|z_0| > 1$. We are interested in finding another vector $a = [a_1, \dots, a_n]^T$ such that its associated polynomial

$$A(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n, \tag{2}$$

has all its zeros z satisfying $|z| \leq 1$, and which is close to $B(z)$ in some sense. The measure of error we will use is the standard Euclidean distance

$$J = \sum_{i=1}^{n}(a_i - b_i)^2 \tag{3}$$

Consider the set of coefficients corresponding to stable polynomials:

$$S = \{a | A(z) = 0 \Rightarrow |z| \leq 1\}. \tag{4}$$

The stabilization problem can then be stated as follows:

> **Problem SP:** *Given a vector $b \notin S$, find the vector $a^\circ \in S$ such that $J = (a^\circ - b)^T(a^\circ - b)$ is minimized over all $a \in S$.*

III. Characterization of the Stability Set

In order to solve the above stabilization problem, it is useful to first first establish some basic properties of the stability set S.

Theorem 1:

a) *S is a closed, compact, subset of R^n.*

b) *Let B_S denote the boundary of S. Then if $a \in B_S$, there is at least one zero z_0 of (2) satisfying $|z_0| = 1$.*

Note that S is not necessary a convex set, so in general there may not be a unique solution to the stabilization problem. Figure 1 shows the stability region for $n = 2$.

Theorem 1 provides a means of obtaining candidate solutions to the stabilization problem. Since S is closed, any solution to the stabilization problem will lie on the boundary of S, so its corresponding polynomial $A(z)$ will have at least one zero on the unit circle. We can write this zero as $z = e^{j\omega}$. From equation (2), we have

$$\begin{aligned} c(a,\omega) &= 0 & (5)\\ c(a,\omega) &\triangleq \cos n\omega + a_1 \cos(n-1)\omega + \cdots + a_n, & \\ s(a,\omega) &= 0 & (6)\\ s(a,\omega) &\triangleq \sin n\omega + a_1 \sin(n-1)\omega + \cdots + a_{n-1}\sin\omega & \end{aligned}$$

If $\omega = 0$ or $\omega = \pi$, then $s(a,\omega) \equiv 0$.

Note that for $\omega \in (0,\pi)$, equations (5) and (6) each represents a hyperplane of dimension $(n-1)$ in the coefficient space. The intersection of these two hyperplanes is also a hyperplane of dimension $(n-2)$. Let H_1 be the intersection of these two hyperplanes when $\omega \in (0,\pi)$, and be the hyperplane (5) when $\omega = 0$ and $\omega = \pi$. Then we have the following theorem:

Theorem 2: *If $a \in H_1$, then either $a \in S^c$ (the complement of S), or $a \in B_S$.*

Thus, even though the set S is not convex, for any frequency ω, the corresponding hyperplane H_1 intersects S only on the boundary. The theorem is easily proven by noting that at every point a in the interior of S, the corresponding polynomial $A(z)$ can have no zeros on the unit circle.

IV. A Related Minimization Problem

In general, if a° is a solution to the stabilization problem, then $A^\circ(z) = 0$ for k distinct frequencies $\omega_1^\circ, \dots, \omega_k^\circ$ in $[0,\pi]$. It follows that a° lies on the intersection of k hyperplanes defined by equations (5) and (6), for $\omega \in \{\omega_1^\circ, \dots, \omega_k^\circ\}$. Consider the following minimization problem:

> **Problem MP:** *Find $a^* \in R^n$ to minimize J under the constraints $c(a,\omega) = 0$, and $s(a,\omega) = 0$, for $\omega \in \{w_1, \dots, \omega_k\}$.*

The above problem is readily solved using the Lagrange method. To this end, define the functional

$$\begin{aligned} J_k(\underline{\omega}) &= \sum_{i=1}^{n}(a_i - b_i)^2 + 2\sum_{i=1}^{k} A_i c(a,\omega_i) \\ & \sum_{i=1}^{k} B_i s(a,\omega_i) \end{aligned} \tag{7}$$

where

$$\underline{\omega} = [\omega_1, \dots, \omega_k]^T$$

and where A_i and B_i are Lagrange multipliers.[1] For each $\underline{\omega}$, minimization of $J_k(\underline{\omega})$ gives the point $a^* \in R^n$ which is closest to b under the constraint that $a^* \in H_1 \cap \cdots \cap H_k$, where

$$H_i = \{a | c(a,\omega_i) = 0 \text{ and } s(a,\omega_i) = 0\}$$

By Theorem 2, this point lies either on the boundary of S or in the compliment of S. We are interested in values of $\underline{\omega}$ for which $a \in B_S$, because these points are candidate solutions to the stabilization problem SP.

The solution to the minimization problem, for a given $\underline{\omega}$ vector, is readily found by solving a set of linear equations. These equations can be written as

$$\begin{bmatrix} I_n & M \\ M^T & 0 \end{bmatrix} \begin{bmatrix} a \\ C \end{bmatrix} = \begin{bmatrix} b \\ -d \end{bmatrix} \tag{8}$$

[1] If $\omega = 0$ or π, the constraints corresponding to the B_i multipliers are always satisfied. These constraints are omitted from J_k for $\omega = 0$ or π, and corresponding changes are made in equations (8)–(10).

where

$$M^T = \begin{bmatrix} \cos(n-1)\omega_1 & \cos(n-2)\omega_1 & \cdots & 1 \\ \sin(n-1)\omega_1 & \sin(n-2)\omega_1 & \cdots & 0 \\ \vdots & & & \vdots \\ \cos(n-1)\omega_k & \cos(n-2)\omega_k & \cdots & 1 \\ \sin(n-1)\omega_k & \sin(n-2)\omega_k & \cdots & 0 \end{bmatrix}$$

$$C = [A_1, B_1, \ldots, A_k, B_k]^T$$

$$d = [\cos n\omega_1, \sin n\omega_1, \ldots, \cos n\omega_k, \sin n\omega_k]^T$$

The solution to equation (8) is given by

$$a^* = b - M(M^TM)^{-1}f \tag{9}$$

$$J_k^*(\underline{\omega}) = (b-a^*)^T(b-a^*) = f^T(M^TM)^{-1}f \tag{10}$$

$$f = [c(b,\omega_1), s(b,\omega_1), \ldots, c(b,\omega_k), s(b,\omega_k)]^T.$$

The solution a^* to the linear minimization problem will not, in general, be the solution to the stabilization problem. However, for certain values of k and $\underline{\omega}$, the MP solution a^* will be the optimum solution a° to SP.

Finally, if a multiple zero constraint is needed for a frequency ω, two additional constraint hyperplenes (only one if $\omega = 0$ or π) can be obtained by using the derivative hyperplanes $\partial/\partial\omega c(a,\omega) = 0$ and $\partial/\partial\omega s(a,\omega) = 0$. These constraints are incorporated into the minimization problem in the same manner as described above.

V. Solution to the Problem

We are interested in solving the stabilization problem SP by considering certain solutions to the linear minimization problem MP. Because the solution to MP is parameterized on a continuous variable $\underline{\omega}$, it is impractical to check every solution to MP for optimality in SP. Thus, it is important to find restrictions on the set of MP solutions which need to be checked. To this end, we have the following:

Conjecture: *Assume a° is a solution to SP, and that the corresponding polynomial $A^\circ(z)$ has zeroes at values $z_i = e^{\pm j\omega_i^\circ}$ for (distinct) frequencies $\omega_1^\circ, \ldots, \omega_k^\circ$. If $\omega_i^\circ \in (0,\pi)$, then J_k (as a function of ω_i) has a local minimum at ω_i°.*

This conjecture states that a necessary condition for a solution of MP to be a solution to SP is that the functional J_k be at a local minimum. Since there are only a finite number of these local minima, there is a finite number of solutions to MP which are candidate solutions to SP.

The above conjecture can be proven for $n = 2$, and also at points a° where the set S is locally convex. The proof for the general case has not yet been found. However, in all examples considered to date, a brute-force method for finding a° has verified that the conjecture holds. To date, we have checked about 30 examples using n =2–5.

Assuming the conjecture is true, the following algorithm can be used to find the solution a° to the stabilization problem SP. First, consider $J_1(\omega)$, and find any minimal points. Because a simple, closed-form expression for J_k is available, this minimization is not difficult to perform. At each minimum, determine the corresponding solution a^* to equation (9), and check if $a^* \in S$ (using, for example, the Levinson-Durbin algorithm; see [6]). If this point is in S, then $J_1^*(\omega)$ is the value of the error J in equation (3) for this point. If no such points are in S, then repeat the procedure for J_2. Continue until a stable minimum is found.

VI. Examples

Below we present some examples which illustrate the theory discussed above. The solution a° for these examples are described by using the conjecture in the previous section; however, in each case, this solution was verified to be the solution to SP by using a brute force method to check all points $a \in B_S$ for optimality. Thus, for these examples, the conjecture has been verified.

Example 1: $n = 2$

All of the examples for $n = 2$ are quite simple, because there are few cases to consider. Moreover, the minimum point to the stabilization problem SP can be found immediately by inspection of Figure 1. The hyperplane constraint for $\omega = 0$ is the line defined by $1 + a_1 + a_2 = 0$; similarly, for $\omega = \pi$, one obtains the line $1 - a_1 + a_2 = 0$. For each $\omega \in (0,\pi)$, the intersection of the two hyperplane constraints give a point on the third side of the region in Figure 1. In the case that one hyperplane does not give an admissible solution (as is the case when the point is in regions D, E, or F in Figure 1, then two constraints must be used. The only three cases to consider are: $\omega = 0, 0$; $\omega = 0, \pi$; and $\omega = \pi, \pi$. These cases give the three corner points of the region.

Example 2: $n = 3$, $b = [0.8, 1.2, 1.1]^T$

Figure 2 shows $J_1^*(\omega)$ for this case. This curve has a local minimum at $\omega^* = 91.01°$, and $J_1^*(\omega^*) = 0.00112$ at this point. The corresponding solution to equation (9) gives:

$$a^* = [.8181, 1.0277, .7827]^T$$

The local minima for the single constraint $\omega = 0$ or $\omega = \pi$ give unstable solution points. Thus, the above solution point is the optimum point a°. Note that the zeroes of $B(z)$ are at $0.0357 \pm 1.1229j$ and -0.8714. The solution obtained by moving the two unstable roots to $0.0318 \pm 0.9995j$ gives a cost of $J = .1176$, so this solution is suboptimal.

Example 3: $n = 4$, $b = [-2.0213, 1.3179, 1.7524, -1.6200]^T$

For this case, $B(z)$ has zeros at 0.8, -0.9, $1.5e^{j\pm\pi/4}$. No minima of J_1^*, J_2^*, or J_3^* gave a stable solution. The curve for $J_4^*(0,\pi,\omega)$ is shown in Figure 3. The minimum occurs for $\omega^* = 19.091°$, and the corresponding solution is:

$$a^* = [-1.89, 0.0, 1.89, -1]^T \quad J_4^*(0,\pi,\omega^*) = 2.1574$$

Note that the solution has all four zeros on the unit circle, even though two of the zeros of $B(z)$ were inside the unit circle.

VII. Conclusions

We have considered the problem (SP) of finding the closest stable polynomial to a given unstable one. The measure of error between these two polynomials is the Euclidean distance in coefficient space. We approached the problem by considering a related minimization problem (MP) with linear constraints. The latter problem can be solved in closed form given a set of constraint frequencies. We then considered only those solutions to MP which could also be candidate solutions to the desired minimization problem. We hypothesized that there is only a finite number of solutions to MP which can also be solutions to SP; this, in turn, provided an algorithm for finding the solution to SP. While numerical examples suggest that the conjecture may be true in a large number of cases, it remains an open problem to find under what conditions this conjecture is valid.

VIII. References

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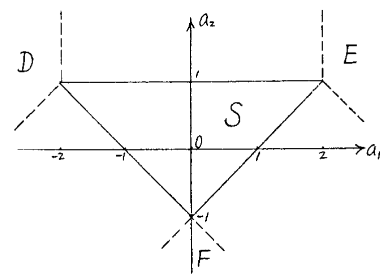

Figure 1: Stability Region S for $n = 2$.

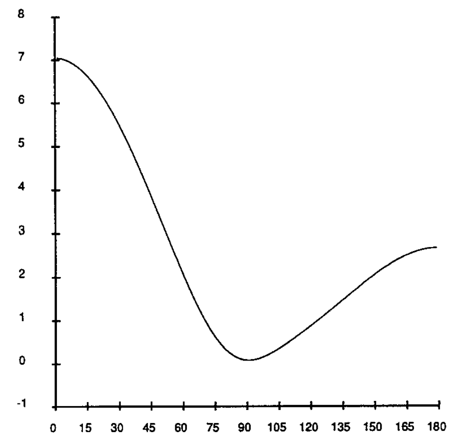

Figure 2: $J_1^*(\omega)$ for Example 2

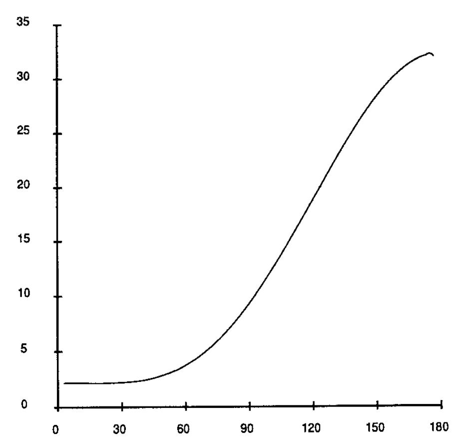

Figure 3: $J_4^*(0, \pi, \omega)$ for Example 3