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ON NON-NEGATIVE DEFINITENESS OF ESTIMATED MOVING AVERAGE AUTOCOVARIANCE SEQUENCES

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Abstract

This paper considers the following problem: Given a normalized (finite-duration) covariance sequence which is not nonnegative definite (NND), find the closest NND sequence to it. Here, closeness is measured by the Euclidean distance in coefficient space. We provide a solution to this problem by considering a set of constrained minimization problems. The solution to the constrained minimization problem does not in general give NND solutions. Properties of NND solutions are established, and used to find the minimizing NND sequence.

I. Introduction

There are many problems in which one is interested in obtaining a parametric model of the spectrum of a time series. The autoregressive (AR), moving average (MA), and autoregressive moving average (ARMA) models are widely used in many engineering problems. In obtaining MA and ARMA spectral estimates, a problem which often arises is that of ensuring that the resulting spectral estimate is nonnegative definite (NND), that is, that the spectral estimate is nonnegative and real on the unit circle [1]. For example, a commonly used method of MA spectral estimation is to estimate the first $n+1$ autocovariances γ_k of a time series from some measurements of that time series. The corresponding spectral estimate is the Fourier transform of the estimated γ_k sequence. Depending on the estimator used for γ_k, the spectral estimate may not be nonnegative and real for all frequencies. A similar problem occurs in ARMA spectral estimation algorithms in which the AR parameters are estimated in a first step, and the MA part of the spectrum is estimated using the AR coefficient estimates [2,3,4,5].

If the MA part of the spectral estimate is not NND, there are various ways in which one can alter the estimate to make it NND. The most common procedure entails multiplying the estimated autocovariances by some window sequence (such as the Bartlett window or an exponential window) [5,1]. For some estimates, the window can be chosen in such a way as to guarantee NND estimates; however, such a window imposes a severe bias on the resulting estimate [1]. An alternate approach is to use a data adaptive window, in which a parameter in the window is chosen to ensure NND estimates, with a minimum of bias for that particular window. An exponential window $w_k = \alpha^{|k|}$ with adjustable α is an example; α is chosen as small as possible so that the sequence $\{\alpha_k \gamma_k\}$ is NND. While the second method is less biased than the first method, it is also suboptimal in the sense that the a particular (suboptimal) window structure is used.

In this paper we consider an alternate approach to obtaining a NND covariance sequence. Given an estimated covariance sequence of a MA time series, we wish to find the optimal NND sequence to that estimate, where optimality is measured in terms of the ℓ_2 error norm in coefficient space.

II. Problem Statement

Let $\{\gamma_k\}_{k=0}^{n}$ denote a sequence of real-valued estimates of the autocovariances of an MA(n) process. Consider the function

$$S_\gamma(z) = \sum_{k=-n}^{n} \gamma_{|k|} z^{-k}. \tag{1}$$

In order to ensure that $S_\gamma(z)$ is a valid spectral density function, we must have

$$S_\gamma(z) \geq 0 \text{ on } |z| = 1. \tag{2}$$

It is clear that equation (2) is satisfied if $\gamma_0 > 0$ and if

$$g(\omega) = 1 + s_1 \cos\omega + \cdots + s_n \cos n\omega \geq 0 \tag{3}$$

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for $\omega \in [0, \pi]$, and where $s_k = \gamma_k/\gamma_0$. (We will not consider the trivial case $\gamma_k \equiv 0$). Nearly all covariance estimators guarantee that $\gamma_0 > 0$, but often do not guarantee that equation (3) is satisfied.

Assume condition (3) is not satisfied. In this case, we are interested in finding a covariance sequence which is NND and which is close to the given sequence. To this end, let $\rho = [\rho_1 \ldots \rho_n]^T$ and define

$$f(\omega, \rho) = 1 + \rho_1 \cos\omega + \cdots + \rho_n \cos n\omega. \tag{4}$$

Define the nonnegative definite set D by

$$D = \{\rho | f(\omega, \rho) \geq 0 \text{ for } \omega \in [0, \pi]\}.$$

Then the problem (P) of finding the closest NND sequence can be stated as follows:

> **Problem P:** *Given a vector $s = [s_1 \ldots s_n]^T$ with $s \notin D$, find the vector $\rho \in D$ such that $Q = (\rho - s)^T(\rho - s)$ is minimized.*

III. Description of the NND Region

The above minimization problem is nontrivial because the set D is a complicated function of the ρ vector. In order to approach the minimization problem, we first establish some properties of D. Some of these properties are summarized below.

Theorem 1:

- **a)** *D is a closed, compact, convex subset of R^n.*
- **b)** *Let B_D denote the boundary of D. Then if $\rho \in B_D$, there is at least one $\omega_0 \in [0, \pi]$ such that $f(\omega_0, \rho) = 0$.*
- **c)** *If $\rho \in B_D$, then for each ω_0 such that $f(\omega_0, \rho) = 0$, $f'(\omega_0, \rho) \triangleq \frac{d}{d\omega} f(\omega_0, \rho) = 0$.*
- **d)** *There is a unique solution ρ^* to the minimization problem P.*

Figure 1 shows the region D for $n = 2$. Also shown are the frequency values for which $f(\omega, \rho) = 0$ at certain boundary points ρ. Note that at the corner point $\rho = [0, 1]$, $f(\omega, \rho) = 0$ for both $\omega = 0$ and $\omega = \pi$; for all other boundary points, $f(\omega, \rho) = 0$ for only one value of ω. These characteristics generalize to higher orders. If a point $\rho \in B_D$ has k zero frequencies (i.e. $f(\omega, \rho) = 0$ for $\omega \in \{\omega_1, \ldots, \omega_k\}$) then ρ lies on at least k supporting hyperplanes, each defined by $H_i = \{\rho | f(\omega_i, \rho) = 0\}$. Moreover, all points $\rho \in B_D$ which have zeroes at these k frequencies are in the set $H_1 \cap H_2 \cdots \cap H_k$. The intersection of these hyperplanes is itself a hyperplane, which is of dimension $(n - k)$. Thus, we have the following theorem:

Theorem 2: *Consider the set*

$$H = \{\rho | f(\omega, \rho) = 0 \Rightarrow w \in \{w_1, \ldots, \omega_k\}\ \}$$

Then

$$H = H_1 \cap H_2 \cdots \cap H_k \cap D$$

where H_i is the hyperplane defined by

$$H_i = \{\rho | f(\omega_i, \rho) = 0\}.$$

IV. Solution to the Minimization Problem

If ρ^* is a solution to the constrained minimization problem, there is at least one ω^* such that $f(\omega^*, \rho^*) = 0$. Assume $f(\omega, \rho^*) = 0$ for k distinct frequencies $w_1, \ldots, w_k$, and consider the following functionals:

$$\begin{aligned} Q_k(\underline{\omega}) &= \sum_{i=1}^{n} (\rho_i - s_i)^2 \\ &\quad + 2\sum_{i=1}^{k} A_i (1 + \rho_1 \cos\omega_i + \cdots + \rho_n \cos n\omega_i) \end{aligned} \tag{5}$$

where $\underline{\omega} = (\omega_1, \ldots, \omega_k)$. Each A_i is a Lagrange multiplier. Let $Q_k^*(\underline{\omega})$ denote the minimum of $Q_k(\omega)$ for that frequency. For each $\underline{\omega}$, minimization of $Q_k(\omega)$ gives the point ρ which is closest to s under the constraint that ρ lies on the hyperplane $H_1 \cap \cdots \cap H_k$. Moreover, $Q_k^*(\underline{\omega}) = (\rho - s)^T(\rho - s)$ for this point ρ. The point ρ_1 in Figure 1 shows this minimum for $n = 2$ and $\omega = \pi/4$. It is important to note that ρ does not necessarily lie in D; if it does lie in D, it lies on the boundary B_D. We are only interested in those values of ω for which the corresponding points ρ are in B_D.

Once $\underline{\omega}$ is fixed, the minimization of $Q_k(\underline{\omega})$ reduces to the solution of a set of linear equations. These equations can be expressed as:

$$\begin{bmatrix} I_n & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \rho \\ A \end{bmatrix} = \begin{bmatrix} s \\ -\ell \end{bmatrix} \tag{6}$$

where

$$\begin{aligned} C^T &= \begin{bmatrix} \cos\omega_1 & \cos 2\omega_1 & \cdots & \cos n\omega_1 \\ \vdots & \vdots & & \vdots \\ \cos\omega_k & \cos 2\omega_k & \cdots & \cos n\omega_k \end{bmatrix} \\ A &= [A_1 \cdots A_k]^T \\ \ell &= [1 \cdots 1]^T \end{aligned}$$

The solution to equation (6) is given by:

$$\rho^* = s - C(C^T C)^{-1} g \tag{7}$$

$$A^* = (C^T C)^{-1} g \tag{8}$$

$$Q_k^*(\underline{\omega}) = g^T (C^T C)^{-1} g \tag{9}$$

where

$$g = [g(w_1) \cdots g(\omega_k)]^T.$$

If the frequencies $\omega_1, \cdots, \omega_k$ are known, the point ρ^* which minimizes $Q_k(\underline{\omega})$ can readily be found from equation (7). Thus, the solution to the minimization problem P reduces to finding these frequencies. To this end, the following theorems are of interest.

Theorem 3: *Let ρ^* be the solution to the minimization problem P, and let ω^* be any frequency for which $f(\omega^*, \rho^*) = 0$. If $\omega^* \in (0, \pi)$, then $g(\omega^*) \leq 0$.*

Theorem 3 provides restrictions to the range of ω values for which NND solutions exists. This theorem can be proven using an argument along the following lines. Consider any separating hyperplane $f(\omega^*, \rho^*) = 0$ at ρ^*. Assume also that $g(\omega^*) > 0$. Then s lies on the same side of the separating hyperplane as D (see Figure 2). Since D is convex, it follows that there is a point $\rho \in D$ which is closer to s than ρ^* is; this is a contradiction, so $g(\omega^*) \leq 0$. Note that for $\omega^* = 0$ or $\omega^* = \pi$, the boundary is not smooth, and the above argument no longer holds. It turns out, though, that for $n = 2$, the result of the theorem is also true when $\omega^* = 0$ and $\omega^* = \pi$.

Theorem 4: *Let ρ^* be the optimum solution to the minimization problem P, and let ω^* be any frequency for which $f(\omega^*, \rho^*) = 0$. Then $\sum_{i=1}^{k} A_k(\omega^*) \leq 0$.*

By manipulation of equations (7)-(9), it can be shown that $\sum_{i=1}^{k} A_k(\omega^*) = -\rho^{*T}(s - \rho^*)$. Now, since $0 \in D$, it can be shown that $\rho^{*T}(s - \rho^*) \geq 0$, and the result follows immediately.

Theorem 5: *Let ρ^* be a solution to the minimization problem, and let $\{w_1^*, \ldots, \omega_k^*\}$ be the set of all frequencies for which $f(\omega, \rho^*) = 0$. Then the functional $Q_k^*(\underline{\omega})$ in equation (9) has a local maximum at the point $\underline{\omega} = (\omega_1^*, \ldots, \omega_k^*)$.*

The proof of Theorem 5 makes use of Theorem 2 along with some properties of convex sets [6].

An algorithm for finding the solution ρ^* to the minimization problem proceeds as follows. First, by considering $g(\omega)$, and using Theorems 3 and 4, we can determine the ranges of possible frequencies where $f(\omega, \rho^*) = 0$. From this information, we can obtain a finite set of possible zero distributions corresponding to ρ^*. The number of possibilities depends on order n and on the particular sequence s. For each case, the appropriate functional Q_k^* is formed, and its local maxima are found. A simple, closed-form expression for Q_k^* is available, so this maximization is not difficult. Each local maximum gives a corresponding point ρ. Of these points, the one which lies on the boundary B_D and whose corresponding Q_k^* value is minimum is the solution to the minimization problem P.

V. Examples

Below we consider three examples which illustrate the theorems in the paper and the resulting algorithm for finding the solution to the minimization problem. Note that in all examples, $f(\omega)$ and $g(\omega)$ are *normalized* spectral densities; in practice, $f(\omega)$ would be multiplied by a constant to make it agree more closely with $g(\omega)$.

Example 1: $n = 2$, $s = [2, 3]^T$
Figure 3 shows a plot of the functions $g(\omega)$, $Q_1^*(\omega)$, and $f(\omega, \rho^*)$ for this example. Since $g(\omega) \leq 0$ for a range of frequencies which does not include 0 or π it follows from Theorem 1b and Theorem 3 that $f(\omega, \rho^*) = 0$ somewhere in this range. Thus, we need only consider $Q_1^*(\omega)$ for ω in this range. This functional has a local maximum at $\omega = 109.2°$, and the corresponding solution to (7), (9) gives:

$$\rho^* = [1.083, 0.822]^T \qquad Q_1^*(\omega^*) = 5.587.$$

Example 2: $n = 3$, $s = [-2, 0, 0]^T$
Figure 4 shows $g(\omega)$ and $f(\omega, \rho^*)$ for this example. From $g(\omega)$ it is clear that the zeros of $f(\omega, \rho^*)$ will be at 0, π, or $\omega \in (0, 60°]$. There are thus several cases to consider; most cases require computation of a solution for only one point, and two cases require maximization of Q with respect to one variable $\omega \in [0, 60°]$. The global minimum is found for one zero at $\omega = 0$ and one zero at $\omega = 2.467°$, and the resulting $Q^*(\omega)$ function is shown in Figure 4. The corresponding solution is:

$$\rho^* = [-1.369, 0.387, -0.018]^T \qquad Q^*(\omega^*) = 0.549.$$

Note that another NND solution is $\rho = [-1, 0, 0]$, giving $Q = 1$. This is an admissible solution, but it is not optimal.

Example 3: $n = 4$, $s = [2, 3, 4, 5]^T$
Figure 5 shows $g(\omega)$ and $f(\omega, \rho^*)$ for this example. Note that $g(\omega) \leq 0$ for two distinct ranges of ω, neither of which include 0 or π. Thus, by Theorem 3, $f(\omega, \rho^*) = 0$ only on these regions. This gives only two possibilities; either $f(\omega, \rho^*)$ has one complex pair of roots, or two complex pairs. By performing both minimizations, it was found that the global minimum occurs for

$$\rho^* = [1.327, 1.011, 0.830, 0.608]^T \qquad Q^*(\omega^*) = 33.75$$

VI. Conclusions

We have considered the problem of finding the closest nonnegative definite MA covariance sequence to a given estimate which is not nonnegative definite. The solution is based on solving a constrained minimization problem, and finding points in the solution which lie on the boundary of the set of all NND sequences. Based on an analysis of this boundary, we are able to considerably restrict the frequency ranges of consideration. Moreover, we obtained closed-form expressions for functionals, and showed that solution points to the minimization problem could only occur at local maxima of these functionals over a restricted set of frequencies.

Future work will focus on obtaining tighter restrictions on the set of candidate solution points, and by generalizing the results to incorporate a broader class of minimization criteria.

VII. References

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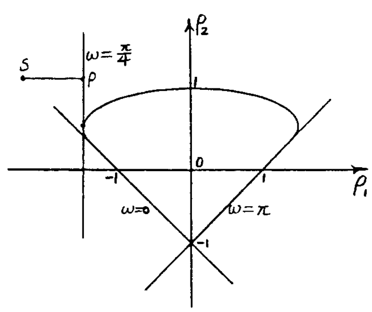

Figure 1: Nonnegativity Region for $n = 2$. The point ρ_1 shown is found by minimizing $Q_1(\pi/2)$.

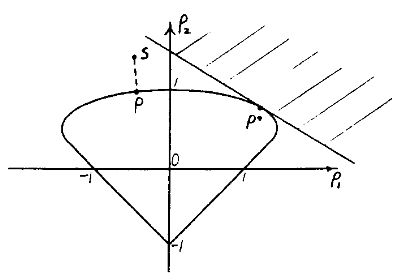

Figure 2: On the Proof of Theorem 3 for $n = 2$. If $g(\omega^*) > 0$, there is a point on D closer to s than is ρ^*.

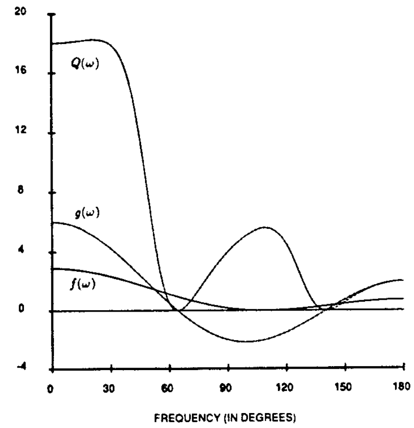

Figure 3: Example 1 Original and NND functions.

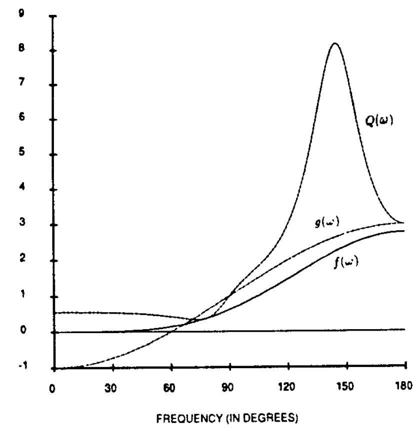

Figure 4: Example 2 Original and NND functions.

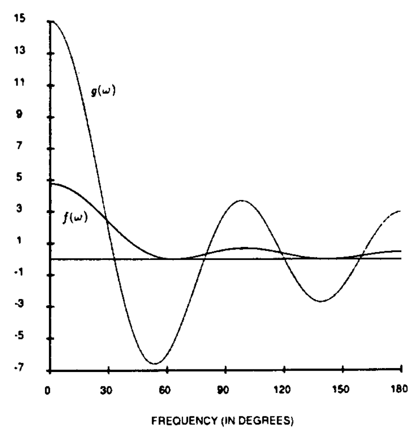

Figure 5: Example 3 Original and NND functions.