NESTED UNIFORM SAMPLING FOR MULTIRESOLUTION 3-D TOMOGRAPHY

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ABSTRACT
A nested uniform sampling over the sphere is presented. The set of sampling points contains a sequence of nested subsets, each providing an approximately uniform covering of the sphere. The proposed Matryoshka covering of the sphere allows multiresolution 3-D imaging. Simulation and electron paramagnetic resonance imaging results illustrate its application to 3-D tomography.

Index Terms—3-D tomography, Radon inversion, Fekete points, binning spherical codes, progressive sampling

1. INTRODUCTION
The problem of generating uniform samples over a sphere has been studied extensively and arises in many scientific fields including robotics, computational biology, computational electromagnetics, and 3-D tomography. For example, sampling based on spherical t-design [1] providing exact integration over the sphere for any integrand expressible by spherical harmonics of degree t or less has been reported. Deterministic sampling schemes using Platonic solids have also been reported. Likewise, generating uniform distributions of points on the sphere by optimizing a generalized energy [2] has been a subject of several studies. In contrast to the circle where any arbitrary number of points can be distributed uniformly, a regular tessellation of the sphere is only achieved by the five Platonic solids. Thus, a perfectly uniform distribution of an arbitrary number of points on the sphere has not been achieved and still remains an active area of research.

Multiresolution grids, for $\mathbb{R}^d$, are routinely used for a variety of applications such as numerical optimization, searching, and integration. Only a handful of such grids exist for $S^d$ [3]. To the best of our knowledge, all these progressive sampling methods work only for a geometric growth, i.e., the number of samples from lower resolution level to next higher resolution level increases by a constant multiplicative factor. For practical applications, where each point on the sphere represents a measurement, geometric increase in the data collection time may not be desirable. Here, we present a method of multiresolution sampling that provides nearly uniform distributions across all levels for any growth rate including linear. Such nested sampling is achieved by minimizing a modified generalized energy expression.

The remainder of the paper is organized as follows: Section 2 provides a brief overview of 3-D tomography and its connection to uniform sampling over $S^2$; Section 3 provides basic concept of Fekete points; Section 4 describes the proposed nested uniform sampling; Section 5 gives simulation and experimental results; and Section 6 states conclusions.

2. 3-D TOMOGRAPHY

Tomographic reconstruction commonly arises in many imaging applications especially biomedical imaging. Our work is applicable to all 3-D imaging modalities in which data, in the form of 1-D projections, are collected by Radon transform of a 3-D object. Example applications include MRI for samples with short echo times [4] and electron paramagnetic resonance (EPR) imaging [5].

Mathematically, the Radon transform of a 3-D object can be written as

$$p_{\eta,\phi}(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{u}) \delta(\langle \vec{u}, \vec{\gamma} \rangle - s) \, d\vec{u} \quad (1)$$

where $s$ is the distance of the integration plane from the origin, $p$ is the 1-D projection along the orientation defined by azimuth angle $\eta$ and polar angle $\phi$, $f$ is the 3-D object, $\vec{u} = [x, y, z]^T$, $\vec{\gamma} = [\cos \eta \sin \phi, \sin \eta \sin \phi, \cos \phi]$, (…) represents inner product, and $\delta$ represents Dirac’s delta function.

For the suggested applications, each projection is typically acquired with dense sampling of the variable $s$. Typical methods for reconstruction from these 1-D projections, such as filtered backprojection (FBP), are based on the direct inversion of the Radon transform which reduces to numerical integration over the sphere as expressed below

$$\hat{f}(\vec{u}) = \int_0^{\pi} \int_0^{\pi} \hat{p}_{\eta,\phi}(s) \delta(\langle \vec{u}, \vec{\gamma} \rangle - s) \sin \phi \, ds \, d\eta \, d\phi \quad (2)$$

where $\hat{p}$ represents filtered projection. For 3-D case, $\hat{p} = \frac{\partial^2 p}{\partial s^2}$.
Several factors affect the quality of 3-D reconstruction from measured projections, including the sampling strategy to distribute the projections over the acquisition space. For an object of arbitrary shape and orientation, a more uniform sampling of the sphere provides better numerical integration.

3. FEKETE POINTS

Fekete points [2] are based on minimizing energy \( U \)

\[
U(t, \omega_N) = \frac{1}{2} \sum_{1 \leq j < k \leq N} |v_j - v_k|^t
\]

where \( \omega_N \) denotes a set of \( N \) points, \( v_1, \ldots, v_N \), on the sphere \( S^d \) in \( d + 1 \) dimensions. For \( t = 1 \), \( U \) represents the electrostatic potential energy of \( N \) charged particles on the sphere that repel each other according to Coulomb’s law, and the minimizing set of points \( \omega_N^* \) is called Fekete points, denoted by \( d_0 \). Results of numerically minimizing \( U \) have been widely reported. Also, bounds on numerically minimizing \( U \) have been widely reported [7].

The energy \( U \) can be minimized using a gradient descent method. In each iteration, the repulsive force \( F \), which is the negative of the potential gradient \( -\nabla U \), is calculated on each particle due to the influence of the rest. The particle is then moved along the sphere surface in the direction of force to reduce the net \( U \) of the system. For example, for point \( v_k \), the displacement during \( n \)th iteration can be summarized as

\[
v_k^{(n)} = v_k^{(n-1)} + \beta F_k^{(n)}
\]

\[
v_k^{(n)} = \frac{\tilde{v}_k^{(n)}}{\|\tilde{v}_k^{(n)}\|}
\]

where \( F_k \) represents the force experienced by \( k \)th particle due the rest of particles on the sphere surface, and the constant \( \beta \) controls the displacement step size. The process is repeated for all points in each iteration. A significant limitation of Fekete points set is that the uniformity is retained only when the entire dataset is considered.

4. MULTiresolution Sampling

In this work, we propose a distribution of points on the sphere such that each subset in a nested sequence provides low potential energy and hence a nearly uniform distribution. The proposed nested point sets allow multiresolution data acquisition, whereby image reconstruction progressively improves at each level. Data collection may be terminated if the reconstruction at level \( i \) achieves a prescribed quality criterion.

For \( L \) levels, we denote the nested sets by

\[
\omega_{N(1)} \subset \omega_{N(2)} \subset \cdots \subset \omega_{N(L)}
\]

where \( N(i) \) is the number of points at level \( i \) and \( N(L) \) is the maximum number of points to be distributed. To design a nested sequence of uniformly distributed point sets, we propose to minimize the objective function which is the weighted sum of energy functions for each nesting level.

\[
\sum_{i=1}^{L} \alpha_i U(t, \omega_{N(i)})
\]

where \( \alpha \in \mathbb{R} \) such that \( \alpha_1 > \alpha_2 > \ldots > \alpha_L > 0 \). Relative values of \( \alpha_i \) provide a trade-off among uniformities at various nesting levels. We denote this proposed nested uniform distribution by \( d_1 \).

To solve the nonconvex minimization of Eq. 6, we adopt a procedure similar to the one presented in Eq. 4, but replace \( F_k \) with \( \sum_i \alpha_i F_{k,i} \), with \( F_{k,i} \) being the force on \( k \)th particle due to rest of particles in \( i \)th subset, and the summation performed over all subsets containing \( k \). This is equivalent to a situation where charged particles in each subset repel each other with a force which is scaled by \( \alpha_i \). Antipodal symmetry, if required, can be maintained by adjusting antipodal points in pairs at each descent step. Clearly, the nesting requirement implies that \( U(t, \omega_{N(L)}) \geq U(t, \omega_{N(L)}} \), and hence may not minimize the potential energy in Eq. 3. By numerical simulation, we demonstrate that every nested subset provides a sampling with surprisingly small impact on the uniformity at each of the \( L \) levels. For larger values of \( L \), however, we expect the added nonuniformity of the nested sampling to increase.

Fig. 1. Sampling over a unit sphere. Levels 1 to 3, from left to right, contain 140, 280, and 420 projections, respectively. Distributions \( d_0 \), \( d_1 \), and \( d_2 \) are displayed from top to bottom, respectively.

Figure 1 shows three distribution schemes for \( L = 3 \) and \( N(i) = 140i \). We selected \( \alpha_1 = 1 \), \( \alpha_2 = 1/3 \), and \( \alpha_3 = 1/9 \).
The antipodal symmetry was maintained. The first row shows independent Fekete point distributions $d_0$ for each level. The uniformity is maximized at each level but the data points from the previous level are not used. Reconstructions based on this distribution are treated as gold standard. The second row shows $d_1$ that is described in this paper where more points are added to the distribution at a previous level without losing the uniformity. For the sake of comparison, we consider another distribution where independent uniform distribution, at each next level, is added to the distribution from the previous level. The data points from previous levels are retained, but distribution uniformity is progressively lost because a uniform distribution added at each level does not automatically interleave with the distribution from a previous level. Such a distribution is shown in the third row. We donate this union of Fekete point sets by $d_2$.

For $L = 3$ with $N(i) = 140i$, the uniformity of the distributions, quantified in terms of the potential energy $U$ and standard deviation of Voronoi cells size $w$, is reported in Table 1, while resolution, in terms of peak-sidelobe of the 3-D point spread function of the corresponding sampling scheme, is reported in Table 2. The results are based on five runs. For each run, the distributions $d_0$, $d_1$, and $d_2$ were obtained from different random initializations.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$U \times 10^4$</th>
<th>$\sigma_w$</th>
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<tbody>
<tr>
<td>$d_0$</td>
<td>1</td>
<td>0.8889 ± 0.0000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.6619 ± 0.0004</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.3456 ± 0.0002</td>
</tr>
<tr>
<td>$d_1$</td>
<td>1</td>
<td>0.8897 ± 0.0001</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.6680 ± 0.0002</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.3534 ± 0.0006</td>
</tr>
<tr>
<td>$d_2$</td>
<td>1</td>
<td>0.8888 ± 0.0001</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.7345 ± 0.0317</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.5408 ± 0.0359</td>
</tr>
</tbody>
</table>

Table 1. Potential energy $U$ and the standard deviation of Voronoi cell size $\sigma_w$ for different distributions across three levels for five runs.

5. RESULTS

5.1. Simulation

To demonstrate the performance of the proposed nested sampling on the sphere, we consider tomographic reconstruction from 1-D projections collected by Radon transform of the 3-D Shepp-Logan phantom (MathWorks, MA). Figure 2 shows three orthogonal central slices of the phantom. The reconstruction was carried out for distributions shown in Fig. 1.

The reconstruction was performed using the single-stage FBP method (Eq. 2). Before applying FBP, each projection was weighted according to the Voronoi cell size computed over the sphere surface. Reconstruction results for all three distributions at each of the three levels are shown in Fig. 3. For brevity, only one central slice is selected for the display. The resulting mean square-error (MSE) values are reported in Table 3. The MSE for $d_1$ is comparable to that of $d_0$, while the $d_2$ has considerably higher MSE. Also the reconstruction based on $d_0$ and $d_1$ exhibit visibly lower artifacts compared to $d_2$ especially for $i = 3$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>MSE</th>
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<tbody>
<tr>
<td>$d_0$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
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<td>3</td>
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<td>2</td>
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<td>3</td>
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</table>

Table 3. Mean-square-error in the reconstruction of 3-D Shepp-Logan phantom shown in Fig. 2. The results are based on five runs.
5.2. EPR Imaging

EPR imaging is an emerging medical imaging modality used to detect free radical distributions in biological samples. In principle, EPR is similar to NMR except that it measures the transition of unpaired elections, as opposed to protons. For EPR imaging, data is collected by irradiating the sample with a fixed frequency RF and sweeping the external magnetic field in the presence of magnetic field gradient. Data collection for 3-D EPR imaging complies with Eq. 1. Each measured 1-D projection, however, is blurred by the Lorentzian (or its first derivative) function.

For validation of the technique, an experimental phantom was constructed with 10 capillary tubes arranged in the shape of equilateral triangle as shown in Fig. 4. The tubes were filled to a height of 7 mm with EPR sensitive Triaryl methyl radical TAM (0.7 mM) dissolved in phosphate buffer saline. The phantom was imaged on an L-Band (1.2 GHz) EPR imaging system. The maximum signal-to-noise was maintained at 21 dB. The FBP reconstruction was carried out for distributions shown in Fig. 1. Figure 5 displays central cross-sectional slice of the 3-D reconstructed image for distributions shown in the last column of Fig. 1. The capillary tubes are more distinctly visible for \( d_0 \) and \( d_1 \) as compared to reconstruction based in \( d_2 \).

6. CONCLUSIONS

In this paper, we have proposed a nested, uniform sampling strategy that permits multiresolution 3-D imaging. Perhaps surprisingly, at each level of the nesting the Coulomb energy of the point set is observed to be no more than 0.2% above the minimum energy for an unconstrained point set. Thus, the nesting is achieved with very little loss of uniformity. The uniformity of distribution across all levels translates to lower MSE, lower reconstruction artifacts and higher spatial resolution, as illustrated by simulation and EPR imaging results. The proposed approach is equally applicable to tomography in higher dimensions.

7. REFERENCES


